AweSums

Taster Lecture, 24 August 2022

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AweSums
Preview of Evening Course, Autumn 2022

Outline

Outline

Maths Taster Lecture

WELCOME TO THE

Taster Lecture on Maths

Maths Taster Lecture

In this talk, I'll tell you something about the course

AweSums: Marvels and Mysteries of Mathematics

I will give some examples of the topics and ideas that will be covered in the course.

Also, we will take a preliminary glance at the *AweSums* website

`https://maths.ucd.ie/~plynch/AweSums/2022/`

Meaning and Content of Mathematics

The word Mathematics comes from Greek $\mu\alpha\theta\eta\mu\alpha$ (máthéma), meaning "knowledge" or "lesson" or "learning".

It is the study of topics such as

- *Quantity* (numbers)
- *Structure* (patterns)
- *Space* (geometry)
- *Change* (analysis).

Outline

The *AweSums* Course

The course AweSums will have eight lectures from 3 October to 28 November, 2022.

Splits into two groups of 4+4 lectures.
Sessions on Mondays, 7:00 – 9:00 pm.

No Lecture on 31 October.

The aim of the course is to show you

- The tremendous beauty of mathematics;
- Its great utility in our daily lives;
- The fun we can have studying maths.

The *AweSums* Course

This is the seventh time I have taught a popular maths course.

The course is broadly similar from year to year, but I always include *new material* each time.

In this *Taster Lecture* I will give a sample of some of the topics to be covered in the course.

If there is a topic you'ld like, please let me know. Maybe, i can include it!

IT'S YOUR COURSE!

Some topics in the 2022 Course

- *Golden Ratio.* Visual Maths I.
- Georg Cantor. Set Theory.
- Cutting the Plane. Infinite Sets.
- Special Lecture on Infinity.
- Topology. The Pythagoreans.
- Numerical Weather Prediction.
- *Astronomy.* Pascal's Triangle.
- Euler's Gem. Parity of Rationals.
- Gauss. *Prime Numbers.* Hilbert.
- Möbius Band. Sieve of Eratosthenes.
- Mathematics and Music.

Outline

Beautiful, Useful and Fun

Ludwig van Beethoven and Carl Friedrich Gauss

Beethoven and Gauss

Beethoven (1770–1827). Gauss (1777–1855).

Beethoven's music is accessible to people with no special knowledge of music.

Gauss produced results of singular genius, great utility and deep aesthetic appeal.

But the brilliance and beauty of his work is hidden from most of us.

Maths Takes Time

Music is accessible to all while maths presents greater obstacles.

Music gets into the soul on a high-speed emotional autobahn.

Maths follows a rational, step-by-step route.

Music has instant appeal; maths takes time.

Gauss’s Work Really Matters

The beauty of maths is difficult to appreciate.
Its significance in our lives is often underestimated.

We all benefit from the power of maths to model our world and enable technological advances.

The work of Gauss has a greater impact on our daily lives than the magnificent creations of Beethoven.

Maths is Important and Useful

Mathematics is essential for modern society:

Smart phones, iPads, SatNavs, Computers, Communications, the Internet.

Pharmaceuticals, Air Transport, Weather Forecasts, Agricultural Production.

Forensic Medicine, Crime Detection, Sporting Performance and Equipment.

Maths now reaches into every corner of our lives.

The Goal of AweSums

Maths has great recreational value, with surprising and paradoxical results that are a source of amusement and delight.

The goal of *AweSums* is to elucidate the beauty, utility and fun of mathematics.

We examine its many uses in modern society and also some aspects of pure mathematics.

Outline

A Splendid Spiral in Booterstown

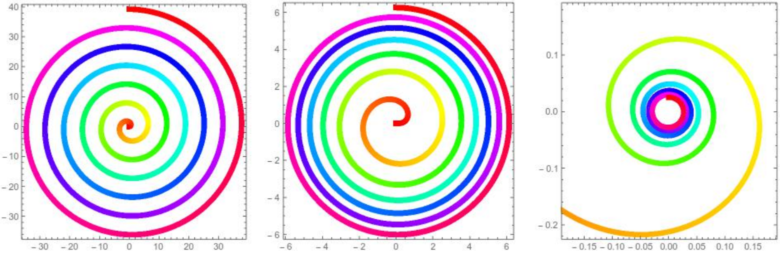
This sandbank, a beautiful spiral form, has slowly built up on the beach near Booterstown Station.

Spirals are found throughout the natural world.

A Splendid Spiral in Booterstown

A recent update (June 2022).

Some Mathematical Spirals

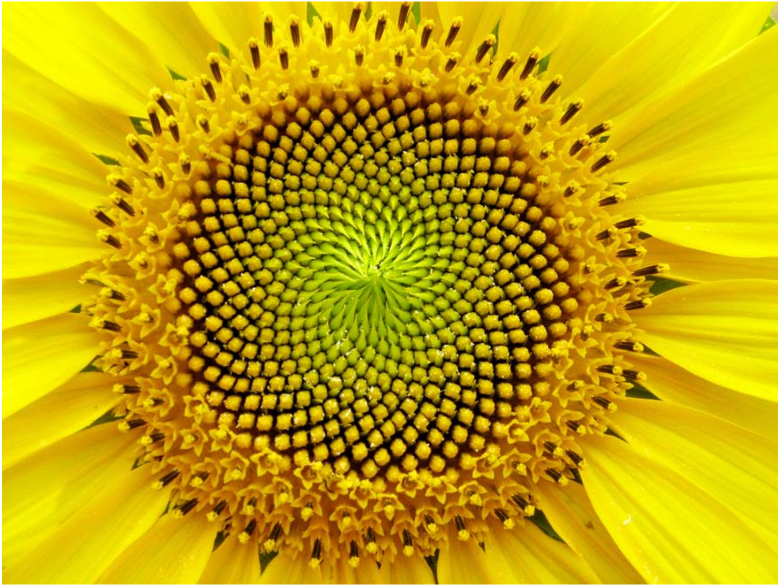


Archimedes Spiral. Fermat Spiral. Hyperbolic Spiral.

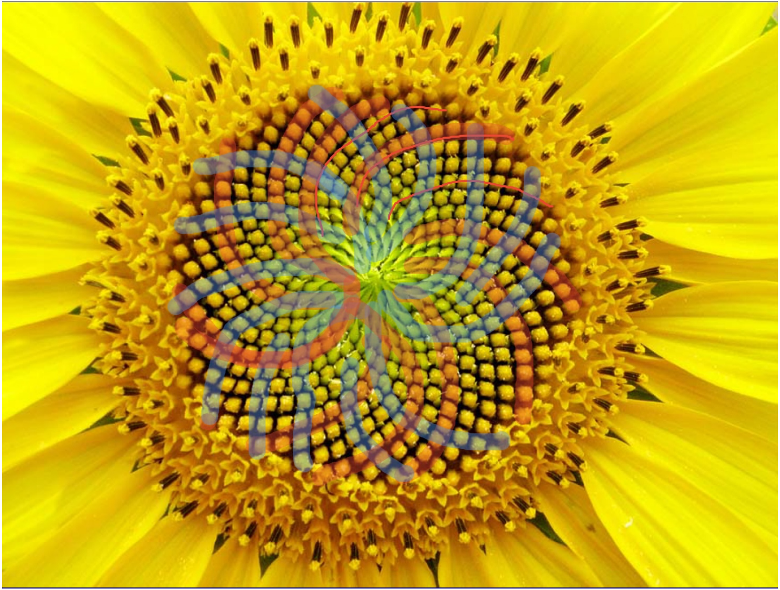
Challenge: Find mathematical equations for these.
Hint: Use polar coordinates (r, θ).

The Nautilus Shell: *a logarithmic Spiral.*

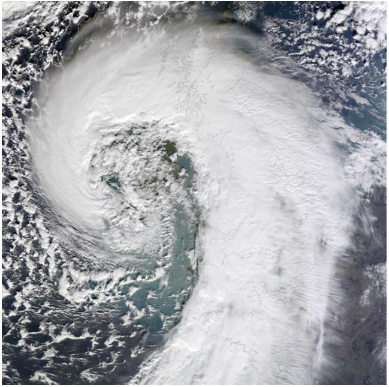
The Sunflower: Groups of Spirals



The Sunflower: Groups of Spirals



Spirals in the Physical World



⋆ ⋆ ⋆

https://thatsmaths.com/

[Search for "Spirals"]

Fibonacci Numbers

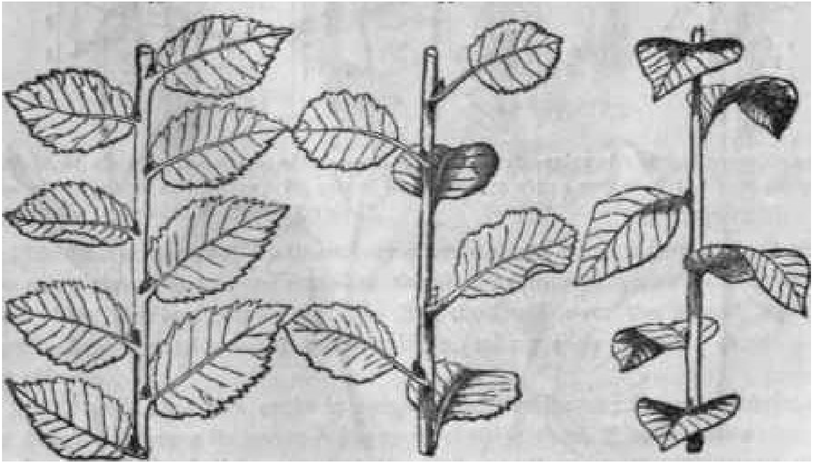
- Count the petals on a flower.
- Count leaves on a stem or
  bumps on an asparagus.
- Look at patterns on pineapples/pine-cones.
- Study the geometry of seeds on sunflowers.

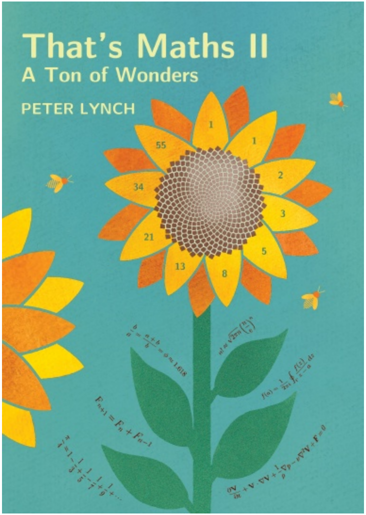
In all cases, we find numbers in the sequence:

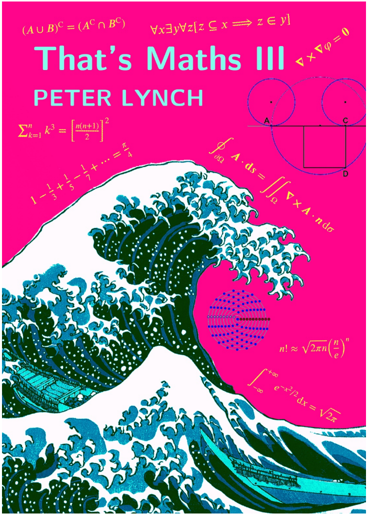
$$1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, \ldots$$

This is the famous Fibonacci sequence.

Fibonacci and Phyllotaxis







Great value at `https://logicpress.ie/authors/lynch/`

Outline

Ubiquity and Beauty of Symmetry

Symmetry is all around us.

- Many buildings are symmetric.
- Our bodies have bilateral symmetry.
- Crystals have great symmetry.
- Deoxyribonucleic acid (DNA).
- Viruses can display stunning symmetries.
- At the sub-atomic scale, symmetry reigns.
- Galaxies have many symmetries.

Like spirals, symmetry is found at all scales.

The Taj Mahal

Does Anyone Know Where This Is?

“Oh, Have you Been to Avondale?”

"Oh, Have you Been to Avondale?"

Buildings with Mathematical Themes

For examples of buildings using interesting mathematical principles, see this website:

`www.mathscareers.org.uk/interesting-buildings-based-on-mathematics/`

A Face with Symmetry: Halle Berry

Halle Berry **Berry Halle**

An Asymmetric Face: You know Who!

Symmetry and Group Theory

Symmetry is an essentially *geometric* concept.

The mathematical theory of symmetry is *algebraic*.

The key concept is that of a group.

A group is a *set of elements* such that any two elements can be combined to produce another.

From 2 to 3 Dimensional Symmetry

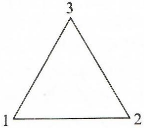


Fig. 4.6

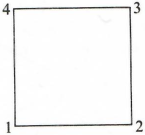
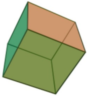
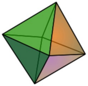
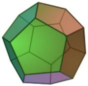
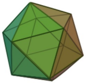


Fig. 4.7

Tetrahedron	Cube	Octahedron	Dodecahedron	Icosahedron
Four faces	Six faces	Eight faces	Twelve faces	Twenty faces
 (Animation)	 (Animation)	 (Animation)	 (Animation)	 (Animation)

Outline

Recreational Mathematics

Recreational mathematics puts the focus on insight, imagination and beauty.

Recreational Maths includes the study of

- The culture of mathematics,
- Its relevance to art, music and literature,
- Its role in technology,
- Mathematical games and puzzles,
- The lives of the great mathematicians.

Many Resources Available

Great variety of books on popular mathematics.

Wealth of literature suitable for a general audience

Magazines available free online.

One of the best is the e-zine Plus:

`https://plus.maths.org/.`

All past content is available and is a valuable resource for school students and teachers.

Content of an Earlier Course

Lecture	*Content*
1	Introduction. Spirals. Golden Ratio. Visual Maths I.
2	Beginning of Numbers. Shackleton. Babylon. Georg Cantor. Set Theory.
3	Set Theory II. Hilbert's Hotel. Topology I.
3A	Extra: Introduction to Numerical Weather Prediction.
4	Quadrivium. Pythagoras Th. Topology II. Archimedes Th. NumberLine 1.
5	Irrational Numbers. Astronomy. Real Numbers. Pascal's Triangle. Euler's Gem.
6	Prime Numbers. Topology III. Random Numbers. Möbius Band. Golden Ratio.

This year's course will be different: Better!

Outline

The Empty Set: Nothing to Worry About

$$\emptyset$$

The Universe may
be as great as they say,
But it wouldn't be missed
If it didn't exist.

Piet Hein

Worlds out of Nothing

From the empty set, we can construct the entire number system:

$$\varnothing \to 0 \qquad \{\varnothing\} \to 1 \qquad \{\varnothing, \{\varnothing\}\} \to 2 \qquad \dots$$

From the Natural Numbers $\mathbb{N}$, we can form the Rational Numbers $\mathbb{Q}$.

From the Rational Numbers $\mathbb{Q}$, we can form the Real Numbers $\mathbb{R}$.

From the Real Numbers $\mathbb{R}$, we can form the Complex Numbers $\mathbb{C}$.

The Liar Paradox

This statement is false

Well-formed sentences can be constructed that cannot consistently be assigned a truth value.

Call the sentence A. A says that A is false.

If A is a true statement, then it implies that A is false.

If A is a false statement, then it implies that A is true.

Russell's Paradox

Let A be a "big set" if $|A| \geq 100$.

Define $X = \{A \mid A \text{ is a big set}\}$

Clearly, X is a big set.

Therefore, X is a member of X. Symbolically, $X \in X$.

Bertrand Russell defined the set

$$R = \{x \mid x \notin x\}$$

He then deduced that

$$[R \in R] \iff [R \notin R]$$

Berry's Paradox

Consider this "definition" of a number:

The smallest positive whole number that cannot be described in fewer than twenty English words.

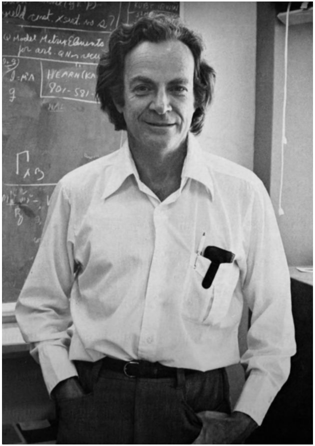
900 Random Digits?

82148086513282306647093844609550582231725359408128
48111745028410270193852110555964462294895493038196
44288109756659334461284756482337867831652712019091
45648566923460348610454326648213393607260249141273
72458700660631558817488152092096282925409171536436
78925903600113305305488204665213841469519415116094
33057270365759591953092186117381932611793105118548
07446237996274956735188575272489122793818301194912
98336733624406566430860213949463952247371907021798
60943702770539217176293176752384674818467669405132
00056812714526356082778577134275778960917363717872
14684409012249534301465495853710507922796892589235
42019956112129021960864034418159813629774771309960
51870721134999999837297804995105973173281609631859
50244594553469083026425223082533446850352619311881
71010003137838752886587533208381420617177669147303
59825349042875546873115956286388235378759375195778
18577805321712268066130019278766111959092164201989

900 Random Digits?

82148086513282306647093844609550582231725359408128
48111745028410270193852110555964462294895493038196
44288109756659334461284756482337867831652712019091
45648566923460348610454326648213393607260249141273
72458700660631558817488152092096282925409171536436
78925903600113305305488204665213841469519415116094
33057270365759591953092186117381932611793105118548
07446237996274956735188575272489122793818301194912
98336733624406566430860213949463952247371907021798
60943702770539217176293176752384674818467669405132
00056812714526356082778577134275778960917363717872
14684409012249534301465495853710507922796892589235
42019956112129021960864034418159813629774771309960
51870721134999999837297804995105973173281609631859
50244594553469083026425223082533446850352619311881
71010003137838752886587533208381420617177669147303
59825349042875546873115956286388235378759375195778
18577805321712268066130019278766111959092164201989

Richard Feynman's π Joke



Richard Feynman said he would like to find the first string of six *9's* in π.

He would then rattle off the sequence up to that point, finishing off with " $\ldots 999999 \ldots$ and so on".

The string of 9's above occurs after about 750 digits.

Regular and Random Binary Strings

1010101010101010101010101
0101010101010101010101010

This 50-bit string can be *described* in a few words.

The string below cannot easily be condensed.

0001110100100010110100100
0101111010100000100111101

Randomness = Compressability

How to Measure Complexity

Information is the resolution of uncertainty

We can convert everything into strings of bits.

Complexity can be defined as:
“the number of bits needed to describe a string”

Kolmogorov Complexity

Kolmogorov complexity may be defined as
“ The length of the shortest string
that describes a string."

Equivalently,
The shortest algorithm that will generate the string.

Kolmogorov complexity is uncomputable.

Order and Information

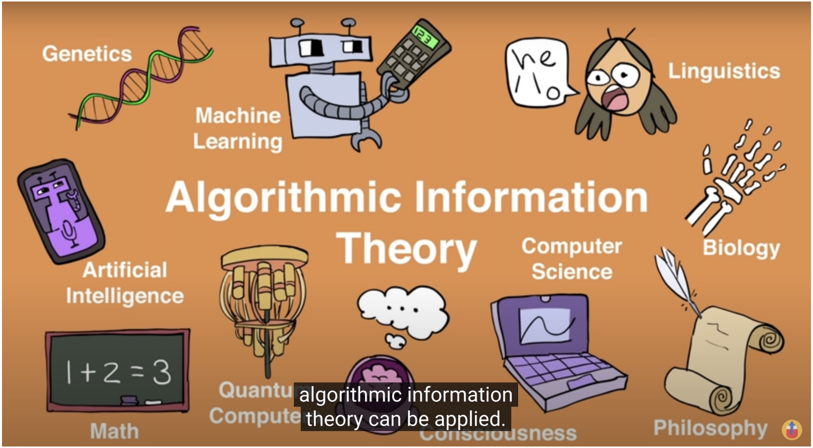
An orderly string has *low information content.*

It can be severely condensed by a short algorithm.

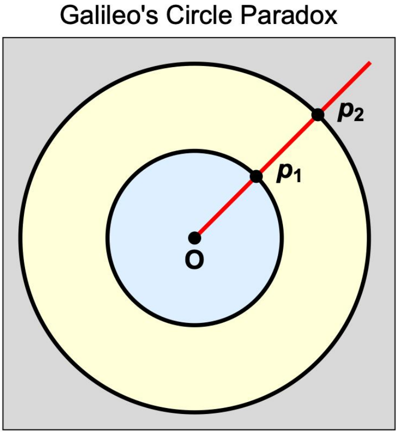
A random string cannot be easily condensed.

An algorithm defining it may be
as long as the number itself.

Algorithmic Information Theory



Galileo's Paradox



Grappling with Infinity

Galileo's result: all is not simple with infinity.

George Cantor **found an amazing result.**
It challenges our conception of dimension.

He found a one-to-one mapping $f : I \to Q$ **between the 1D unit interval** I **and the 2D unit square** Q**.**

Any point t **on the unit interval** I **may be expressed in decimal form,** $t = 0.abcd\ldots$**.**

Cantor separated the odd and even digits of t**,**

$$x = 0.ace\ldots \qquad \textbf{and} \qquad y = 0.bdf\ldots\,,$$

giving a point (x, y) **in the unit square** Q**.**

Grappling with Infinity

Thus, any point in Q can be obtained from some $t \in I$.

It is clear that this argument can be reversed: given the two coordinates of any point in Q,

$$x = 0.abcd\ldots \qquad \textbf{and} \qquad y = 0.ABCD\ldots$$

we can form the number

$$t = 0.aAbBcCdD\cdots \in I$$

This maps Q into I

(We have ignored some *"sticky points"*. For more detail, see Gouvêa, 2011).

Space-Filling Curves

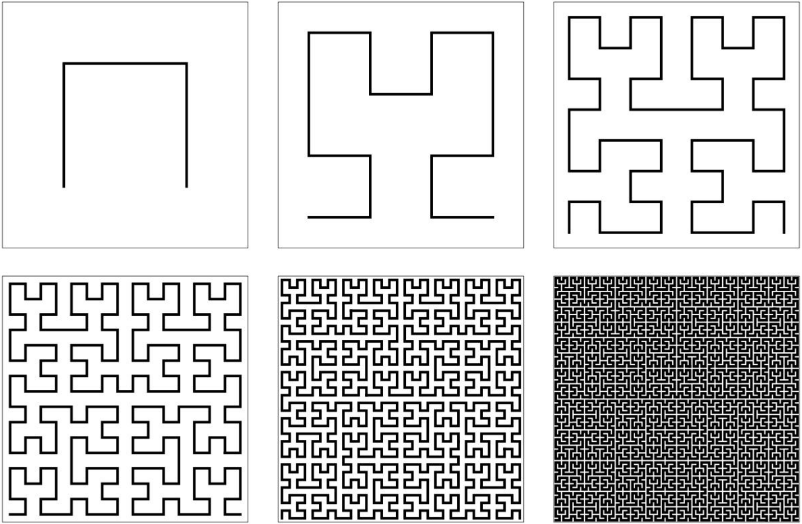
Cantor's map shows that the interval I contains as many points as the square Q.

Also, it is possible to specify any point in Q by a single number. ***Is Q a 1D space ?????***

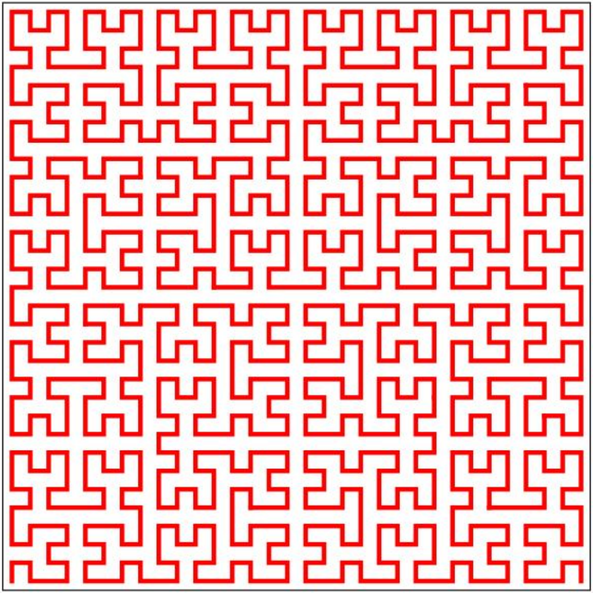
Cantor's function is not continuous. Giuseppe Peano found a continuous function from I onto Q;

A curve that passes through every point of Q.

First Six Approximations to Hilbert Curve



Approximation $H_5(t)$ to the Hilbert Curve



Space-Filling Curves

It is far from obvious that Hilbert's Curve really passes through every point in the unit square.

For some more details see two posts on my blog

`https://thatsmaths.com`

1. **Space-Filling Curves, Part I:**
   "I see it, but I don't believe it"
2. **Space-Filling Curves, Part II:**
   Computing the Limit Function.

Numbers+Geometry: Polygonal Numbers

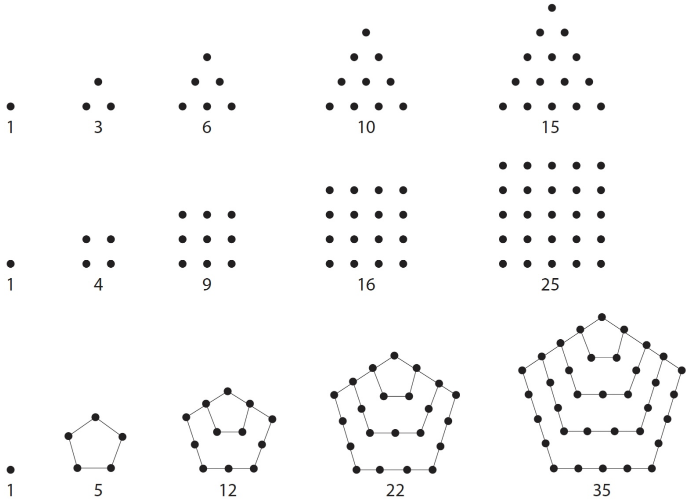


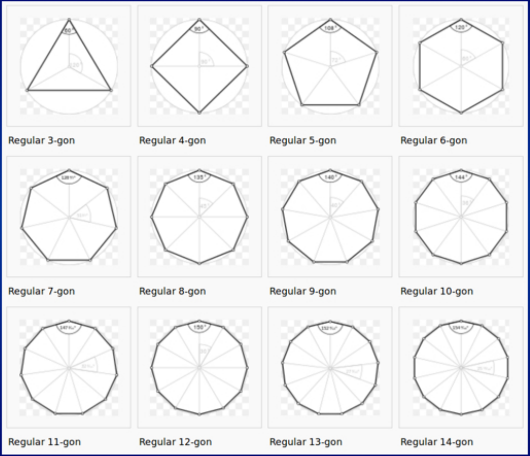
Figure 10.4. Pentagonal numbers. The form $n(3n-1)/2$ gives a pentagonal number; these begin with 1, 5, 12, 22, 35, 51, ... Each is one-third of a triangular number.

Outline

Euler's polyhedron formula.

Carving up the globe.

Regular Polygons



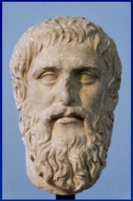
The Platonic Solids
(polyhedra)

Tetrahedron (four faces)	Cube or hexahedron (six faces)	Octahedron (eight faces)	Dodecahedron (twelve faces)	Icosahedron (twenty faces)

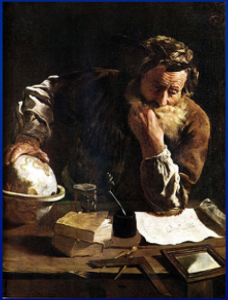
These five regular polyhedra were discovered in ancient Greece, perhaps by Pythagoras.

Plato used them as models of the universe.

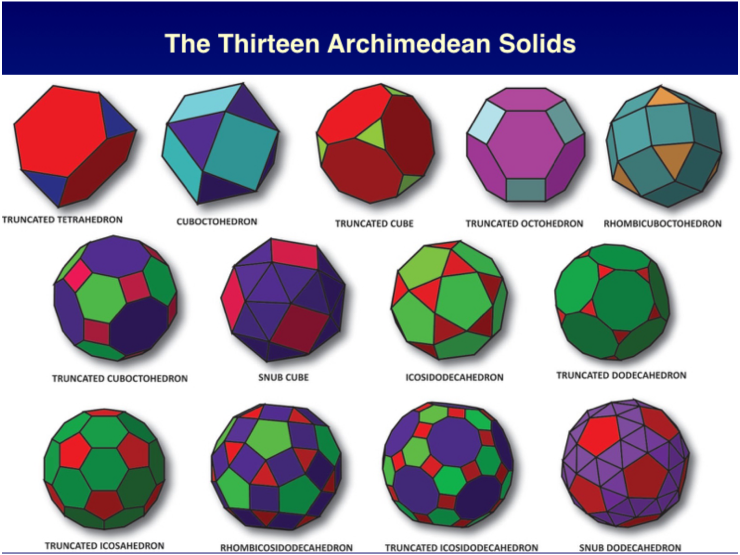
They are analysed in Book XIII of Euclid's Elements.



There are only five **Platonic** solids.



But **Archimedes** found, using different types of polygons, that he could construct 13 new solids.



Check V - E + F for the Truncated Cube

Euler's Polyhedron Formula

The great Swiss mathematician, Leonard Euler, noticed that, for all (convex) polyhedra,

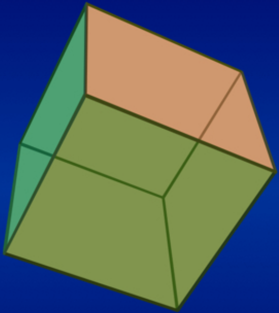
$$V - E + F = 2$$

where

- V = Number of vertices
- E = Number of edges
- F = Number of faces

Mnemonic: *Very Easy Formula*

For example, a Cube

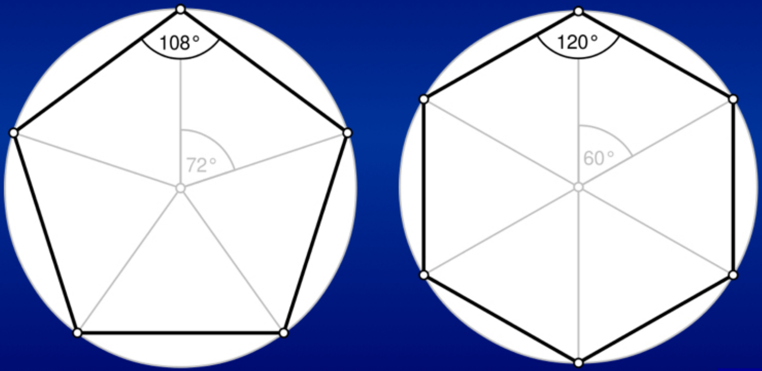


Number of vertices: V = 8
Number of edges: E = 12
Number of faces: F = 6

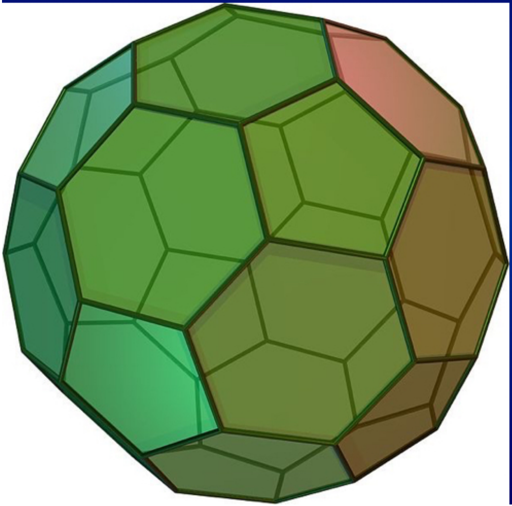
$$(V - E + F) = (8 - 12 + 6) = 2$$

Mnemonic: *Very Easy Formula*

Pentagons and Hexagons



The Truncated Icosahedron

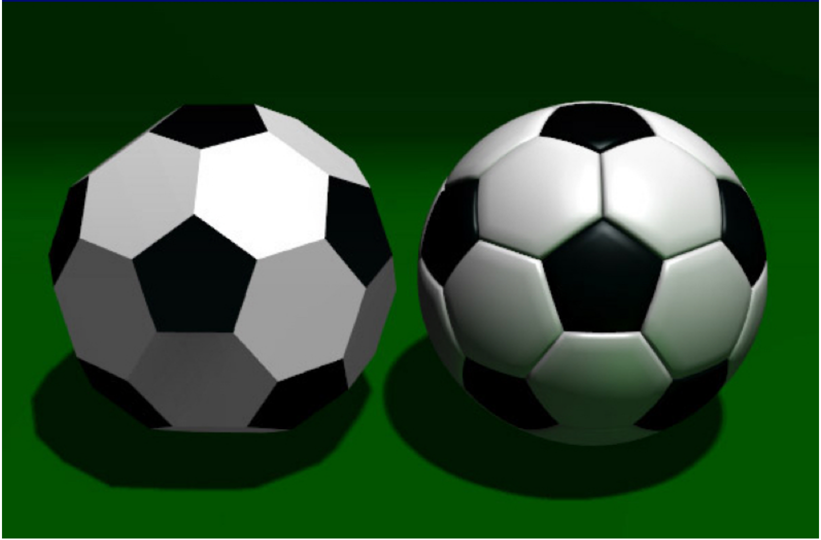


An Archimedean solid
with
pentagonal and
hexagonal faces.

The Truncated Icosahedron

Whare have you seen this before?

The Truncated Icosahedron

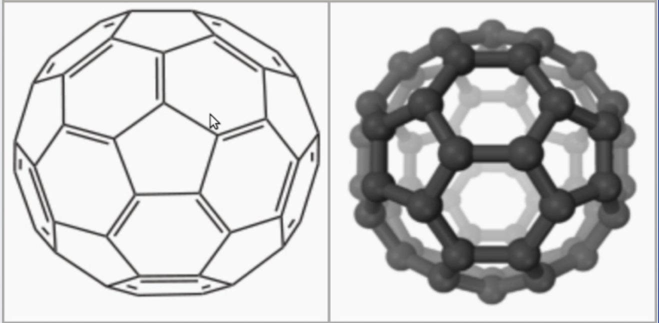


The "Buckyball", introduced at the
1970 World Cup Finals in Mexico.

It has 32 panels: 20 hexagons and 12 pentagons.

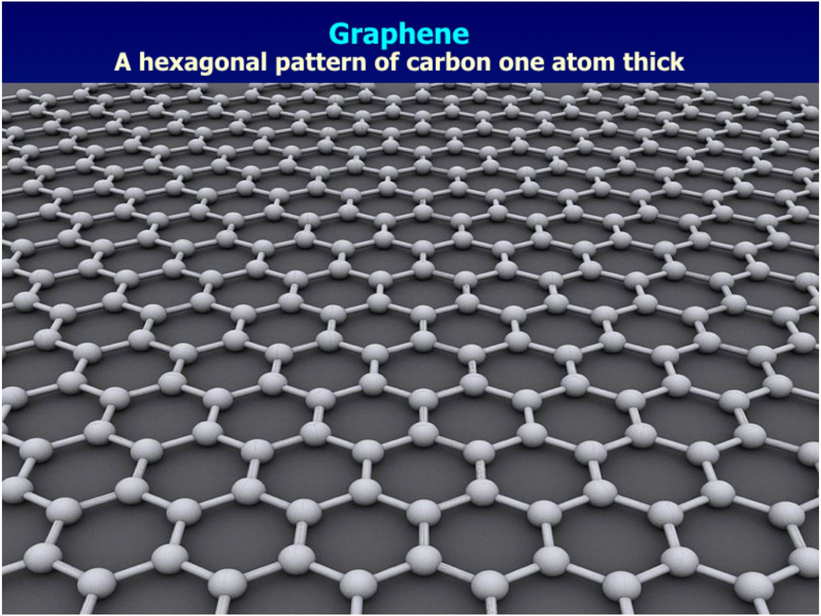

A Geodesic Dome designed by the American architect
Richard Buckminster "Bucky" Fuller.

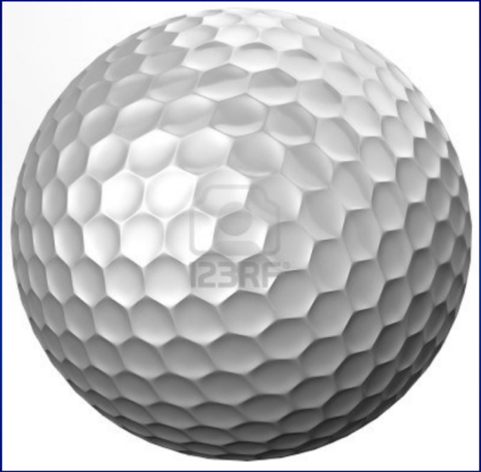
Buckminsterfullerene

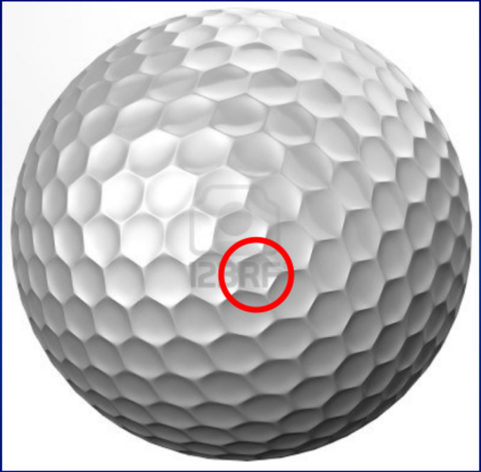


Buckminsterfullerene is a molecule with formula C_{60}

It was first synthesized in 1985.


Graphene
A hexagonal pattern of carbon one atom thick

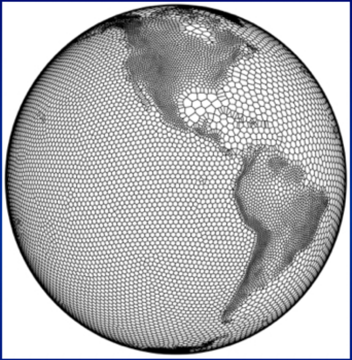


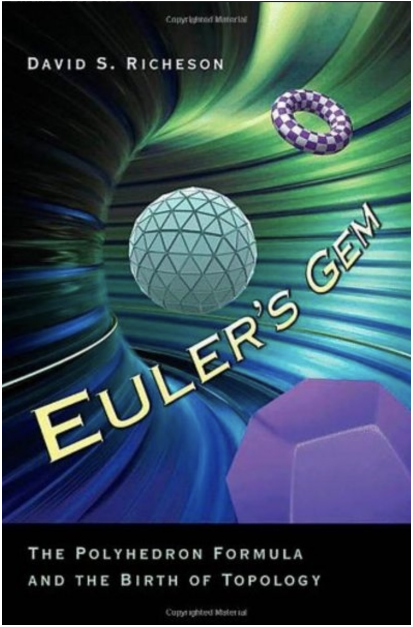


Euler's Polyhedron Formula

$$V - E + F = 2$$

still holds.

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