REPRESENTATIONS OF INTEGERS BY CERTAIN POSITIVE DEFINITE BINARY QUADRATIC FORMS

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Abstract. We prove part of a conjecture of Borwein and Choi concerning an estimate on the square of the number of solutions to $n = x^2 + Ny^2$ for a squarefree integer N.

1. Introduction

We consider the positive definite quadratic form $Q(x, y) = x^2 + Ny^2$ for a squarefree integer N. Let $r_{2,N}(n)$ denote the number of solutions to $n = Q(x, y)$ (counting signs and order). In this note, we estimate

$$\sum_{n \le x} r_{2,N}(n)^2.$$

A positive squarefree integer N is called solvable if $x^2 + Ny^2$ has one form per genus. Note that this means the class number of the form class group of discriminant $-4N$ equals the number of genera, 2^t, where t is the number of distinct prime factors of N. Concerning $r_{2,N}(n)$, Borwein and Choi [2] proved the following:

Theorem 1.1. *Let N be a solvable squarefree integer. Let $x > 1$ and $\epsilon > 0$. We have*

$$\sum_{n \le x} r_{2,N}(n)^2 = \frac{3}{N}\Big(\prod_{p|2N} \frac{2p}{p+1}\Big)(x \log x + \alpha(N)x) + O(N^{\frac{1}{4}+\epsilon} x^{\frac{3}{4}+\epsilon})$$

where the product is over all primes dividing $2N$ and

$$\alpha(N) = -1 + 2\gamma + \sum_{p|2N} \frac{\log p}{p+1} + \frac{2L'(1, \chi_{-4N})}{L(1, \chi_{-4N})} - \frac{12}{\pi^2}\zeta'(2)$$

where γ is the Euler-Mascheroni constant and $L(1, \chi_{-4N})$ is the L-function corresponding to the quadratic character mod $-4N$.

Based on this result, Borwein and Choi posed the following:

Conjecture 1.2. For any squarefree N,

$$\sum_{n \le x} r_{2,N}(n)^2 \sim \frac{3}{N}\Big(\prod_{p|2N} \frac{2p}{p+1}\Big) x \log x$$

Our main result is the following.

Theorem 1.3. *Let $Q(x, y) = x^2 + Ny^2$ for a squarefree integer N with $-N \not\equiv 1 \bmod 4$. Let $r_{2,N}(n)$ denote the number of solutions to $n = Q(x, y)$ (counting signs and order). Then*

$$\sum_{n \le x} r_{2,N}(n)^2 \sim \frac{3}{N}\Big(\prod_{p|2N} \frac{2p}{p+1}\Big) x \log x.$$

2. Preliminaries

We first discuss two key estimates and a result of Kronecker on genus characters. Then using Kronecker's result, we prove a proposition relating genus characters to poles of the Rankin-Selberg convolution of L-functions. The first estimate is a recent result of Kühleitner and Nowak [13], namely

Theorem 2.1. *Let $a(n)$ be an arithmetic function satisfying $a(n) \ll n^{\epsilon}$ for every $\epsilon > 0$, with a Dirichlet series*

$$F(s) = \sum_{n=1}^{\infty} \frac{a(n)}{n^s} = \frac{(\zeta_K(s))^2}{(\zeta(2s))^{m_1}(\zeta_K(2s))^{m_2}} G(s)$$

where $\Re(s) > 1$ and $\zeta_K(s)$ is the Dedekind zeta function of some quadratic number field K, $G(s)$ is holomorphic and bounded in some half plane $\Re(s) \geq \theta$, $\theta < \frac{1}{2}$, and m_1, m_2 are nonnegative integers. Then for x large,

$$\sum_{n \leq x} a(n) = Res_{s=1}\Big(F(s)\frac{x^s}{s}\Big) + O(x^{\frac{1}{2}}(\log x)^3(\log\log x)^{m_1+m_2})$$
$$= Ax\log x + Bx + O(x^{\frac{1}{2}}(\log x)^3(\log\log x)^{m_1+m_2})$$

where A and B are computable constants.

For an arbitrary quadratic number field K with discriminant d_K, let $\mathcal{O}_K$ denote the ring of integers in K, and $r_K(n)$ the number of integral ideals $\mathcal{I}$ in $\mathcal{O}_K$ of norm $N(\mathcal{I}) = n$. From (4.1) in [13], we have

$$\sum_{n=1}^{\infty} \frac{(r_K(n))^2}{n^s} = \frac{(\zeta_K(s))^2}{\zeta(2s)} \prod_{p \mid d_K} (1 + p^{-s})^{-1}.$$

Applying Theorem 2.1 with $m_1 = 1$ and $m_2 = 0$, we obtain

Corollary 2.2. *For any quadratic field K of discriminant d_K and x large,*

$$\sum_{n \leq x} (r_K(n))^2 = A_1 x\log x + B_1 x + O(x^{\frac{1}{2}}(\log x)^3 \log\log x),$$

with $A_1 = \frac{6}{\pi^2} L(1, \chi_{d_K})^2 \prod_{p \mid d_K} \frac{p}{p+1}$ and $B_1 = A_1\alpha(N)$ with $\alpha(N)$ as in Theorem 1.1.

The second estimate is a classical result of Rankin [16] and Selberg [17] which estimates the size of Fourier coefficients of a modular form. Specifically, if $f(z) = \sum_{n=1}^{\infty} a(n)e^{2\pi i n z}$ is a nonzero cusp form of weight k on $\Gamma_0(N)$, then

$$\sum_{n \leq x} |a(n)|^2 = \alpha\langle f, f\rangle x^k + O(x^{k-\frac{2}{5}})$$

where $\alpha > 0$ is an absolute constant and $\langle f, f\rangle$ is the Petersson scalar product. In particular, if f is a cusp form of weight 1, then $\sum_{n \leq x} |a(n)|^2 = O(x)$. One can adapt their result to say the following. Given two cusp forms of weight k on a suitable congruence subgroup of $\Gamma = SL_2(\mathbb{Z})$, say $f(z) = \sum_{n=1}^{\infty} a(n)e^{2\pi i n z}$ and $g(z) = \sum_{n=1}^{\infty} b(n)e^{2\pi i n z}$, then

$$\sum_{n \leq x} a(n)\overline{b(n)} n^{1-k} = Ax + O(x^{\frac{3}{5}})$$

where A is a constant. In particular, if f and g are cusp forms of weight 1, then $\sum_{n\le x} a(n)\overline{b(n)} = O(x)$.

We will also use a result of Kronecker on genus characters. Let us first explain some terminology. Let $K = \mathbb{Q}(\sqrt{d})$ be a quadratic field of discriminant d_K. d_K is said to be a prime discriminant if it only has one prime factor. Thus it must be of the form: -4, ± 8, $\pm p \equiv 1 \bmod 4$ for an odd prime p. Every discriminant can be written uniquely as a product of prime discriminants, say $d_K = P_1 \dots P_k$. Here k denotes the number of distinct prime factors of d_K. Thus d_K can be written as a product of two discriminants, say $d_K = D_1 D_2$ in 2^{k-1} distinct ways (excluding order). Now, for any such decomposition we define a character χ_{D_1,D_2} on ideals by

$$\chi_{D_1,D_2}(\mathfrak{p}) = \begin{cases} \chi_{D_1}(N\mathfrak{p}) & \text{if } \mathfrak{p} \nmid D_1 \\ \chi_{D_2}(N\mathfrak{p}) & \text{if } \mathfrak{p} \nmid D_2 \end{cases}$$

where $\chi_d(n)$ is the Kronecker symbol. This is well defined on prime ideals because $\chi_D(N\mathfrak{a}) = 1$ if $(\mathfrak{a}, D) = 1$. χ_{D_1,D_2} extends to all fractional ideals by multiplicativity. Hence we have

$$\chi_{D_1,D_2} : I \to \{\pm 1\}$$

where I is the group of nonzero fractional ideals of $\mathcal{O}_K$. Thus χ_{D_1,D_2} has order two, except for the trivial character corresponding to $d_K = d_K \cdot 1 = 1 \cdot d_K$. Every such character χ_{D_1,D_2} is called the genus character of discriminant d_K. As these are different for distinct factorizations of d_K (into a product of two discriminants), we have 2^{k-1} genus characters. Kronecker's theorem (see Theorem 12.7 in [11]) is as follows.

Theorem 2.3. *The L-function of K associated with the genus character χ_{D_1,D_2} factors into the Dirichlet L-functions,*

$$L(s, \chi_{D_1,D_2}) = L(s, \chi_{D_1}) L(s, \chi_{D_2}).$$

Let $K = \mathbb{Q}(\sqrt{-N})$, N squarefree, I as above, and P the subgroup of I of principal ideals. For a non-zero integral ideal $\mathfrak{m}$ of $\mathcal{O}_K$, define

$$I(\mathfrak{m}) = \{\mathfrak{a} \in I : (\mathfrak{a}, \mathfrak{m}) = 1\}$$
$$P(\mathfrak{m}) = \{\langle a \rangle \in P : a \equiv 1 \bmod \mathfrak{m}\}.$$

A group homomorphism $\chi : I_{\mathfrak{m}} \to S^1$ is an ideal class character if it is trivial on $P(\mathfrak{m})$, i.e.

$$\chi(\langle a \rangle) = 1$$

for $a \equiv 1 \bmod \mathfrak{m}$. Thus an ideal class character is a character on the ray class group $I(\mathfrak{m}) \big/ P(\mathfrak{m})$. Taking the trivial modulus $\mathfrak{m} = 1$, we obtain a character on the ideal class group of K. Note that for $K = \mathbb{Q}(\sqrt{-N})$ a genus character is an ideal class character of order at most two.

Let us also recall the notion of the Rankin-Selberg convolution of two L-functions. For squarefree N, consider two ideal class characters χ_1, χ_2 for $\mathbb{Q}(\sqrt{-N})$ and their associated Hecke L-series

$$L(s, \chi_1) = \sum_{n=1}^{\infty} \frac{\chi_1(n)}{n^s}$$
$$L(s, \chi_2) = \sum_{n=1}^{\infty} \frac{\chi_2(n)}{n^s}$$

which converge absolutely in some right half-plane. We form the convolution L-series by multiplying the coefficients,

$$L(s,\chi_1\otimes\chi_2)=\sum_{n=1}^{\infty}\frac{\chi_1(n)\chi_2(n)}{n^s}.$$

The following result describes a relationship between genus characters χ and the orders of poles of $L(s,\chi\otimes\chi)$. Precisely,

Proposition 2.4. *Let χ be an ideal class character of $\mathbb{Q}(\sqrt{-N})$, $-N\not\equiv 1 \bmod 4$, and $L(s,\chi)$ the associated Hecke L-series. Then χ is a genus character if and only if $L(s,\chi\otimes\chi)$ has a double pole at $s=1$.*

Proof. Suppose χ_{D_1,D_2} is a genus character of discriminant $-4N$, and $L(s,\chi_{D_1,D_2})=\sum_{n=1}^{\infty}\frac{b_i(n)}{n^s}$. By Theorem 2.3 and Exercise 1.2.8 in [14] (see the solution), we have

$$\sum_{n=1}^{\infty}\frac{b_i(n)^2}{n^s}=\frac{L(s,\chi_{D_1}^2)L(s,\chi_{D_2}^2)L(s,\chi_{D_1}\chi_{D_2})^2}{L(2s,\chi_{D_1}^2\chi_{D_2}^2)}.$$

Note that

$$L(s,\chi_{D_1}^2)=\zeta(s)\cdot\prod_{p\mid D_1}(1-p^{-s}),$$

$$L(s,\chi_{D_2}^2)=\zeta(s)\cdot\prod_{p\mid D_2}(1-p^{-s}),$$

$$L(s,\chi_{D_1}\chi_{D_2})^2=L(s,\chi_{-4N})^2,$$

and

$$L(2s,\chi_{D_1}^2\chi_{D_2}^2)=\zeta(2s)\cdot\prod_{p\mid D_1D_2}(1-p^{-2s}).$$

We have

$$\sum_{n=1}^{\infty}\frac{b_i(n)^2}{n^s}=\frac{\zeta(s)^2L(s,\chi_{-4N})^2}{\zeta(2s)}\prod_{p\mid 2N}(1+p^{-s})^{-1}$$

and thus a double pole at $s=1$.

Conversely, let χ be an ideal class character of $K=\mathbb{Q}(\sqrt{-N})$ and suppose $L(s,\chi\otimes\chi)$ has a double pole at $s=1$. Now χ is an automorphic form on $GL_1(\mathbb{A}_K)$. By automorphic induction (see [1]), χ is mapped to π, a cuspidal automorphic representation of $GL_2(\mathbb{A}_{\mathbb{Q}})$. Note that π is reducible as, otherwise, $L(s,\pi\otimes\pi)$ has a simple pole at $s=1$ ([1], page 200). As K is a quadratic extension of $\mathbb{Q}$, we must have $\pi=\chi_1+\chi_2$ where χ_i are Dirichlet characters. As $L(s,\chi)=L(s,\pi)$ (see [1]) and thus $L(s,\chi\otimes\chi)=L(s,\pi\otimes\pi)$,

$$L(s,\pi\otimes\pi)=L(s,\chi\otimes\chi)=\frac{L(s,\chi_1^2)L(s,\chi_2^2)L(s,\chi_1\chi_2)^2}{L(2s,\chi_1^2\chi_2^2)}.$$

Now $L(s,\chi\otimes\chi)$ has a double pole at $s=1$ if and only if either $\chi_1=\overline{\chi_2}$, $\chi_2^2\neq 1$, and $\chi_1^2\neq 1$ or $\chi_1^2=1$, $\chi_2^2=1$, and $\chi_1\chi_2\neq 1$. The latter implies χ is a genus character. We now need to show that the former also implies that χ is a genus character. Note that

$$L(s,\chi)=\prod_{\mathfrak{p}}\Big(1-\frac{\chi(\mathfrak{p})}{N(\mathfrak{p})^s}\Big)^{-1}$$

and

$$L(s,\chi_1+\chi_2)=\prod_p\Big(1-\frac{\chi_1(p)}{p^s}\Big)^{-1}\prod_p\Big(1-\frac{\chi_2(p)}{p^s}\Big)^{-1}.$$

As $L(s,\chi)=L(s,\pi)$ and $L(s,\pi)=L(s,\chi_1+\chi_2)$, we compare Euler factors to get

$$\chi_1(p)+\chi_2(p)=\begin{cases}0 & \text{if } p \text{ is inert in } K\\ \chi(\mathfrak{p})+\overline{\chi(\mathfrak{p})} & \text{if } p \text{ splits in } K.\end{cases}$$

For p inert in K, this yields $\chi_1(p)=-\chi_2(p)$ and so $\overline{\chi_2(p)}=\chi_1(p)=-\chi_2(p)$ which implies $\chi_2^2(p)=-1$ and so $\chi_2(p)=\pm i$. Now consider the following equation whose sum sieves the inert primes

$$\frac{1}{2}\sum_{\substack{p\le x\\ p\ \text{prime}}}\Big(1-\Big(\frac{-4N}{p}\Big)\Big)\chi_2^2(p)=-\pi(x).$$

Here $\pi(x)$ is the number of primes between 1 and x. Thus

$$\frac{1}{2}\sum_{\substack{p\le x\\ p\ \text{prime}}}\chi_2^2(p)-\frac{1}{2}\sum_{\substack{p\le x\\ p\ \text{prime}}}\Big(\frac{-4N}{p}\Big)\chi_2^2(p)=-\pi(x).$$

As $\chi_2^2\neq 1$, we have by the prime ideal theorem, $\sum_{p\le x}\chi_2^2(p)=o(\pi(x))$ and so

$$\sum_{p\le x}\Big(\frac{-4N}{p}\Big)\chi_2^2(p)\sim\pi(x).$$

This implies $\Big(\frac{-4N}{p}\Big)\chi_2^2(p)=1$. If p splits in K, then $\chi_2^2(p)=1$ and so $\chi_2(p)=\pm 1$. A similar argument works for χ_1 and so we also have $\chi_1(p)=\pm 1$ if p splits in K.

Again comparing the Euler factors in $L(s,\chi)$ and $L(s,\pi)$, the values of $\chi(\mathfrak{p})$ must coincide with the values of $\chi_1(p)$ and $\chi_2(p)$, that is, $\chi(\mathfrak{p})=\pm 1$. Now $\chi(\mathfrak{p})=\chi([\mathfrak{p}])$ where $[\mathfrak{p}]$ is the class of $\mathfrak{p}$ in the ideal class group of K. By the analog of Dirichlet's theorem for ideal class characters, we know that in each ideal class $\mathfrak{C}$ there are infinitely many prime ideals which split. Thus $\chi(\mathfrak{C})=\pm 1$ and hence is of order 2. This implies χ is a genus character.

□

Remark 2.5. By Proposition 2.4, if χ is a non-genus character, then $L(s,\chi\otimes\chi)$ has at most a simple pole at $s=1$.

3. Proof of Theorem 1.3

Proof. As $-N\not\equiv 1 \bmod 4$, the discriminant of $K=\mathbb{Q}(\sqrt{-N})$ is $-4N$. We also assume that t is the number of distinct prime factors of N and so the discriminant $-4N$ has $t+1$ distinct prime factors.

Given the quadratic form $Q(x,y)=x^2+Ny^2$, we consider the associated Epstein zeta function (see [7], [12], [18], or [19])

$$\zeta_Q(s)=\sum_{x,y\neq 0}\frac{1}{(x^2+Ny^2)^s}=\sum_{n=1}^{\infty}\frac{r_{2,N}(n)}{n^s}.$$

for $\Re(s) > 1$. Now for $K = \mathbb{Q}(\sqrt{-N})$, we have Dedekind's zeta function

$$\zeta_K(s) = \sum_{\mathfrak{a}} \frac{1}{N(\mathfrak{a})^s} = \sum_{n=1}^{\infty} \frac{a_n}{n^s}$$

where the sum is over all nonzero ideals $\mathfrak{a}$ of $\mathcal{O}_K$. We now split up $\zeta_K(s)$, according to the classes c_i of the ideal class group $C(K)$, into the partial zeta functions (see page 458 of [15])

$$\zeta_{c_i}(s) = \sum_{\mathfrak{a} \in c_i} \frac{1}{N(\mathfrak{a})^s}$$

so that $\zeta_K(s) = \sum_{i=0}^{h-1} \zeta_{c_i}(s)$ where h is the class number of K. In our case $K = \mathbb{Q}(\sqrt{-N})$ is an imaginary quadratic field and so by [6] (Theorem 7.7, page 137), we may write

$$\zeta_K(s) = \sum_{i=0}^{h-1} \zeta_{Q_i}(s)$$

where Q_i is a class in the form class group. Note that in this context, $Q(x, y)$ corresponds to the trivial class c_0 in $C(K)$ and so $\zeta_{c_0}(s) = \zeta_{Q(x,y)}(s)$. Now let χ be an ideal class character and consider the Hecke L-function for χ, namely

$$L(s, \chi) = \sum_{\mathfrak{a}} \frac{\chi(\mathfrak{a})}{N(\mathfrak{a})^s}$$

where $\mathfrak{a}$ again runs over all nonzero ideals of $\mathcal{O}_K$. We may now rewrite the Hecke L-function as

$$L(s, \chi) = \sum_{i=0}^{h-1} \chi(c_i)\zeta_{c_i}(s).$$

And so summing over all ideal class characters of $C(K)$, we have

$$\sum_{\chi} \overline{\chi}(c_0)L(s, \chi) = \sum_{i=0}^{h-1} \zeta_{c_i}(s)\Big(\sum_{\chi} \overline{\chi}(c_0)\chi(c_i)\Big).$$

The inner sum is nonzero precisely when $i = 0$. As $\overline{\chi}(c_0) = 1$ we have $\zeta_{c_0}(s) = \frac{1}{h}\sum_{\chi} L(s, \chi)$. Thus

$$\zeta_{c_0}(s) = \tfrac{1}{h}(L(s, \chi_0) + L(s, \chi_1) + \cdots + L(s, \chi_{h-1})).$$

As χ_0 is the trivial character, $L(s, \chi_0) = \zeta_K(s)$. We now compare n^{th} coefficients, yielding

$$r_{2,N}(n) = \tfrac{1}{h}(a_n + b_1(n) + \cdots + b_{h-1}(n))$$

where a_n is the number of integral ideals of $\mathcal{O}_K$ of norm n and the b_i's are coefficients of weight 1 cusp forms (see the classical work of Hecke [9], [10] or [3]). From the modern perspective, this is straightforward. Each $L(s, \chi_i)$, $1 \leq i \leq h-1$, can be viewed as an automorphic L-function of $GL_1(\mathbb{A}_K)$ and by automorphic induction (see [1]) they are essentially Mellin transforms of (holomorphic) cusp forms, in the classical sense. We now have

$$\sum_{n\le x} r_{2,N}(n)^2 = \frac{1}{h^2}\Big(\sum_{n\le x} a_n{}^2 + \sum_{\substack{i\\ n\le x}} b_i(n)^2 + 2\sum_{\substack{i\\ n\le x}} a_n b_i(n) + \sum_{\substack{i\neq j\\ n\le x}} b_i(n)b_j(n)\Big).$$

By the Rankin-Selberg estimate, $2\sum_{\substack{i\\ n\le x}} a_n b_i(n)$, $\sum_{\substack{i\neq j\\ n\le x}} b_i(n)b_j(n)$ are equal to $O(x)$. By Corollary 2.2,

$$\frac{1}{h^2}\sum_{n\le x} a_n{}^2 = \frac{1}{h^2}\Big(A_1 x\log x + B_1 x + O(x^{\frac{1}{2}}(\log x)^3\log\log x)\Big).$$

We now must estimate $\sum_{\substack{i\\ n\le x}} b_i(n)^2$. Let us now assume that the first $2^t - 1$ terms arise from L-functions associated to genus characters. By Proposition 2.4 and Nowak's proof of Theorem 2.1 (which uses Perron's formula and the residue theorem), we obtain

$$\sum_{n\le x} b_i(n)^2 = A_1 x\log x + B_1 x + O(x)$$

with A_1 and B_1 as in Corollary 2.2. As this estimate holds for each i such that $1 \le i \le 2^t - 1$, the term $A_1 x\log x$ appears 2^t times in the estimate of $\sum_{n\le x} r_{2,N}(n)^2$. By Remark 2.5, the remaining terms $\sum_{n\le x} b_i(n)^2$ for $2^t - 1 < i \le h-1$ are all $O(x)$. Thus

$$\sum_{n\le x} r_{2,N}(n)^2 = \frac{1}{h^2}\Big[\Big(2^t\frac{6}{\pi^2}L(1,\chi_{-4N})^2\prod_{p|2N}\frac{p}{p+1}\Big)x\log x + O(x)\Big] + O(x).$$

By (4.11) in [8] (or equation (8), page 171 in [5]), we have $L(1,\chi_{-4N}) = \frac{h\pi}{\sqrt{N}}$ and so

$$\sum_{n\le x} r_{2,N}(n)^2 = \frac{3}{N}\Big(\prod_{p|2N}\frac{2p}{p+1}\Big)x\log x + O(x).$$

The result then follows.

□

Remark 3.1. It should be possible to obtain the second term in the asymptotic formula. By a careful application of the Rankin-Selberg method, one should obtain an error term of the form $O(x^\theta)$ with $\theta < 1$. The remaining case $-N \equiv 1 \bmod 4$ requires more subtle analysis due to the fact that for $K = \mathbb{Q}(\sqrt{-N})$, $\mathbb{Z}[\sqrt{-N}]$ is not the maximal order of K. It involves the study of L-series attached to orders. Using the techniques in [4] and [12], we will take this and sharper error terms up in some detail in a forthcoming paper.

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