Knot invariants and modular forms

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Outline

- Motivation
- Knots
- Results
- Future Directions

Partitions

- A *partition* of a natural number n is a non-increasing sequence of positive integers, or parts, whose sum is n.
- There are 7 partitions of 5, namely

$$5,\ 4+1,\ 3+2,\ 3+1+1,\ 2+2+1,\ 2+1+1+1,\ 1+1+1+1+1.$$

Theorem (Rogers (1894), Ramanujan (1913))

The number of partitions of n such that all parts differ by at least 2 is equal to the number of partitions of n such that all parts are congruent to 1 or 4 modulo 5.

q-series

- The analytic version is

$$\sum_{n\geq 0}\frac{q^{n^2}}{(q)_n}=\frac{1}{(q;q^5)_\infty(q^4;q^5)_\infty}$$

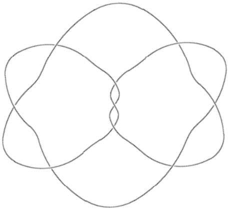
where

$$(a)_n=(a;q)_n=\prod_{k=1}^{n}(1-aq^{k-1})$$

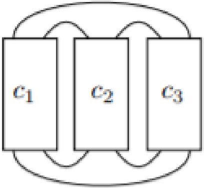
valid for $n\in\mathbb{N}\cup\{\infty\}$.

Knots

- A *knot* K is an embedding of a circle in $\mathbb{R}^3$. For example,



-9_{35}



Pretzel Knot

- We consider *alternating* knots and their connection to q-series.

Knot Invariants

- K knot $\rightsquigarrow$ "colored Jones polynomial" $J_{K,N}(q) \in \mathbb{Z}[q, q^{-1}]$.
- For $J_{-9_{35},N}(q)$, we have

  $N = 1 : 1$
  $N = 2 : 1 - q + 3q^2 - 4q^3 + 3q^4 - 5q^5 + 4q^6 + \cdots - q^9$
  $N = 3 : 1 - q - q^2 + 4q^3 - q^4 + \cdots + q^{27}$
  $N = 4 : 1 - q - q^2 + 0q^3 + 4q^4 + \cdots + q^{54}$
  $N = 5 : 1 - q - q^2 + 0q^3 + 0q^4 + 5q^5 - \cdots + q^{90}$
  $N = 6 : 1 - q - q^2 + 0q^3 + 0q^4 + q^5 + 4q^6 + \cdots + q^{135}$

- (Dasbach, Lin, 2006) The *tail* of $J_{K,N}(q)$ (if it exists) is a power series $T_K(q)$ whose first N coefficients agree with the first N coefficients of $J_{K,N}(q)$ for all $N \geq 1$.
- For example,

$$T_{-9_{35}}(q) = 1 - q - q^2 + q^5 + q^7 + \ldots = (q)_\infty$$

Tails

Theorem (Armond (2011), Garoufalidis, Lê (2011), Hajij (2015))

Let K be an alternating knot. Then $T_K(q)$ exists and equals a q-multisum $\Phi_K(q)$.

- The q-multisum $\Phi_K(q)$ is explicitly computable. For example,

$$\begin{aligned}\Phi_{-9_{35}}(q) &= (q)_\infty^9 \sum_{a,b,c,d,e,g,h,i,j\geq 0} (-1)^c \frac{q^{\frac{c(3c+1)}{2}+d^2+e^2+g^2+h^2+i^2+j^2+3ab+ac}}{(q)_a(q)_b(q)_c(q)_d(q)_e(q)_g(q)_h(q)_{a+c}} \\ &\quad \times \frac{q^{ad+ae+ai+aj+bc+bg+bh+bi+bj+a+b}}{(q)_{a+d}(q)_{a+e}(q)_{a+i}(q)_{a+j}(q)_{b+c}(q)_{b+g}(q)_{b+h}(q)_{b+i}(q)_{b+j}} \\ &= (q)_\infty\end{aligned}$$

GLZ conjectures

- Garoufalidis, Lê and Zagier (2011) conjectured Rogers-Ramanujan type identities between $\Phi_K(q)$ and products of h_b where

$$h_b = h_b(q) = \sum_{n \in \mathbb{Z}} \epsilon_b(n) q^{\frac{bn(n+1)}{2} - n}$$

and

$$\epsilon_b(n) = \begin{cases} (-1)^n & \text{if } b \text{ is odd,} \\ 1 & \text{if } b \text{ is even and } n \geq 0, \\ -1 & \text{if } b \text{ is even and } n < 0. \end{cases}$$

- One can show that $h_1 = 0$, $h_2 = 1$ and $h_3 = (q)_\infty$.

GLZ conjectures

K	$\Phi_K(q)$	$\Phi_{-K}(q)$	K	$\Phi_K(q)$	$\Phi_{-K}(q)$
3_1	h_3	1	7_5	h_4	$h_3 h_4$
4_1	h_3	h_3	7_6	h_3^2	$h_3 h_4$
5_1	1	h_5	7_7	h_3^2	h_3^3
5_2	h_4	h_3	8_1	h_7	h_3
6_1	h_5	h_3	8_2	h_3	$h_3 h_6$
6_2	h_3	$h_3 h_4$	8_3	h_5	h_5
6_3	h_3^2	h_3^2	8_4	h_3	$h_4 h_5$
7_1	1	h_7	8_5	?	h_3
7_2	h_6	h_3	$K_p,\ p > 0$	h_{2p}	h_3
7_3	h_5	h_4	$K_p,\ p < 0$	$h_{2\|p\|+1}$	h_3
7_4	h_3	h_4^2	$T(2,p),\ p > 0$	h_{2p+1}	1

Theorem (Andrews, 2012)

$\Phi_{3_1}(q) = h_3$, $\Phi_{4_1}(q) = h_3$ *and* $\Phi_{6_3}(q) = h_3^2$.

Theorem (Keilthy, Osburn, 2016)

All of the GLZ identities are true.

Main Result

- (Zagier's question) Is the list of 43 conjectural identities complete? No!

Theorem (Beirne, Osburn, 2017)

The following identities are true.

8 and 9 crossings

K	$\Phi_K(q)$	$\Phi_{-K}(q)$	K	$\Phi_K(q)$	$\Phi_{-K}(q)$	K	$\Phi_K(q)$	$\Phi_{-K}(q)$
8_6	h_5	$h_3 h_4$	9_6	h_4	$h_3 h_6$	9_{24}	?	?
8_7	$h_3 h_5$	h_3^2	9_7	h_6	$h_3 h_4$	9_{25}	?	h_3^3
8_8	h_3^2	$h_3 h_5$	9_8	h_3^2	$h_3 h_6$	9_{26}	$h_3^2 h_4$	h_3^3
8_9	$h_3 h_4$	$h_3 h_4$	9_9	h_4	$h_4 h_5$	9_{27}	$h_3^2 h_4$	h_3^3
8_{10}	?	h_3^2	9_{10}	h_5	h_4^2	9_{28}	?	$h_3^2 h_4$
8_{11}	$h_3 h_4$	$h_3 h_4$	9_{11}	$h_4 h_5$	h_3^2	9_{29}	?	?
8_{12}	$h_3 h_4$	$h_3 h_4$	9_{12}	$h_3 h_5$	$h_3 h_4$	9_{30}	?	h_3^3
8_{13}	h_3^2	$h_3^2 h_4$	9_{13}	$h_3 h_4$	h_4^2	9_{31}	h_3^3	h_3^4
8_{14}	$h_3 h_4$	h_3^3	9_{14}	h_3^2	$h_3^2 h_5$	9_{32}	?	?
8_{15}	?	h_3^3	9_{15}	$h_3 h_4$	$h_3 h_5$	9_{33}	?	?
8_{16}	?	?	9_{16}	?	h_4	9_{34}	?	?
8_{17}	?	?	9_{17}	h_3^2	$h_3^2 h_5$	9_{35}	?	h_3
8_{18}	?	?	9_{18}	h_4^2	$h_3 h_4$	9_{36}	?	h_3^2
9_1	1	h_9	9_{19}	$h_3 h_5$	h_3^3	9_{37}	?	h_3^3
9_2	h_8	h_3	9_{20}	h_3^2	$h_3 h_4^2$	9_{38}	?	?
9_3	h_7	h_4	9_{21}	$h_3 h_4$	$h_3^2 h_4$	9_{39}	?	?
9_4	h_6	h_5	9_{22}	?	h_3^2	9_{40}	?	?
9_5	h_3	$h_4 h_6$	9_{23}	h_4^2	h_3^3	9_{41}	?	?

10 crossings

K	$\Phi_K(q)$	$\Phi_{-K}(q)$	K	$\Phi_K(q)$	$\Phi_{-K}(q)$	K	$\Phi_K(q)$	$\Phi_{-K}(q)$
10_1	h_9	h_3	10_{19}	h_3^2	$h_3 h_4 h_5$	10_{37}	$h_3 h_5$	$h_3 h_5$
10_2	h_3	$h_3 h_8$	10_{20}	h_7	$h_3 h_4$	10_{38}	$h_4 h_5$	h_3^3
10_3	h_7	h_5	10_{21}	$h_3 h_4$	$h_3 h_6$	10_{39}	$h_3 h_4$	$h_3^2 h_5$
10_4	h_3	$h_4 h_7$	10_{22}	$h_3 h_4$	$h_4 h_5$	10_{40}	$h_3^2 h_4$	$h_3^2 h_4$
10_5	$h_3 h_7$	h_3^2	10_{23}	$h_3 h_5$	$h_3^2 h_4$	10_{41}	h_3^3	$h_3 h_4^2$
10_6	h_5	$h_3 h_6$	10_{24}	$h_4 h_5$	$h_3 h_4$	10_{42}	$h_3^2 h_4$	h_3^4
10_7	$h_3 h_6$	$h_3 h_4$	10_{25}	$h_3 h_4$	$h_3 h_4^2$	10_{43}	$h_3^2 h_4$	$h_3^2 h_4$
10_8	h_3	$h_5 h_6$	10_{26}	$h_3 h_4$	$h_3 h_4^2$	10_{44}	h_3^3	$h_3^3 h_4$
10_9	$h_3 h_6$	$h_3 h_4$	10_{27}	$h_3 h_5$	$h_3^2 h_4$	10_{45}	h_3^4	h_3^4
10_{10}	h_3^2	$h_3^2 h_6$	10_{28}	h_3^2	$h_3 h_4 h_5$	10_{46}	?	h_3
10_{11}	h_5	$h_4 h_5$	10_{29}	$h_3 h_4$	$h_3 h_4^2$	10_{47}	?	h_3^2
10_{12}	$h_3 h_5$	$h_3 h_5$	10_{30}	$h_3 h_4^2$	h_3^3	10_{48}	?	$h_3 h_5$
10_{13}	$h_4 h_5$	$h_3 h_4$	10_{31}	$h_3 h_5$	$h_3^2 h_4$	10_{49}	?	$h_3^2 h_5$
10_{14}	$h_3 h_4$	$h_3^2 h_5$	10_{32}	$h_3 h_4^2$	h_3^3	10_{50}	?	$h_3 h_4$
10_{15}	h_5^2	h_3^2	10_{33}	$h_3^2 h_4$	$h_3^2 h_4$	10_{51}	?	$h_3^2 h_4$
10_{16}	$h_4 h_5$	$h_3 h_4$	10_{34}	h_3^2	$h_3 h_7$	10_{52}	?	h_3^2
10_{17}	$h_3 h_5$	$h_3 h_5$	10_{35}	$h_3 h_4$	$h_3 h_6$	10_{53}	?	h_3^3
10_{18}	$h_3 h_4$	$h_3^2 h_5$	10_{36}	$h_3 h_6$	h_3^3	10_{54}	?	h_3^2

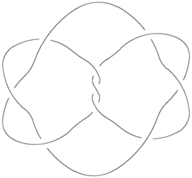
10 crossings

K	$\Phi_K(q)$	$\Phi_{-K}(q)$	K	$\Phi_K(q)$	$\Phi_{-K}(q)$
10_{55}	?	h_3^3	10_{67}	?	h_3^3
10_{56}	?	$h_3 h_4$	10_{68}	?	h_3^2
10_{57}	?	$h_3^2 h_4$	10_{69}	?	h_3^4
10_{58}	?	h_3^3	10_{70}	?	$h_3 h_4$
10_{59}	?	h_3^3	10_{71}	?	$h_3^2 h_4$
10_{60}	?	h_3^3	10_{72}	?	$h_3 h_4$
10_{61}	?	h_3	10_{73}	?	$h_3^2 h_4$
10_{62}	?	h_3^2	10_{74}	?	$h_3 h_4$
10_{63}	?	$h_3 h_4$	10_{75}	?	$h_3^3 h_4$
10_{64}	?	$h_3 h_4$	10_{76}	?	h_5
10_{65}	?	$h_3^2 h_4$	10_{77}	?	$h_3 h_5$
10_{66}	?	$h_3^2 h_5$	10_{78}	?	$h_3^2 h_5$

- $10_{79}, \ldots, 10_{123} \rightsquigarrow$?

Future Directions

- The remaining "?" cases



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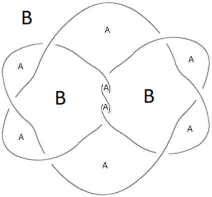

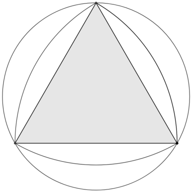

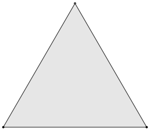

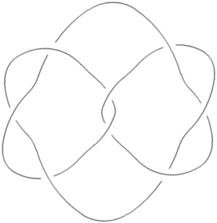

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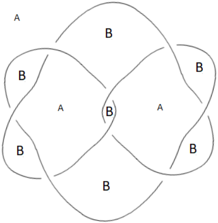

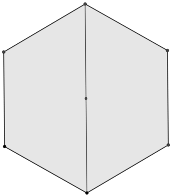

Future Directions

- First order stability for $J_{K,N}(q)$. Recall that for $J_{-9_{35},N}(q)$ we have

$$
\begin{aligned}
N=1:&\quad 1\\
N=2:&\quad 1-q+3q^2-4q^3+3q^4-5q^5+4q^6+\ldots-q^9\\
N=3:&\quad 1-q-q^2+4q^3-q^4+\ldots+q^{27}\\
N=4:&\quad 1-q-q^2+0q^3+4q^4+\ldots+q^{54}\\
N=5:&\quad 1-q-q^2+0q^3+0q^4+5q^5-\ldots+q^{90}\\
N=6:&\quad 1-q-q^2+0q^3+0q^4+q^5+4q^6+\ldots+q^{135}
\end{aligned}
$$

- Subtracting $\Phi_{-9_{35}}(q)=1-q-q^2+0q^3+0q^4+q^5+0q^6+q^7+\ldots$ we have

$$
\begin{aligned}
N=2:&\ 4-4q+3q^2-6q^3+4q^4-4q^5+\ldots\\
N=3:&\ 4-q-7q^2+7q^3-q^4-11q^5+8q^6+\ldots\\
N=4:&\ 4-q-4q^2-6q^3+7q^4+6q^5-4q^6-\ldots\\
N=5:&\ 4-q-4q^2-3q^3-6q^4+11q^5+5q^6+\ldots\\
N=6:&\ 4-q-4q^2-3q^3-3q^4-2q^5+10q^6+\ldots
\end{aligned}
$$

- Prove identities for second order stability in the case of pretzel knots.

Go raibh míle maith agaibh.