

# RANK DIFFERENCES FOR OVERPARTITIONS

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ABSTRACT. In 1954, Atkin and Swinnerton-Dyer proved Dyson's conjectures on the rank of a partition by establishing formulas for the generating functions for rank differences in arithmetic progressions. In this paper, we prove formulas for the generating functions for rank differences for overpartitions. These are in terms of modular functions and generalized Lambert series.

## 1. INTRODUCTION

The rank of a partition is the largest part minus the number of parts. This statistic was introduced by Dyson [14], who observed empirically that it provided a combinatorial explanation for Ramanujan's congruences  $p(5n + 4) \equiv 0 \pmod{5}$  and  $p(7n + 5) \equiv 0 \pmod{7}$ . Here  $p(n)$  denotes the usual partition function. Specifically, Dyson conjectured that if  $N(s, m, n)$  denotes the number of partitions of  $n$  whose rank is congruent to  $s$  modulo  $m$ , then for all  $0 \leq s \leq 4$  and  $0 \leq t \leq 6$  we have

$$N(s, 5, 5n + 4) = \frac{p(5n + 4)}{5}$$

and

$$N(t, 7, 7n + 5) = \frac{p(7n + 5)}{7}.$$

Atkin and Swinnerton-Dyer proved these assertions in 1954 [3]. In fact, they proved much more, establishing generating functions for every rank difference  $N(s, \ell, \ell n + d) - N(t, \ell, \ell n + d)$  with  $\ell = 5$  or  $7$  and  $0 \leq d, s, t < \ell$ . Many of these turned out to be non-trivially 0, while others were infinite products and still others were generalized Lambert series related to Ramanujan's third order mock theta functions. Such formulas in the case  $\ell = 11$  were subsequently given by Atkin and Hussain [2] in a similar (though technically far more difficult) manner.

Dyson's rank extends in the obvious way to overpartitions. Recall that an overpartition [10] is simply a partition in which the first occurrence of each distinct number may be overlined. For example, the 14 overpartitions of 4 are

$$4, \bar{4}, 3 + 1, \bar{3} + 1, 3 + \bar{1}, \bar{3} + \bar{1}, 2 + 2, \bar{2} + 2, 2 + 1 + 1, \bar{2} + 1 + 1, 2 + \bar{1} + 1, \bar{2} + \bar{1} + 1, \\ 1 + 1 + 1 + 1, \bar{1} + 1 + 1 + 1.$$

Overpartitions (often under other names) naturally arise in diverse areas of mathematics where partitions already occur, such as mathematical physics [15, 16], symmetric functions [4, 13], representation theory [20], and algebraic number theory [21, 23]. With basic hypergeometric series, Dyson's rank for overpartitions and its generalizations play an important role in combinatorial studies of Rogers-Ramanujan type identities [11, 12, 22].

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*Date:* May 28, 2007.

*2000 Mathematics Subject Classification.* Primary: 11P81, 05A17; Secondary: 33D15.

The first author was partially supported by an ACI "Jeunes Chercheurs et Jeunes Chercheuses".

As for arithmetic properties, let  $\overline{N}(s, m, n)$  be the number of overpartitions of  $n$  with Dyson's rank congruent to  $s$  modulo  $m$ . It has recently been shown [5] that if  $u$  is any natural number,  $m$  is odd and  $\ell \geq 5$  is a prime such that  $\ell \nmid 6m$  or  $\ell^j = m$ , then there are infinitely many non-nested arithmetic progressions  $An + B$  such that

$$\overline{N}(s, m, An + B) \equiv 0 \pmod{\ell^u}$$

for all  $0 \leq s < m$ . This is analogous to congruences involving Dyson's rank for partitions [6].

On the other hand, there are no congruences of the form  $\overline{p}(\ell n + d) \equiv 0 \pmod{\ell}$  for primes  $\ell \geq 3$  [9]. Here  $\overline{p}(n)$  denotes the number of overpartitions of  $n$ . The generating functions for the rank differences  $\overline{N}(s, \ell, \ell n + d) - \overline{N}(t, \ell, \ell n + d)$  then provide a measure of the extent to which the rank fails to produce a congruence  $\overline{p}(\ell n + d) \equiv 0 \pmod{\ell}$ . In this paper, we find formulas for these generating functions for  $\ell = 3$  and  $5$  in terms of modular functions and generalized Lambert series. Using the notation

$$(1.1) \quad R_{st}(d) = \sum_{n \geq 0} (\overline{N}(s, \ell, \ell n + d) - \overline{N}(t, \ell, \ell n + d)) q^n,$$

where the prime  $\ell$  will always be clear, the main results are summarized in Theorems 1.1 and 1.2 below.

**Theorem 1.1.** *For  $\ell = 3$ , we have*

$$(1.2) \quad R_{01}(0) = -1 + \frac{(q^3; q^3)_\infty^2 (-q; q)_\infty}{(q)_\infty (-q^3; q^3)_\infty^2},$$

$$(1.3) \quad R_{01}(1) = \frac{2(q^3; q^3)_\infty (q^6; q^6)_\infty}{(q)_\infty},$$

$$(1.4) \quad R_{01}(2) = \frac{4(-q^3; q^3)_\infty^2 (q^6; q^6)_\infty^2}{(q^2; q^2)_\infty} - \frac{6(-q^3; q^3)_\infty}{(q^3; q^3)_\infty} \sum_{n \in \mathbb{Z}} \frac{(-1)^n q^{3n^2+3n}}{1 - q^{3n+1}}.$$

**Theorem 1.2.** *For  $\ell = 5$ , we have*

$$(1.5) \quad R_{12}(0) = \frac{2q(q^{10}; q^{10})_\infty}{(q^3, q^4, q^6, q^7; q^{10})_\infty},$$

$$(1.6) \quad R_{12}(1) = \frac{-2q(-q^5; q^5)_\infty}{(q^5; q^5)_\infty} \sum_{n \in \mathbb{Z}} \frac{(-1)^n q^{5n^2+5n}}{1 - q^{5n+2}},$$

$$(1.7) \quad R_{12}(2) = \frac{2(q^{10}; q^{10})_\infty}{(q, q^4; q^5)_\infty},$$

$$(1.8) \quad R_{12}(3) = \frac{-2(q^{10}; q^{10})_\infty}{(q^2, q^3; q^5)_\infty},$$

$$(1.9) \quad R_{12}(4) = \frac{6(-q^5; q^5)_\infty}{(q^5; q^5)_\infty} \sum_{n \in \mathbb{Z}} \frac{(-1)^n q^{5n^2+5n}}{1 - q^{5n+1}} - \frac{4(q^2, q^8, q^{10}; q^{10})_\infty}{(q^4, q^6; q^{10})_\infty^2 (q, q^9; q^{10})_\infty},$$

$$(1.10) \quad R_{02}(0) = -1 + \frac{(-q^2, -q^3; q^5)_\infty (q^5; q^5)_\infty}{(q^2, q^3; q^5)_\infty (-q^5; q^5)_\infty},$$

$$(1.11) \quad R_{02}(1) = \frac{2(q^4, q^6, q^{10}; q^{10})_\infty}{(q^2, q^8; q^{10})_\infty^2 (q^3, q^7; q^{10})_\infty} + \frac{4q(-q^5; q^5)_\infty}{(q^5; q^5)_\infty} \sum_{n \in \mathbb{Z}} \frac{(-1)^n q^{5n^2+5n}}{1 - q^{5n+2}},$$

$$(1.12) \quad R_{02}(2) = 0,$$

$$(1.13) \quad R_{02}(3) = \frac{2(q^{10}; q^{10})_\infty}{(q^2, q^3; q^5)_\infty},$$

$$(1.14) \quad R_{02}(4) = \frac{2(q^2, q^8, q^{10}; q^{10})_\infty}{(q^4, q^6; q^{10})_\infty^2 (q, q^9; q^{10})_\infty} - \frac{2(-q^5; q^5)_\infty}{(q^5; q^5)_\infty} \sum_{n \in \mathbb{Z}} \frac{(-1)^n q^{5n^2+5n}}{1 - q^{5n+1}}.$$

Here we have employed the standard basic hypergeometric series notation [17],

$$(a_1, a_2, \dots, a_j; q)_n = \prod_{k=0}^{n-1} (1 - a_1 q^k)(1 - a_2 q^k) \cdots (1 - a_j q^k),$$

following the custom of dropping the “;  $q$ ” unless the base is something other than  $q$ . We should also remark that if the number of overpartitions of  $n$  with rank  $m$  is denoted by  $\overline{N}(m, n)$ , then conjugating Ferrers diagrams shows that  $\overline{N}(m, n) = \overline{N}(-m, n)$  [22]. Hence the values of  $s$  and  $t$  considered in Theorems 1.1 and 1.2 are sufficient to find any rank difference generating function  $R_{st}(d)$ .

To prove our main theorems we adapt the method of Atkin and Swinnerton-Dyer [3]. This may be generally described as regarding groups of identities as equalities between polynomials of degree  $\ell - 1$  in  $q$  whose coefficients are power series in  $q^\ell$ . Specifically, we first consider the expression

$$(1.15) \quad \sum_{n=0}^{\infty} \left\{ \overline{N}(s, \ell, n) - \overline{N}(t, \ell, n) \right\} q^n \frac{(q)_\infty}{2(-q)_\infty}.$$

By (2.6), (2.7), and (5.3), we write (1.15) as a polynomial in  $q$  whose coefficients are power series in  $q^\ell$ . We then alternatively express (1.15) in the same manner using Theorem 1.2 and Lemma 3.1. Finally, we use various  $q$ -series identities to show that these two resulting polynomials are the same for each pair of values of  $s$  and  $t$ . We should stress that this technique requires knowing all of the generating function formulas for the rank differences beforehand. We cannot prove a generating function for  $R_{st}(d)$  for some  $d$  without proving them for all  $d$ .

The paper is organized as follows. In Section 2 we collect some basic definitions, notations and generating functions. In Section 3 we record a number of equalities between an infinite

product and a sum of infinite products. These are ultimately required for the simplification of identities that end up being more complex than we would like, principally because there is only one 0 in Theorem 1.2. In Section 4 we prove two key  $q$ -series identities relating generalized Lambert series to infinite products, and in Section 5 we give the proofs of Theorems 1.1 and 1.2.

Before proceeding, it is worth mentioning that there is a theoretical reason why some of the rank differences in the main theorems are modular and others are not. We do not go into great detail here, but the weak Maass forms that lie behind the rank differences  $\overline{N}(s, \ell, \ell n + d) - \overline{N}(t, \ell, \ell n + d)$  [5] can be shown in many cases (such as when  $-d$  is not a square modulo  $\ell$ ) to be weakly holomorphic modular forms. This has been carried out in detail for the rank differences for ordinary partitions in [7]. For now, however, it seems that the groups involved are too small (and the number of inequivalent cusps too large) to justify pursuing proofs of identities for rank differences using this framework. In any case, the present technique will always have the advantage of providing formulas for all of the generating functions for rank differences, modular or not.

## 2. PRELIMINARIES

We begin by introducing some notation and definitions, essentially following [3]. With  $y = q^\ell$ , let

$$r_s(d) := \sum_{n=0}^{\infty} \overline{N}(s, \ell, \ell n + d) y^n$$

and

$$r_{st}(d) := r_s(d) - r_t(d).$$

Thus we have

$$\sum_{n=0}^{\infty} \overline{N}(s, \ell, n) q^n = \sum_{d=0}^{\ell-1} r_s(d) q^d.$$

To abbreviate the sums occurring in Theorems 1.1 and 1.2, we define

$$\Sigma(z, \zeta, q) := \sum_{n \in \mathbb{Z}} \frac{(-1)^n \zeta^n q^{n^2+n}}{1 - zq^n}.$$

Henceforth we assume that  $a$  is not a multiple of  $\ell$ . We write

$$\Sigma(a, b) := \Sigma(y^a, y^b, y^\ell) = \sum_{n \in \mathbb{Z}} \frac{(-1)^n y^{bn + \ell n(n+1)}}{1 - y^{\ell n + a}}.$$

and

$$\Sigma(0, b) := \sum'_{n \in \mathbb{Z}} \frac{(-1)^n y^{bn + \ell n(n+1)}}{1 - y^{\ell n}},$$

where the prime means that the term corresponding to  $n = 0$  is omitted.

To abbreviate the products occurring in Theorems 1.1 and 1.2, we define

$$P(z, q) := \prod_{r=1}^{\infty} (1 - zq^{r-1})(1 - z^{-1}q^r),$$

$$P(a) := P(y^a, y^\ell),$$

and

$$P(0) := \prod_{r=1}^{\infty} (1 - y^{\ell r}).$$

Note that  $P(0)$  is not  $P(a)$  evaluated at  $a = 0$ . We also have the relations

$$(2.1) \quad P(z^{-1}q, q) = P(z, q)$$

and

$$(2.2) \quad P(zq, q) = -z^{-1}P(z, q).$$

From (2.1) and (2.2), we have

$$(2.3) \quad P(\ell - a) = P(a)$$

and

$$(2.4) \quad P(-a) = P(\ell + a) = -y^{-a}P(a).$$

In [22], it is shown that the two-variable generating function for  $\overline{N}(m, n)$  is

$$(2.5) \quad \sum_{n=0}^{\infty} \overline{N}(m, n)q^n = \frac{2(-q)_{\infty}}{(q)_{\infty}} \sum_{n=1}^{\infty} (-1)^{n-1} q^{n^2 + |m|n} \frac{1 - q^n}{1 + q^n}.$$

From this we may easily deduce that the generating function for  $\overline{N}(s, m, n)$  is

$$(2.6) \quad \sum_{n=0}^{\infty} \overline{N}(s, m, n)q^n = \frac{2(-q)_{\infty}}{(q)_{\infty}} \sum'_{n \in \mathbb{Z}} \frac{(-1)^n q^{n^2 + n} (q^{sn} + q^{(m-s)n})}{(1 + q^n)(1 - q^{mn})}.$$

Hence it will be beneficial to consider sums of the form

$$(2.7) \quad \overline{S}(b) := \sum'_{n \in \mathbb{Z}} \frac{(-1)^n q^{n^2 + bn}}{1 - q^{\ell n}}.$$

We will require the relation

$$(2.8) \quad \overline{S}(b) = -\overline{S}(\ell - b),$$

which follows from the substitution  $n \rightarrow -n$  in (2.7). We shall also exploit the fact that the functions  $\overline{S}(\ell)$  are essentially infinite products.

**Lemma 2.1.** *We have*

$$\overline{S}(\ell) = \frac{-(q)_\infty}{2(-q)_\infty} + \frac{1}{2}.$$

*Proof.* Using the relation (2.8), we have

$$\begin{aligned} -2S(\ell) &= -2 \sum'_{n \in \mathbb{Z}} \frac{(-1)^n q^{n^2 + \ell n}}{1 - q^{\ell n}} \\ &= \sum'_{n \in \mathbb{Z}} \frac{(-1)^n q^{n^2}}{1 - q^{\ell n}} - \sum'_{n \in \mathbb{Z}} \frac{(-1)^n q^{n^2 + \ell n}}{1 - q^{\ell n}} \\ &= \sum'_{n \in \mathbb{Z}} (-1)^n q^{n^2}. \end{aligned}$$

The lemma now follows upon applying the case  $z = -1$  of Jacobi's triple product identity,

$$(2.9) \quad \sum_{n \in \mathbb{Z}} z^n q^{n^2} = (-zq, -q/z, q^2; q^2)_\infty.$$

□

### 3. INFINITE PRODUCT IDENTITIES

In this section we record some identities involving infinite products. These will be needed later on for simplification and verification of certain identities. First, we have a result which is the analogue of Lemma 6 in [3].

**Lemma 3.1.** *We have*

$$(3.1) \quad \frac{(q)_\infty}{(-q)_\infty} = \frac{(q^9; q^9)_\infty}{(-q^9; q^9)_\infty} - 2q(q^3, q^{15}, q^{18}; q^{18})_\infty$$

and

$$(3.2) \quad \frac{(q)_\infty}{(-q)_\infty} = \frac{(q^{25}; q^{25})_\infty}{(-q^{25}; q^{25})_\infty} - 2q(q^{15}, q^{35}, q^{50}; q^{50})_\infty + 2q^4(q^5, q^{45}, q^{50}; q^{50})_\infty.$$

*Proof.* This really just amounts to two special cases of [1, Theorem 1.2]. Indeed, (3.1) is [1, Eq. (1.18)]. We give the details for (3.2). Beginning with Jacobi's triple product identity (2.9), we have

$$\begin{aligned} \frac{(q)_\infty}{(-q)_\infty} &= \sum_{\substack{n \in \mathbb{Z} \\ n \equiv 0 \pmod{5}}} (-1)^n q^{n^2} + \sum_{\substack{n \in \mathbb{Z} \\ n \equiv \pm 1 \pmod{5}}} (-1)^n q^{n^2} + \sum_{\substack{n \in \mathbb{Z} \\ n \equiv \pm 2 \pmod{5}}} (-1)^n q^{n^2} \\ &= \sum_{n \in \mathbb{Z}} (-1)^n q^{25n^2} + 2 \sum_{n \in \mathbb{Z}} (-1)^{5n+1} q^{(5n+1)^2} + 2 \sum_{n \in \mathbb{Z}} (-1)^{5n+2} q^{(5n+2)^2} \\ &= \sum_{n \in \mathbb{Z}} (-1)^n q^{25n^2} - 2q \sum_{n \in \mathbb{Z}} (-1)^n q^{25n^2 + 10n} + 2q^4 \sum_{n \in \mathbb{Z}} (-1)^n q^{25n^2 + 20n}. \end{aligned}$$

Again using (2.9), we obtain the right hand side of (3.2). □

Next, we quote a result of Hickerson [18, Theorem 1.1] along with some of its corollaries.

**Lemma 3.2.**

$$P(x, q)P(z, q)(q)_\infty^2 = P(-xz, q^2)P(-qz/x, q^2)(q^2; q^2)_\infty^2 - xP(-xzq, q^2)P(-z/x, q^2)(q^2; q^2)_\infty^2.$$

The first corollary was recorded by Hickerson [18, Theorem 1.2]. It follows by applying Lemma 3.2 twice, once with  $x$  replaced by  $-x$  and once with  $z$  replaced by  $-z$ , and then subtracting.

**Lemma 3.3.**

$$P(-x, q)P(z, q)(q)_\infty^2 - P(x, q)P(-z, q)(q)_\infty^2 = 2xP(z/x, q^2)P(xzq, q^2)(q^2; q^2)_\infty^2.$$

The second corollary follows just as the first, except we add instead of subtract in the final step.

**Lemma 3.4.**

$$P(-x, q)P(z, q)(q)_\infty^2 + P(x, q)P(-z, q)(q)_\infty^2 = 2P(xz, q^2)P(qz/x, q^2)(q^2; q^2)_\infty^2.$$

For the third corollary we subtract Lemma 3.2 with  $z$  replaced by  $-z$  from three times Lemma 3.2 with  $x$  replaced by  $-x$ .

**Lemma 3.5.**

$$\begin{aligned} 3P(-x, q)P(z, q)(q)_\infty^2 - P(x, q)P(-z, q)(q)_\infty^2 \\ = 2P(xz, q^2)P(zq/x, q^2)(q^2; q^2)_\infty^2 + 4xP(xzq, q^2)P(z/x, q^2)(q^2; q^2)_\infty^2. \end{aligned}$$

Finally we record the addition theorem as stated in [3, Eq. (3.7)].

**Lemma 3.6.**

$$P^2(z, q)P(\zeta t, q)P(\zeta/t, q) - P^2(\zeta, q)P(zt, q)P(z/t, q) + \zeta/tP^2(t, q)P(z\zeta, q)P(z/\zeta, q) = 0.$$

## 4. TWO LEMMAS

The proofs of Theorems 1.1 and 1.2 will follow from identities which relate the sums  $\Sigma(a, b)$  to the products  $P(a)$  and  $P(0)$ . The key steps are Lemmas 4.1 and 4.2 below, which are similar to Lemmas 7 and 8 in [3].

**Lemma 4.1.** *We have*

$$(4.1) \quad \sum_{n=-\infty}^{\infty} (-1)^n q^{n^2+n} \left[ \frac{\zeta^{-2n}}{1 - z\zeta^{-1}q^n} + \frac{\zeta^{2n+2}}{1 - z\zeta q^n} \right] = \frac{\zeta(\zeta^2, q\zeta^{-2}, -1, -q)_\infty}{(\zeta, q\zeta^{-1}, -\zeta, -q\zeta^{-1})_\infty} \sum_{n=-\infty}^{\infty} (-1)^n \frac{q^{n^2+n}}{1 - zq^n} \\ + \frac{(\zeta, q\zeta^{-1}, \zeta^2, q\zeta^{-2}, -z, -qz^{-1}, q, q)_\infty}{(z, qz^{-1}, z\zeta, qz^{-1}\zeta^{-1}, z\zeta^{-1}, q\zeta z^{-1}, -\zeta, -q\zeta^{-1})_\infty}.$$

*Proof.* This may be deduced from [19, Eq. (4), p.236] by making the substitutions  $a_3 = \sqrt{a_1 q}$ ,  $a_4 = z\sqrt{a_1/q}$ ,  $a_{10} = \sqrt{a_1 q}/z$ ,  $a_9 = -\sqrt{a_1 q}$ , letting  $a_7$  and  $a_8$  tend to infinity, writing  $\zeta = \sqrt{q/a_1}$  and simplifying. One might also argue as in [3, pp. 94-96]. But the simplest way to establish

the truth of (4.1), kindly pointed out to us by S.H. Chan, is to observe that it is essentially the case  $r = 1$ ,  $s = 3$ ,  $a_1 = -z$ ,  $b_1 = z/\zeta$ ,  $b_2 = z\zeta$ , and  $b_3 = z$  of [8, Theorem 2.1],

$$(4.2) \quad \begin{aligned} & \frac{P(a_1, q) \cdots P(a_r, q)(q)_\infty^2}{P(b_1, q) \cdots P(b_s, q)} \\ &= \frac{P(a_1/b_1, q) \cdots P(a_r/b_1, q)}{P(b_2/b_1, q) \cdots P(b_s/b_1, q)} \sum_{n \in \mathbb{Z}} \frac{(-1)^{(s-r)n} q^{(s-r)n(n+1)/2}}{1 - b_1 q^n} \left( \frac{a_1 \cdots a_r b_1^{s-r-1}}{b_2 \cdots b_s} \right)^n \\ &+ \text{idem}(b_1; b_2, \dots, b_s). \end{aligned}$$

Here we use the notation

$$\begin{aligned} & F(b_1, b_2, \dots, b_m) + \text{idem}(b_1; b_2, \dots, b_m) \\ &:= F(b_1, b_2, \dots, b_m) + F(b_2, b_1, b_3, \dots, b_m) + \cdots + F(b_m, b_2, \dots, b_{m-1}, b_1). \end{aligned}$$

□

We shall use the following specialization of Lemma 4.1, which is the case  $\zeta = y^a$ ,  $z = y^b$ , and  $q = y^\ell$ :

$$(4.3) \quad \begin{aligned} & y^{2a} \Sigma(a + b, 2a) + \Sigma(b - a, -2a) - y^a \frac{P(2a)P(-1, y^\ell)}{P(a)P(-y^a, y^\ell)} \Sigma(b, 0) \\ & - \frac{P(a)P(2a)P(-y^b, y^\ell)P(0)^2}{P(b+a)P(b-a)P(b)P(-y^a, y^\ell)} = 0. \end{aligned}$$

We now define

$$\begin{aligned} g(z, q) &:= z \frac{P(z^2, q)P(-1, q)}{P(z, q)P(-z, q)} \Sigma(z, 1, q) - z^2 \Sigma(z^2, z^2, q) \\ & - \sum_{n=-\infty}^{\infty} \frac{(-1)^n z^{-2n} q^{n(n+1)}}{1 - q^n} \end{aligned}$$

and

$$(4.4) \quad g(a) := g(y^a, y^\ell) = y^a \frac{P(2a)P(-1, y^\ell)}{P(a)P(-y^a, y^\ell)} \Sigma(a, 0) - y^{2a} \Sigma(2a, 2a) - \Sigma(0, -2a).$$

The second key lemma is the following.

**Lemma 4.2.** *We have*

$$(4.5) \quad 2g(z, q) - g(z^2, q) + \frac{1}{2} = \frac{(q)_\infty^2 P(-z^4, q)}{P(z^4, q)P(-1, q)} + z \frac{P(-1, q)^2 (q)_\infty^2 P(z^2, q)}{P(z, q)^2 P(-z, q)^2}$$

and

$$(4.6) \quad g(z, q) + g(z^{-1}q, q) = 1.$$



*Proof.* We first require a short computation involving  $\Sigma(z, \zeta, q)$ . Note that

$$\begin{aligned}
(4.7) \quad z^2 \Sigma(z, \zeta, q) + \zeta \Sigma(zq, \zeta, q) &= \sum_{n=-\infty}^{\infty} (-1)^n \frac{z^2 \zeta^n q^{n(n+1)}}{1 - zq^n} + \sum_{n=-\infty}^{\infty} (-1)^n \frac{\zeta^{n+1} q^{n(n+1)}}{1 - zq^{n+1}} \\
&= \sum_{n=-\infty}^{\infty} \zeta^n q^{n(n-1)} \left( \frac{z^2 q^{2n} - 1}{1 - zq^n} \right) \\
&= - \sum_{n=-\infty}^{\infty} (-1)^n \zeta^n q^{n(n-1)} (1 + zq^n)
\end{aligned}$$

upon writing  $n - 1$  for  $n$  in the second sum of the first equation. Taking  $\zeta = 1$  yields

$$\begin{aligned}
(4.8) \quad z^2 \Sigma(z, 1, q) + \Sigma(zq, 1, q) &= - \sum_{n=-\infty}^{\infty} (-1)^n q^{n(n-1)} (1 + zq^n) \\
&= -z \sum_{n=-\infty}^{\infty} (-1)^n q^{n^2} \\
&= -z \prod_{r=1}^{\infty} \frac{1 - q^r}{1 + q^r}
\end{aligned}$$

where (2.9) was used for the last step. Now write  $g(z, q)$  in the form

$$g(z, q) = f_1(z) - f_2(z) - f_3(z)$$

where

$$f_1(z) := z \frac{P(z^2, q)P(-1, q)}{P(z, q)P(-z, q)} \Sigma(z, 1, q),$$

$$f_2(z) := z^2 \Sigma(z^2, z^2, q),$$

and

$$f_3(z) := \sum_{n=-\infty}^{\infty} \frac{(-1)^n z^{-2n} q^{n(n+1)}}{1 - q^n}.$$

By (2.1), (2.2), and (4.8),

$$\begin{aligned}
(4.9) \quad f_1(zq) - f_1(z) &= \frac{P(z^2, q)P(-1, q)}{P(z, q)P(-z, q)} \prod_{r=1}^{\infty} \frac{1 - q^r}{1 + q^r} \\
&= 2 \sum_{n=-\infty}^{\infty} (-1)^n z^{2n} q^{n^2}.
\end{aligned}$$

A similar argument as in (4.7) yields

$$(4.10) \quad f_2(zq) - f_2(z) = \sum_{n=-\infty}^{\infty} (-1)^n z^{2n-2} q^{n(n-1)} + \sum_{n=-\infty}^{\infty} (-1)^n z^{2n} q^{n^2}$$

and

$$(4.11) \quad f_3(zq) - f_3(z) = -2 + \sum_{n=-\infty}^{\infty} (-1)^n z^{-2n} q^{n(n-1)} + \sum_{n=-\infty}^{\infty} (-1)^n z^{-2n} q^{n^2}.$$

Adding (4.10) and (4.11), then subtracting from (4.9) gives

$$(4.12) \quad g(z, q) - g(zq, q) = -2.$$

If we now define

$$f(z) := 2g(z, q) - g(z^2, q) + \frac{1}{2} - \frac{(q)_\infty^2 P(-z^4, q)}{P(z^4, q)P(-1, q)} - z \frac{P(-1, q)^2 (q)_\infty^2 P(z^2, q)}{P(z, q)^2 P(-z, q)^2},$$

then from (2.1), (2.2), and (4.12), one can verify that

$$(4.13) \quad f(zq) - f(z) = 0.$$

Now, the only possible poles of  $f(z)$  are simple ones at points equivalent (under  $z \rightarrow zq$ ) to those given by  $z^4 = 1, q, q^2$ , or  $q^3$ . For each such point  $w$  it is easy to calculate  $\lim_{z \rightarrow w} (z - w)f(z)$  and see that there are, in fact, no poles. Hence  $f(z)$  is analytic except at  $z = 0$ , and applying (4.13) to its Laurent expansion around this point shows that  $f(z)$  is constant. Next, let us show that  $f(-1) = 0$ . Since the non-constant terms in  $f(z)$  have simple poles at  $z = -1$ , we must consider  $\lim_{z \rightarrow -1} \frac{d}{dz}(z + 1)f(z)$ . We omit the computation, but mention that the term  $g(z, q)$  gives  $-1/4$ ,  $g(z^2, q)$  gives  $7/8$ , and the last two terms give  $-1/8$  and  $1$ , respectively. Then  $2(-1/4) - 7/8 + 1/2 + 1/8 - 1 = 0$ , and we conclude that  $f(z)$  is identically 0. This proves (4.5).

To prove (4.6), it suffices to show, after (4.12),

$$(4.14) \quad g(z^{-1}, q) + g(z, q) = -1.$$

Note that

$$(4.15) \quad \begin{aligned} \Sigma(z, 1, q) + z^{-2}\Sigma(z^{-1}, 1, q) &= \sum_{n=-\infty}^{\infty} (-1)^n \frac{q^{n(n+1)}}{1 - zq^n} - z^{-1} \sum_{n=-\infty}^{\infty} (-1)^n \frac{q^{n^2}}{1 - zq^n} \\ &= -z^{-1} \sum_{n=-\infty}^{\infty} (-1)^n q^{n^2} \end{aligned}$$

where we have written  $-n$  for  $n$  in the second sum in the first equation. Thus, by (2.1), (2.2), and (4.15), we have

$$(4.16) \quad f_1(z) + f_1(z^{-1}) = -z^{-1} \sum_{n=-\infty}^{\infty} (-1)^n q^{n^2} \frac{P(z^2, q)P(-1, q)}{P(z, q)P(-z, q)}.$$

Again, a similar argument as in (4.15) gives

$$(4.17) \quad f_2(z) + f_2(z^{-1}) = - \sum_{n=-\infty}^{\infty} (-1)^n z^{2n} q^{n^2}$$

and

$$(4.18) \quad f_3(z) + f_3(z^{-1}) = 1 - \sum_{n=-\infty}^{\infty} (-1)^n z^{2n} q^{n^2}.$$

Adding (4.17) and (4.18), then subtracting from (4.16) yields (4.14). □

Letting  $z = y^a$  and  $q = y^\ell$  in Lemma 4.2, we get

$$(4.19) \quad 2g(a) - g(2a) + \frac{1}{2} = \frac{P(-y^{4a}, y^\ell)P(0)^2}{P(4a)P(-1, y^\ell)} + y^a \frac{P(-1, y^\ell)^2 P(0)^2 P(2a)}{P(a)^2 P(-y^a, y^\ell)^2}$$

and

$$(4.20) \quad g(a) + g(q - a) = 1.$$

These two identities will be of key importance in the proofs of Theorems 1.1 and 1.2.

## 5. PROOFS OF THEOREMS 1.1 AND 1.2

We now compute the sums  $\overline{S}(\ell - 2m)$ . The reason for this choice is two-fold. First, we would like to obtain as simple an expression as possible in the final formulation (5.3). Secondly, to prove Theorem 1.2, we will need to compute  $\overline{S}(1)$  and  $\overline{S}(3)$ . For  $\ell = 5$ , we can then choose  $m = 2$  and  $m = 1$  respectively. As this point, we follow the idea of Section 6 in [3]. Namely, we write

$$(5.1) \quad n = \ell r + m + b,$$

where  $-\infty < r < \infty$ . The idea is to simplify the exponent of  $q$  in  $\overline{S}(\ell - 2m)$ . Thus

$$\ell n - 2mn + n^2 = \ell^2 r(r + 1) + 2b\ell r + (b + m)(b - m + \ell).$$

We now substitute (5.1) into (2.7) and let  $b$  take the values  $0, \pm a$ , and  $\pm m$ . Here  $a$  runs through  $1, 2, \dots, \frac{\ell-1}{2}$  where the value  $a \equiv \pm m \pmod{\ell}$  is omitted. As in [3], we use the notation  $\sum_a''$  to denote the sum over these values of  $a$ . We thus obtain

$$\begin{aligned}\bar{S}(\ell - 2m) &= \sum'_{n=-\infty}^{\infty} (-1)^n \frac{q^{(\ell-2m)n+n^2}}{1-y^n} \\ &= \sum_b \sum'_{r=-\infty}^{\infty} (-1)^{r+m+b} q^{(b+m)(b-m+\ell)} \frac{y^{\ell r(r+1)+2br}}{1-y^{\ell r+m+b}},\end{aligned}$$

where  $b$  takes values  $0, \pm a$ , and  $\pm m$  and the term corresponding to  $r = 0$  and  $b = -m$  is omitted. Thus

$$(5.2) \quad \begin{aligned}\bar{S}(\ell - 2m) &= (-1)^m q^{m(\ell-m)} \Sigma(m, 0) + \Sigma(0, -2m) + y^{2m} \Sigma(2m, 2m) \\ &\quad + \sum_a'' (-1)^{m+a} q^{(a+m)(a-m+\ell)} \left\{ \Sigma(m+a, 2a) + y^{-2a} \Sigma(m-a, -2a) \right\}.\end{aligned}$$

Here the first three terms arise from taking  $b = 0, -m$ , and  $m$  respectively. We now can use (4.3) to simplify this expression. By taking  $b = m$  and dividing by  $y^{2a}$  in (4.3), the sum of the two terms inside the curly brackets becomes

$$y^{-a} \frac{P(2a)P(-1, y^\ell)}{P(a)P(-y^a, y^\ell)} \Sigma(m, 0) + y^{-2a} \frac{P(a)P(2a)P(-y^m, y^\ell)P(0)^2}{P(m)P(m+a)P(m-a)P(-y^a, y^\ell)}.$$

Similarly, upon taking  $a = m$  in (4.4), then the sum of the second and third terms in (5.2) is

$$y^m \frac{P(2m)P(-1, y^\ell)}{P(m)P(-y^m, y^\ell)} \Sigma(m, 0) - g(m).$$

In total, we have

$$(5.3) \quad \begin{aligned}\bar{S}(\ell - 2m) &= -g(m) \\ &\quad + \sum_a'' (-1)^{m+a} q^{(a+m)(a-m+\ell)} y^{-2a} \frac{P(a)P(2a)P(-y^m, y^\ell)P(0)^2}{P(m)P(m+a)P(m-a)P(-y^a, y^\ell)} \\ &\quad + \Sigma(m, 0) \left\{ (-1)^m q^{m(\ell-m)} + y^m \frac{P(2m)P(-1, y^\ell)}{P(m)P(-y^m, y^\ell)} \right. \\ &\quad \left. + \sum_a'' (-1)^{m+a} q^{(a+m)(a-m+\ell)} y^{-a} \frac{P(2a)P(-1, y^\ell)}{P(a)P(-y^a, y^\ell)} \right\}.\end{aligned}$$

We can simplify some of the terms appearing in (5.3) as we are interested in certain values of  $\ell$ ,  $m$ , and  $a$ . To this end, we prove the following result. Let  $\{ \}$  denote the coefficient of  $\Sigma(m, 0)$  in (5.3).

**Proposition 5.1.** *If  $\ell = 3$  and  $m = 1$ , then*

$$\{ \} = -q^2 \frac{(q)_\infty (-q^9; q^9)_\infty}{(-q)_\infty (q^9; q^9)_\infty}.$$

*If  $\ell = 5$ ,  $m = 2$ , and  $a = 1$ , then*

$$\{ \quad \} = q^6 \frac{(q)_\infty (-q^{25}; q^{25})_\infty}{(-q)_\infty (q^{25}; q^{25})_\infty}.$$

If  $\ell = 5$ ,  $m = 1$ ,  $a = 2$ , then

$$\{ \quad \} = -q^4 \frac{(q)_\infty (-q^{25}; q^{25})_\infty}{(-q)_\infty (q^{25}; q^{25})_\infty}.$$

*Proof.* This is a straightforward application of Lemma 3.1.  $\square$

We are now in a position to prove Theorems 1.1 and 1.2. We begin with Theorem 1.1.

*Proof.* By (2.6), (2.7), and (2.8), we have

$$(5.4) \quad \sum_{n=0}^{\infty} \left\{ \bar{N}(0, 3, n) - \bar{N}(1, 3, n) \right\} q^n \frac{(q)_\infty}{2(-q)_\infty} = 3\bar{S}(1) + \bar{S}(3).$$

By (2.3), (2.4), (5.3), and Proposition 5.1 we have

$$(5.5) \quad \bar{S}(1) = -g(1) + -q^2 \Sigma(1, 0) \frac{(q)_\infty (-q^9; q^9)_\infty}{(-q)_\infty (q^9; q^9)_\infty}.$$

By Lemma 2.1 we have

$$(5.6) \quad \bar{S}(3) = \frac{-(q)_\infty}{2(-q)_\infty} + \frac{1}{2}.$$

We need to prove that

$$-3g(1) + -3q^2 \Sigma(1, 0) \frac{(q)_\infty (-q^9; q^9)_\infty}{(-q)_\infty (q^9; q^9)_\infty} - \frac{(q)_\infty}{2(-q)_\infty} + \frac{1}{2} = \left\{ r_{01}(0)q^0 + r_{01}(1)q + r_{01}(2)q^2 \right\} \frac{(q)_\infty}{2(-q)_\infty}.$$

We now multiply the right hand side of the above expression using Lemma 3.1 and the  $r_{01}(d)$  from Theorem 1.2 (recall that  $r_{01}(d)$  is just  $R_{01}(d)$  with  $q$  replaced by  $q^3$ ). We then equate coefficients of powers of  $q$  and verify the resulting identities. The only power of  $q$  for which the resulting equation does not follow easily upon cancelling factors in infinite products is the constant term. We obtain

$$-3g(1) + \frac{1}{2} = \frac{(q^9; q^9)_\infty^3 (-q^3; q^3)_\infty}{2(q^3; q^3)_\infty (-q^9; q^9)_\infty^3} - 4q^3 \frac{(-q^9; q^9)_\infty^3 (q^{18}; q^{18})_\infty^3}{(q^6; q^6)_\infty (-q^3; q^3)_\infty}.$$

But this follows from (4.19) and some simplification. This then completes the proof of Theorem 1.1.  $\square$

We now turn to Theorem 1.2.

*Proof.* We begin with the rank differences  $R_{12}(d)$ . By (2.6), (2.7), and (2.8), we have

$$(5.7) \quad \sum_{n=0}^{\infty} \left\{ \bar{N}(1, 5, n) - \bar{N}(2, 5, n) \right\} q^n \frac{(q)_\infty}{2(-q)_\infty} = -\bar{S}(1) - 3\bar{S}(3)$$

and by (2.3), (2.4), (5.3), and Proposition 5.1,

$$(5.8) \quad \bar{S}(1) = -g(2) + qy\Sigma(2, 0) \frac{(q)_\infty(-q^{25}; q^{25})_\infty}{(-q)_\infty(q^{25}; q^{25})_\infty} - q^2 \frac{(q^{25}; q^{25})_\infty^2(-q^{10}, -q^{15}; q^{25})_\infty}{(q^{10}, q^{15}; q^{25})_\infty(-q^5, -q^{20}; q^{25})_\infty}$$

and

$$(5.9) \quad \bar{S}(3) = -g(1) - q^4\Sigma(1, 0) \frac{(q)_\infty(-q^{25}; q^{25})_\infty}{(-q)_\infty(q^{25}; q^{25})_\infty} + q^3 \frac{(q^{25}; q^{25})_\infty^2(-q^5, -q^{20}; q^{25})_\infty}{(q^5, q^{20}; q^{25})_\infty(-q^{10}, -q^{15}; q^{25})_\infty}.$$

By (5.7), (5.8), and (5.9), we need to prove

$$\begin{aligned} & g(2) - qy\Sigma(2, 0) \frac{(q)_\infty(-q^{25}; q^{25})_\infty}{(-q)_\infty(q^{25}; q^{25})_\infty} + q^2 \frac{(q^{25}; q^{25})_\infty^2(-q^{10}, -q^{15}; q^{25})_\infty}{(q^{10}, q^{15}; q^{25})_\infty(-q^5, -q^{20}; q^{25})_\infty} \\ & + 3g(1) + 3q^4\Sigma(1, 0) \frac{(q)_\infty(-q^{25}; q^{25})_\infty}{(-q)_\infty(q^{25}; q^{25})_\infty} - 3q^3 \frac{(q^{25}; q^{25})_\infty^2(-q^5, -q^{20}; q^{25})_\infty}{(q^5, q^{20}; q^{25})_\infty(-q^{10}, -q^{15}; q^{25})_\infty} \\ & = \left\{ r_{12}(0)q^0 + r_{12}(1)q + r_{12}(2)q^2 + r_{12}(3)q^3 + r_{12}(4)q^4 \right\} \frac{(q)_\infty}{2(-q)_\infty}. \end{aligned}$$

We now multiply the right hand side of the above expression using Lemma 3.1 and the  $R_{12}(d)$  from Theorem 1.2, equating coefficients of powers of  $q$ . The coefficients of  $q^0, q^1, q^2, q^3, q^4$  give us, respectively,

$$(5.10) \quad g(2) + 3g(1) = y \frac{(q^{25}; q^{25})_\infty^2}{(q^{15}, q^{20}, q^{30}, q^{35}; q^{50})_\infty} + 4y \frac{(q^{10}, q^{15}, q^{35}, q^{40}; q^{50})_\infty (q^{50}; q^{50})_\infty^2}{(q^{20}, q^{30}; q^{50})_\infty^2 (q^5, q^{45}; q^{50})_\infty},$$

$$(5.11) \quad y \frac{(q^{50}; q^{50})_\infty (q^{15}, q^{35}, q^{50}; q^{50})_\infty}{(q^{15}, q^{20}, q^{30}, q^{35}; q^{50})_\infty} = y \frac{(q^{50}; q^{50})_\infty (q^5, q^{45}, q^{50}; q^{50})_\infty}{(q^5, q^{20}; q^{25})_\infty},$$

$$(5.12) \quad \frac{(q^{25}; q^{25})_\infty^2(-q^{10}, -q^{15}; q^{25})_\infty}{(q^{10}, q^{15}; q^{25})_\infty(-q^5, -q^{20}; q^{25})_\infty} = \frac{(q^{25}; q^{25})_\infty^2}{(q^5, q^{20}; q^{25})_\infty} - 2y \frac{(q^{50}; q^{50})_\infty^2 (q^5, q^{45}; q^{50})_\infty}{(q^{10}, q^{15}; q^{25})_\infty},$$

$$(5.13) \quad \frac{3(q^{25}; q^{25})_\infty^2(-q^5, -q^{20}; q^{25})_\infty}{(q^5, q^{20}; q^{25})_\infty(-q^{10}, -q^{15}; q^{25})_\infty} = \frac{(q^{25}; q^{25})_\infty^2}{(q^{10}, q^{15}; q^{25})_\infty} + \frac{2(q^{50}; q^{50})_\infty^2 (q^{15}, q^{35}; q^{50})_\infty}{(q^5, q^{20}; q^{25})_\infty} \\ + 4y \frac{(q^{10}, q^{40}; q^{50})_\infty (q^{50}; q^{50})_\infty^2}{(q^{20}, q^{30}; q^{50})_\infty^2},$$

$$(5.14) \quad \frac{(q^{10}, q^{40}, q^{50}; q^{50})_\infty (q^{25}; q^{25})_\infty}{(q^{20}, q^{30}; q^{50})_\infty^2 (q^5; q^{45}; q^{50})_\infty (-q^{25}; q^{25})_\infty} = \frac{(q^{50}; q^{50})_\infty^2 (q^{15}, q^{35}; q^{50})_\infty}{(q^{10}, q^{15}; q^{25})_\infty} \\ + y \frac{(q^{50}; q^{50})_\infty^2 (q^5, q^{45}; q^{50})_\infty}{(q^{15}, q^{20}, q^{30}, q^{35}; q^{50})_\infty}.$$

Equation (5.11) is immediate. Upon clearing denominators in (5.12)-(5.14) and simplifying, we see that (5.12) is equivalent to the case  $(x, z, q) = (-q^5, -q^{10}, q^{25})$  of Lemma 3.3, (5.13) is the case  $(x, z, q) = (q^5, q^{10}, q^{25})$  of Lemma 3.5, and (5.14) follows from the case  $(z, \zeta, t, q) = (q^{20}, q^{10}, q^5, q^{50})$  of Lemma 3.6.

As for (5.10), let us take  $a = 1$  and  $a = 2$  in (4.19), and then replace  $g(4)$  by  $1 - g(1)$  using (4.20). This gives

$$3g(1) + g(2) = \frac{(-q^5, -q^{20}, q^{25}, q^{25}; q^{25})_\infty}{2(q^5, q^{20}, -q^{25}, -q^{25}; q^{25})_\infty} - \frac{4y(q^{15}, q^{35}, q^{50}, q^{50}; q^{50})_\infty}{(q^{10}, q^{25}, q^{25}, q^{40}; q^{50})_\infty} \\ - \frac{(-q^{10}, -q^{15}, q^{25}, q^{25}; q^{25})_\infty}{2(q^{10}, q^{15}, -q^{25}, -q^{25}; q^{25})_\infty} + \frac{4y^2(q^5, q^{45}, q^{50}, q^{50}; q^{50})_\infty}{(q^{20}, q^{25}, q^{25}, q^{30}; q^{50})_\infty}.$$

Now after making a common denominator in the first and third terms, we may apply the case  $(x, z, q) = (q^5, q^{10}, q^{25})$  of Lemma 3.3 to these two terms, the result being precisely the first term in (5.10). For the second and fourth terms, we make a common denominator and multiply top and bottom by  $(q^5, q^{20}; q^{25})_\infty$ . Then the case  $(z, \zeta, t, q) = (q^{20}, q^{10}, q^5, q^{50})$  of Lemma 3.6 applies and we obtain the second term in (5.10).

We now turn to the rank differences  $R_{02}(d)$ , proceeding as above. Again by (2.6), (2.7), and (2.8), we have

$$(5.15) \quad \sum_{n=0}^{\infty} \left\{ \bar{N}(0, 5, n) - \bar{N}(2, 5, n) \right\} q^n \frac{(q)_\infty}{2(-q)_\infty} = -\bar{S}(5) + 2\bar{S}(1) + \bar{S}(3).$$

By Lemma 2.1 (with  $\ell = 5$ ), (5.15), (5.8), and (5.9), it suffices to prove

$$\frac{-(q)_\infty}{2(-q)_\infty} + \frac{1}{2} - 2g(2) + 2qy\Sigma(2, 0) \frac{(q)_\infty(-q^{25}; q^{25})_\infty}{(-q)_\infty(q^{25}; q^{25})_\infty} - 2q^2 \frac{(q^{25}; q^{25})_\infty^2(-q^{10}, -q^{15}; q^{25})_\infty}{(q^{10}, q^{15}; q^{25})_\infty(-q^5, -q^{20}; q^{25})_\infty} \\ - g(1) - q^4\Sigma(1, 0) \frac{(q)_\infty(-q^{25}; q^{25})_\infty}{(-q)_\infty(q^{25}; q^{25})_\infty} + q^3 \frac{(q^{25}; q^{25})_\infty^2(-q^5, -q^{20}; q^{25})_\infty}{(q^5, q^{20}; q^{25})_\infty(-q^{10}, -q^{15}; q^{25})_\infty} \\ = \left\{ r_{02}(0)q^0 + r_{02}(1)q + r_{02}(2)q^2 + r_{02}(3)q^3 + r_{02}(4)q^4 \right\} \frac{(q)_\infty}{2(-q)_\infty}.$$

Again, equating coefficients of powers of  $q$  yields the following identities.

$$(5.16) \quad \frac{1}{2} - 2g(2) - g(1) = \frac{1}{2} \frac{(-q^{10}, -q^{15}; q^{25})_\infty (q^{25}; q^{25})_\infty^2}{(q^{10}, q^{15}; q^{25})_\infty (-q^{25}; q^{25})_\infty^2} - 2y \frac{(q^{10}, q^{40}; q^{50})_\infty (q^{15}, q^{35}; q^{50})_\infty (q^{50}; q^{50})_\infty^2}{(q^{20}, q^{30}; q^{50})_\infty^2 (q^5, q^{45}; q^{50})_\infty} \\ + 2y \frac{(q^{20}, q^{30}; q^{50})_\infty (q^5, q^{45}; q^{50})_\infty (q^{50}; q^{50})_\infty^2}{(q^{10}, q^{40}; q^{50})_\infty^2 (q^{15}, q^{35}; q^{50})_\infty},$$

$$(5.17) \quad \frac{(q^{20}, q^{30}, q^{50}; q^{50})_\infty (q^{25}; q^{25})_\infty}{(q^{10}, q^{40}; q^{50})_\infty^2 (q^{15}, q^{35}; q^{50})_\infty (-q^{25}; q^{25})_\infty} = \frac{(-q^{10}, -q^{15}; q^{25})_\infty (q^{25}; q^{25})_\infty (q^{15}, q^{35}, q^{50}; q^{50})_\infty}{(q^{10}, q^{15}; q^{25})_\infty (-q^{25}; q^{25})_\infty},$$

$$(5.18) \quad \frac{(q^{25}; q^{25})_\infty^2 (-q^{10}, -q^{15}; q^{25})_\infty}{(-q^5, -q^{20}; q^{25})_\infty (q^{10}, q^{15}; q^{25})_\infty} \\ = \frac{(q^{50}; q^{50})_\infty^2 (q^{20}, q^{30}; q^{50})_\infty}{(q^{10}, q^{40}; q^{50})_\infty^2} - y \frac{(q^{50}; q^{50})_\infty^2 (q^5, q^{45}; q^{50})_\infty}{(q^{10}, q^{15}; q^{25})_\infty},$$

$$(5.19) \quad \frac{(q^{25}; q^{25})_{\infty}^2 (-q^5, -q^{20}; q^{25})_{\infty}}{(-q^{10}, -q^{15}; q^{25})_{\infty} (q^5, q^{20}; q^{25})_{\infty}} = \frac{(q^{25}; q^{25})_{\infty}^2}{(q^{10}, q^{15}; q^{25})_{\infty}} + 2y \frac{(q^{50}; q^{50})_{\infty}^2 (q^{10}, q^{40}; q^{50})_{\infty}}{(q^{20}, q^{30}; q^{50})_{\infty}^2},$$

$$(5.20) \quad \frac{(q^{10}, q^{40}, q^{50}; q^{50})_{\infty} (q^{25}; q^{25})_{\infty}}{(q^{20}, q^{30}; q^{50})_{\infty}^2 (q^5; q^{45}; q^{50})_{\infty} (-q^{25}; q^{25})_{\infty}} + \frac{(-q^{10}, -q^{15}; q^{25})_{\infty} (q^{25}; q^{25})_{\infty} (q^5, q^{45}, q^{50}; q^{50})_{\infty}}{(q^{10}, q^{15}; q^{25})_{\infty} (-q^{25}; q^{25})_{\infty}} \\ = 2 \frac{(q^{50}; q^{50})_{\infty}^2 (q^{15}, q^{35}; q^{50})_{\infty}}{(q^{10}, q^{15}; q^{25})_{\infty}}.$$

Now, (5.17) is immediate. After clearing denominators and simplifying, (5.18) follows from the case  $(z, \zeta, t, q) = (q^{20}, q^{15}, q^{10}, q^{50})$  of Lemma 3.6 and (5.19) is the case  $(x, z, q) = (q^5, q^{10}, q^{25})$  of Lemma 3.3. For (5.20), we simplify the first term and apply the case  $(x, z, q) = (q^5, q^{10}, q^{25})$  of Lemma 3.4.

As for (5.16), taking the case  $a = 2$  of (4.19) together with an application of (4.20) gives

$$\frac{1}{2} - 2g(2) - g(1) = \frac{(-q^{10}, -q^{15}; q^{25})_{\infty} (q^{25}; q^{25})_{\infty}^2}{2(q^{10}, q^{15}; q^{25})_{\infty} (-q^{25}; q^{25})_{\infty}^2} - \frac{4y^2 (q^5, q^{45}; q^{50})_{\infty} (q^{50}; q^{50})_{\infty}^2}{(q^{20}, q^{30}; q^{50})_{\infty} (q^{25}; q^{50})_{\infty}^2}.$$

Now the first terms of the above equation and (5.16) match up, while after some simplification of the final two terms of (5.16) we may apply the case  $(x, z, q) = (q^5, q^{10}, q^{25})$  of Lemma 3.3 to obtain the final term above.  $\square$

#### ACKNOWLEDGEMENTS

The second author would like to thank the Institut des Hautes Études Scientifiques and the Max-Planck-Institut für Mathematik for their hospitality and support during the preparation of this paper.

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