Joins of graphs realisable by orthogonal matrices

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1 Introduction

1.1 Background and overview.

Given a simple graph G with vertex set $V(G) = \{1, \ldots, n\}$ and edge set $E(G)$, consider the set of matrices

$$\widetilde{S}(G) = \left\{A = (a_{ij}) \in \mathbb{R}^{n\times n} : \text{for } i \neq j,\ a_{ij} \neq 0 \iff \{i,j\} \in E(G)\right\}.$$

Define also the set of symmetric matrices $S(G) \subseteq \widetilde{S}(G)$, where $S(G) = \{A : A = A^T,\ A \in \widetilde{S}(G)\}$. Given a square matrix A, let $q(A)$ be the number of distinct eigenvalues of A. Define

$$q(G) = \min\{q(A) : A \in S(G)\}.$$

It has been shown that for a graph G, $q(G) = 2$ if and only if there exists an orthogonal matrix X in $S(G)$ [1, Section 4]. The value of $q(G)$ has been widely studied as part of the *Inverse Eigenvalue Problem for Graphs (IEP-G)*.

In this report, we focus on graphs of the form $G \vee H$, the join of G and H, and ask more generally whether an orthogonal matrix exists in $\widetilde{S}(G \vee H)$. We say a graph G is *realisable by* an orthogonal matrix if we can find an orthogonal matrix $X \in \widetilde{S}(G)$.

In Section 2, we introduce the concept of *compatible singular value multiplicity matrices*, inspired by the compatible multiplicity matrices introduced in [2, Section 2]. We show that two graphs G and H having compatible singular value multiplicity matrices is a necessary condition for $G \vee H$ being realisable by an orthogonal matrix. Under certain conditions, we will show that this requirement is also sufficient.

Section 3 focuses on applications of the theory developed in Section 2. We show a necessary and sufficient condition for the join of G and H to be realisable by an orthogonal matrix when the connected components of G and H are complete graphs. The result is as follows:

Theorem 1.1. *Let k, $l \in \mathbb{N}$, with $k \leq l$. Let G and H be two graphs with k and l connected components respectively, where the connected components of both G and H are complete graphs. Then $G \vee H$ is realisable by an orthogonal matrix if and only if $l \leq |G|$.*

The remainder of Section 3 focuses on paths and some examples. We finish with some possible future directions and questions.

1.2 Notation.

Let $\mathbb{N} = \{1, 2, \ldots\}$ be the set of positive integers, $\mathbb{N}_0 = \{0, 1, \ldots\}$ the set of non-negative integers, and $[n] = \{1, \ldots n\}$ for $n \in \mathbb{N}$.

We will consider matrices over $\mathbb{R}$, and denote by $\mathbb{R}^{m\times n}$ the set of $m \times n$ matrices with real entries, for m, $n \in \mathbb{N}$. Given a matrix X, we write $X \geq a$ for $a \in \mathbb{R}$ if all entries of X are greater than or equal to a.

Let I_n denote the $n \times n$ identity matrix, and let $0_{m\times n}$ denote the $m \times n$ matrix with all entries 0. Given a matrix $X \in \mathbb{R}^{m\times n}$, let $X^\top \in \mathbb{R}^{n\times m}$ be its transpose. Let $\mathbf{e}_i$ denote the i-th standard basis vector in $\mathbb{R}^n$ for $i = 1, \ldots, n$. Let $\mathbf{0}_n$ be the column-vector of length n with all zero entries, and let $\mathbf{1}_n$ be the column-vector of length n with all one entries. Let $\bigoplus_{i\in[k]} A_i$ denote the direct sum of square matrices A_i for $i \in [k]$, allowing for the possibility of A_i being empty.

Let $\mathrm{diag}\{a_1, \ldots, a_n\}$ denote the $n \times n$ diagonal matrix with diagonal entries $a_1, \ldots a_n$. We denote the diagonal matrix with entries $\Lambda = (\sigma_1, \ldots, \sigma_r) \in \mathbb{R}^r$ occurring with multiplicities $\mathbf{v} = (v_1, \ldots, v_r)^\top \in \mathbb{N}_0^r$ by $D_{\Lambda,\mathbf{v}} := \bigoplus_{i\in[r]} \sigma_i \mathrm{I}_{v_i}$ for $r \in \mathbb{N}$. We call $\mathbf{v}$ a *multiplicity list.*

Given a matrix $X \in \mathbb{R}^{m\times n}$, and $A \subseteq [m], B \subseteq [n]$, let $X[A, B]$ be the submatrix of X with rows A and columns B. Given a graph G and $X \in \widetilde{S}(G)$, let $X[H]$ denote the principal submatrix of X with the rows and columns of X that correspond to the vertices of a subgraph $H \subseteq G$.

Let $O(n)$ denote the set of $n \times n$ *orthogonal* matrices with real entries. That is, the set of matrices in $\mathbb{R}^{n\times n}$ with $A^\top A = AA^\top = \mathrm{I}_n$. Let $SO(n) \subseteq O(n)$ be the subset of *special orthogonal* matrices, which are matrices with the additional property of having determinant 1.

Let the number of vertices of the graph G, or the *size* of G, be denoted as $|G|$. Given two graphs G, H let the join $G \vee H$ be the graph with vertex set $V(G) \cup V(H)$ and edge set $E(G) \cup E(H) \cup F$, where F denotes all possible edges between vertices in G and vertices in H. Let $G \cup H$ denote the disjoint union of graphs.

Let $\mathrm{mult}(\sigma, A)$ for $\sigma \in \mathbb{R}_{\geq 0}$ and $A \in \mathbb{R}^{m\times n}$ with $m, n \in \mathbb{N}$ denote the multiplicity of the singular value σ in the matrix A. By *singular value list* we will always mean a list of strictly increasing non-negative real numbers, and by *eigenvalue list* we will always mean a list of strictly increasing real numbers.

2 Orthogonal Matrices and Compatibility of Singular Values

2.1 Singular value decomposition and blocks of orthogonal matrices.

Firstly, we establish some preparatory results about the singular values of blocks of orthogonal matrices. The first result is proven in [3].

Lemma 2.1. *[3, Theorem 2.1] Let U be an $n \times n$ unitary matrix, and let A be its $p \times q$ submatrix. Then A has all singular values less than or equal to one, and the number of singular values less than one (counting also zero singular values) does not exceed $n - \max(p, q)$.*

Proposition 2.2. *Let p, q, $n \in \mathbb{N}$ be such that $p+q = n$. Consider $X = \begin{pmatrix} A & B \\ C^\top & -D \end{pmatrix} \in \mathbb{R}^{n\times n}$ orthogonal with $A \in \mathbb{R}^{p\times p}$, $D \in \mathbb{R}^{q\times q}$, and B, $C \in \mathbb{R}^{p\times q}$. Then the singular values not equal to one of A and D are equal, including the multiplicities of these singular values, and likewise the singular values not equal to one of B and C are equal.*

Proof. By computing $XX^\top$ and $X^\top X$, we yield the four following identities:

1. $AA^\top + BB^\top = \mathrm{I}_p$,
2. $C^\top C + DD^\top = \mathrm{I}_q$,
3. $A^\top A + CC^\top = \mathrm{I}_p$,
4. $B^\top B + D^\top D = \mathrm{I}_q$.

For a matrix Y, $YY^\top$ and $Y^\top Y$ have the same non-zero eigenvalues, including the same multiplicities of these eigenvalues, and the non-zero singular values of a matrix Y are precisely the square roots of the non-zero eigenvalues of $Y^\top Y$ or $YY^\top$. By 1. and 4., A and D have the same singular values distinct from 1. By 1. and 3., B and C have the same singular values distinct from 1. By Lemma 2.1, these singular values distinct from 1 are less than 1. □

Proposition 2.3. *Let $X = \begin{pmatrix} D & B \\ C^\top & -D \end{pmatrix}$ be a $2n \times 2n$ orthogonal matrix, with D an $n \times n$ diagonal matrix with $m \in \mathbb{N}$ distinct non-negative entries $A := (\alpha_1, \ldots, \alpha_m)$, written as $D = D_{A,\boldsymbol{s}}$ for $\boldsymbol{s} = (s_1, \ldots, s_m) \in \mathbb{N}_0^m$ a multiplicity list. Suppose that $\alpha_i < 1$ for all $i \in [m]$. Then $B = \bigoplus_{i\in[m]}(1 - \alpha_i^2)^{\frac{1}{2}} U_i$, $C = \bigoplus_{i\in[m]}(1 - \alpha_i^2)^{\frac{1}{2}} V_i$, with U_i, V_i both $s_i \times s_i$ orthogonal matrices. Further, if $\alpha_i \neq 0$, then $U_i = V_i$ for each $i \in [m]$.*

Proof. We have

$$BB^\top = B^\top B = CC^\top = C^\top C = \mathrm{I}_n - D^2 = \bigoplus_{i\in[m]} (1 - \alpha_i^2)\, \mathrm{I}_{s_i} .$$

Let $B = U_B \Sigma_B V_B^\top$ be the singular value decomposition (SVD) of B. It follows that $\Sigma_B = \bigoplus_{i\in[m]}(1 - \alpha_i^2)^{\frac{1}{2}}\, \mathrm{I}_{s_i}$. Then $BB^\top = U_B(\mathrm{I}_n - D^2)U_B^\top = \mathrm{I}_n - D^2$. Thus, $U_B(\mathrm{I}_n - D^2) = (\mathrm{I}_n - D^2)U_B$. Then U_B commutes with D^2, and so has a block structure coinciding with that of D^2. Indeed, $(U_B D^2)_{ij} =$

$U_{Bij}D_{jj}^2 = D_{ii}^2 U_{Bij} = (D^2 U_B)_{ij}$. Therefore if $D_{ii} \neq D_{jj}$ then $D_{ii}^2 \neq D_{jj}^2$ as D is non-negative, and so $U_{Bij} = 0$. By looking at $B^\top B$ we conclude that V_B has this same block structure. Since, $B = U_B \Sigma_B V_B^\top$, B is a block-diagonal matrix. The block structure of C follows similarly.

Thus we can write $B = \bigoplus_{i\in[m]} B_i$ and $C = \bigoplus_{i\in[m]} C_i$ where each B_i, C_i is a $s_i \times s_i$ matrix. Then, $BB^\top = \bigoplus_{i\in[m]} B_i B_i^\top = \bigoplus_{i\in[m]} (1-\alpha_i)^2\, \mathrm{I}_{s_i}$, so that $B_i B_i^\top = (1-\alpha_i^2)\, \mathrm{I}_{s_i}$ for all $i \in [m]$, and likewise $C_i C_i^\top = C_i^\top C_i = B_i^\top B_i = (1-\alpha_i^2)\, \mathrm{I}_{s_i}$ for all $i \in [m]$. Therefore $(1-\alpha_i^2)^{-\frac{1}{2}} B_i$ is orthogonal for all $i \in [m]$, or $B_i = (1-\alpha_i^2)^{\frac{1}{2}} U_i$ for U_i an orthogonal matrix, and similarly $C_i = (1-\alpha_i^2)^{\frac{1}{2}} V_i$ for V_i an orthogonal $s_i \times s_i$ matrix.

The orthogonality of X yields $DB - CD = 0$, that is, $\bigoplus_{i\in[m]} \alpha_i (1-\alpha_i^2)^{\frac{1}{2}} U_i = \bigoplus_{i\in[m]} \alpha_i (1-\alpha_i^2)^{\frac{1}{2}} V_i$. Since $\alpha_i < 1$, $1 - \alpha_i^2 \neq 0$, so that if $\alpha_i \neq 0$, $U_i = V_i$. If $\alpha_i = 0$, then U_i, V_i can be distinct $s_i \times s_i$ orthogonal matrices. □

2.2 Compatibility of Singular Values.

This sub-section is heavily influenced by [2, Section 2]. The following definitions and Theorem 2.8, 2.15 are stated and proven in [2] for the case of finding graphs G, H such that $G \vee H$ is realisable by a symmetric orthogonal matrix. In the context of symmetric matrices, the results in [2] discuss compatibility of eigenvalues. However, the proofs presented here for the non-symmetric case and consideration of singular values are very similar.

Definition 2.4. Let G be a connected graph on n vertices, and let $r \in \mathbb{N}$. We call a multiplicity list $\mathbf{v} = (v_1 \;\; v_2 \;\; \cdots \;\; v_r)^\top \in \mathbb{N}_0^r$ a *singular value (SV) multiplicity vector* if $\sum_{i\in[r]} v_i = n$, and there is a singular value list $\Sigma = (\sigma_1, \dots, \sigma_r) \in \mathbb{R}^r$ with $0 \leq \sigma_1 < \cdots < \sigma_r$ and orthogonal matrices U, $V \in O(n)$ such that $UD_{\Sigma,\mathbf{v}}V^\top \in \widetilde{S}(G)$.

Definition 2.5. Let $G = G_1 \cup \cdots \cup G_k$ be a graph with k connected components, and let $r \in \mathbb{N}$. We call a matrix $V \in \mathbb{N}_0^{r\times k}$ a *singular value (SV) multiplicity matrix* if the i-th column of V is a SV multiplicity vector for the connected component G_i for all $i \in [k]$.

Definition 2.6. Let G, H be graphs with k and l connected components respectively for k, $l \in \mathbb{N}$. G and H are said to have *compatible singular value (SV) multiplicity matrices* if there exists SV multiplicity matrices $V \in \mathbb{N}_0^{r\times k}$ and $W \in \mathbb{N}_0^{r\times l}$ for G, H respectively with $r \in \mathbb{N}$, $r \geq 2$ such that $\widetilde{V}\mathbf{1}_k = \widetilde{W}\mathbf{1}_l$ and $\widetilde{V}^\top \widetilde{W} > 0$, where $\widetilde{V}, \widetilde{W}$ denote the submatrices of V, W obtained by deleting the last row. We call the matrices V, W *SV compatible.*

Remark 2.7. Definitions 2.4, 2.5, 2.6, are non-symmetric versions of [2, Definition 2.1], [2, Definition 2.2], [2, Definition 2.3] respectively. In particular, we will reference the definitions of [2] by adding the prefix EV for eigenvalue. For instance, $\mathbf{v}$ an *EV* multiplicity vector for G means there exists an eigenvalue list Λ so that $UD_{\Lambda,\mathbf{v}}U^\top \in S(G)$ for U orthogonal.

The following theorem shows that compatible SV multiplicity matrices for two graphs G, H are a necessary condition for $G \vee H$ to be realisable by an orthogonal matrix.

Theorem 2.8. *Let G and H be two graphs. If there exists an orthogonal matrix in $\widetilde{S}(G \vee H)$, then G and H have compatible SV multiplicity matrices.*

Proof. Let $G = \cup_{i\in[k]} G_i$ and $H = \cup_{j\in[l]} H_j$ be the decomposition of G and H into their connected components. Suppose that $X \in \widetilde{S}(G \vee H)$ is an orthogonal matrix. Write

$$X := \begin{pmatrix} A & B \\ C^\top & -D \end{pmatrix}.$$

Then $A = \bigoplus_{i\in[k]} A_i$ for $A_i \in \widetilde{S}(G_i)$ and $D = \bigoplus_{j\in[l]} D_j$ for $D_j \in \widetilde{S}(H_j)$. B, C are nowhere-zero matrices of size $|G| \times |H|$.

Being submatrices of the orthogonal matrix X, each A_i, D_j will have singular values less than or equal to one by Lemma 2.1. We note that the singular values of each A_i, D_j are not all one. Indeed, if some A_i has singular values all equal to one, then A_i is orthogonal, so that each row of A_i has norm 1. But then the corresponding rows of X (which have norm 1) must have all other entries equal to zero, contradicting the nowhere-zero-ness of B. The argument for each D_j is similar.

Write the distinct non-one singular values of A and D as $0 \leq \sigma_1 < \cdots < \sigma_{r-1} < 1$ for $r \in \mathbb{N}$, $r > 1$, and let $\sigma_r = 1$. We define SV multiplicity matrices for G and H. Let $V = (v_{ij}) \in \mathbb{N}_0^{r\times k}$ have entries $v_{ij} = \mathrm{mult}(\sigma_i, A_j)$, that is the multiplicity of the singular value σ_i in the SVD of A_j. Define the singular value multiplicity matrix $W = (w_{ij}) \in \mathbb{N}_0^{r\times l}$ for H as $w_{ij} = \mathrm{mult}(\sigma_i, D_j)$. We show that V and W are compatible SV multiplicity matrices.

Each A_i has a SVD $A_i = U_{A_i}\Sigma_{A_i}V_{A_i}^\top$, and likewise each D_j has a SVD $D_j = U_{D_j}\Sigma_{D_j}V_{D_j}^\top$. Let $U_A = \bigoplus_{i\in[k]} U_{A_i}$, $U_D = \bigoplus_{j\in[l]} U_{D_j}$, $V_A = \bigoplus_{i\in[k]} V_{A_i}$, and $V_D = \bigoplus_{j\in[l]} V_{D_j}$. Let $X' = (U_A \oplus U_D)^\top X(V_A \oplus V_D)$. Then,

$$X' = \begin{pmatrix} \bigoplus_{i\in[k]} \Sigma_{A_i} & U_A^\top BV_D \\ U_D^\top C^\top V_A & -\bigoplus_{j\in[l]} \Sigma_{D_j} \end{pmatrix}.$$

X' is orthogonal as the product of orthogonal matrices. Any row of X' with a ± 1 on the diagonal will have all other entries in the corresponding row and column equal to zero. Let X'' be the submatrix of X' obtained after removing these rows and columns. Then X'' is orthogonal. Write $X'' := \begin{pmatrix} \Sigma_1 & F \\ G^\top & -\Sigma_2 \end{pmatrix}$. By Proposition 2.2, the singular values distinct from 1 of A and D are equal, so that Σ_1 and Σ_2 have equal diagonal entries. Since Σ_1 and Σ_2 have the same entries (up to re-ordering) it follows that $\widetilde{V}\mathbf{1}_k = \widetilde{W}\mathbf{1}_l$, that is, the multiplicities of each singular value distinct from 1 in A and D are equal.

We show also that $\widetilde{V}^\top \widetilde{W} > 0$. This corresponds to every pair of matrices A_i, D_j containing at least one common singular value distinct from 1 for all $i \in [k]$, $j \in [l]$. Let $Q = U_A^\top BV_D$. Since $U_A Q V_D^\top$ is nowhere-zero, it follows that Q is non-zero. More specifically, since $U_A = \bigoplus_{i\in[k]} U_{A_i}, V_D = \bigoplus_{j\in[l]} V_{D_j}$ are-block diagonal matrices, the block partition of $Q = (Q_{ij})_{i\in[k],j\in[l]}$ corresponding to each pair $U_{A_i}, V_{D_j}^\top$ must have each Q_{ij} non-zero. Likewise, $R = (U_D^\top C^\top V_A)^\top$ has each block partition non-zero for $R = (R_{ij})$, where each R_{ij} is the block corresponding to $V_{A_i}^\top, U_{D_j}$. These blocks R_{ij}, Q_{ij} will contain zero rows and vectors corresponding to the singular value 1, so that the non-zero entries of Q_{ij}, R_{ij} will correspond to a singular value less than one.

The orthogonality of X' means that $(\bigoplus_{i\in[k]} \Sigma_{A_i})R = Q(\bigoplus_{j\in[l]} \Sigma_{D_j})$, or that $\Sigma_{A_i}R_{ij} = Q_{ij}\Sigma_{D_j}$ for all $i \in [k]$, $j \in [l]$. We also yield $(\bigoplus_{i\in[k]} \Sigma_{A_i})Q = R(\bigoplus_{j\in[l]} \Sigma_{D_j})$, that is $\Sigma_{A_i}Q_{ij} = R_{ij}\Sigma_{D_j}$ for all $i \in [k]$, $j \in [l]$. But then $\Sigma_{A_i}R_{ij}\Sigma_{D_j} = Q_{ij}\Sigma_{D_j}^2 = \Sigma_{A_i}^2 Q_{ij}$. Since Q_{ij} contains a non-zero entry, it follows that some squared diagonal entry of Σ_{A_i} equals some squared diagonal entry of Σ_{D_j}, and so Σ_{A_i} and Σ_{D_j} contain a common diagonal entry, as their entries are all non-negative. Therefore each pair of matrices A_i, D_j share a singular value that is not 1, meaning that $(\widetilde{V}^\top\widetilde{W})_{ij} = \sum_{s\in[r-1]} \mathrm{mult}(\sigma_s, A_i)\,\mathrm{mult}(\sigma_s, D_j) > 0$ for all $i \in [k]$, $j \in [l]$, as required. □

In establishing a converse for Theorem 2.8, the most difficult step will be to ensure that we can satisfy the nowhere-zero pattern of the off-diagonal blocks of a matrix in $\widetilde{S}(G \vee H)$. To do this, we introduce some stricter conditions on our graphs G, H.

Definition 2.9. Let $\mathbf{v} \in \mathbb{N}_0^r$ be a singular value multiplicity vector for a connected graph G on n vertices, with $n, r \in \mathbb{N}$. We say that $\mathbf{v}$ is *SV sane* if for any singular value list $\Sigma = (\sigma_1, \ldots, \sigma_r) \in \mathbb{R}^r$ with $0 \leq \sigma_1 < \cdots < \sigma_r$ and $D_{\Sigma,\mathbf{v}} \neq 0$, there are nowhere-zero orthogonal matrices U, $V \in O(n)$ such that $UD_{\Sigma,\mathbf{v}}V^\top \in \widetilde{S}(G)$. Further, we call $\mathbf{v}$ *SV generically realisable* if $\mathbf{v}$ is SV sane with the additional property that for any finite set $\mathcal{Y} \subseteq \mathbb{R}^n \setminus \{0\}$, and any appropriate singular value list Σ, the orthogonal matrices U, V can be chosen so that $U\mathbf{y}, V\mathbf{y}$ are nowhere-zero for all $\mathbf{y} \in \mathcal{Y}$.

Definition 2.10. Let V be a singular value multiplicity matrix for a graph G. If every column of V is SV sane/SV generically realisable for the corresponding connected component of G, then we say that V is *SV sane/ SV generically realisable* for G.

Definition 2.11. If every singular value multiplicity matrix V for a graph G is SV sane/SV generically realisable, then we call G *SV sane/ SV generically realisable.*

Remark 2.12. Once more, the reader can compare Definitions 2.9, 2.10, 2.11 with their corresponding definitions [2, Definition 2.8], [2, Definition 2.9], [2, Definition 2.10] in the symmetric case, which again we will reference by using the prefix EV. We will relate these EV/SV compatibilities in Section 3.1.

In Section 3, we show that the complete graph K_n and path P_n on n vertices are SV generically realisable. Next, we adapt the proof of [2, Theorem 2.14] to prove Theorem 2.15.

We remark preliminarily that for $q \in \mathbb{N}$, $SO(q)$ is an irreducible algebraic variety (see [4]). Let $\mathbf{p} = (p_1, \dots, p_r) \in \mathbb{N}^r$, and write $SO(\mathbf{p}) := SO(p_1) \times \cdots \times SO(p_r)$. Then $SO(\mathbf{p})$ is an irreducible algebraic variety as the product of irreducible varieties. We will use the following two technical lemmas from [2] in our proof of Theorem 2.15.

Lemma 2.13. *[2, Lemma 2.12] Let $r \in \mathbb{N}$, $\boldsymbol{p} = (p_1, \dots, p_r)^\top \in \mathbb{N}^r$ and for $s \in [r]$, let $A_s \in \mathbb{R}^{p_s \times p_s}$ with $\operatorname{rank} A_s = 1$. If $\sum_{s\in[r]} \operatorname{tr}(A_s X_s) = 0$ for all $(X_1, \dots X_r) \in SO(\boldsymbol{p})$, then $r \geq 2$ and $p_s = 1$ for each $s \in [r]$.*

Lemma 2.14. *[2, Lemma 2.13] Let $k, m, n \in \mathbb{N}$. For $s \in [k]$, let $p_s \in \mathbb{N}$ and $\varnothing \neq \mathcal{R}_s, \mathcal{C}_s \subseteq [p_s]$ where $m' := \sum_{s\in[k]} |\mathcal{R}_s| \leq m$ and $n' := \sum_{s\in[k]} |\mathcal{C}_s| \leq n$. Let $F : \times_{s=1}^{k} \mathbb{R}^{p_s \times p_s} \to \mathbb{R}^{n\times m}$ be given by*

$$F(X_1, \dots, X_k) := \begin{pmatrix} \bigoplus_{s\in[k]} X_s[\mathcal{R}_s, \mathcal{C}_S] & 0_{m'\times(n-n')} \\ 0_{(m-m')\times n'} & 0_{(m-m')\times(n-n')} \end{pmatrix}.$$

Fix $a \in [m]$, $b \in [n]$ and invertible nowhere-zero matrices $S \in \mathbb{R}^{n\times n}$ and $T \in \mathbb{R}^{m\times m}$. Then there exist matrices $A_s \in \mathbb{R}^{p_s\times p_s}$ with $\operatorname{rank} A_s = 1$ for $s \in [k]$ so that $(S^\top F(X_1, \dots, X_k)T)_{ab} = \sum_{s\in[k]} \operatorname{tr}(A_s X_s)$ for any $X_1 \in \mathbb{R}^{p_1\times p_1}, \dots, X_k \in \mathbb{R}^{p_k\times p_k}$.

Theorem 2.15. *Let G and H be two graphs on k and l vertices respectively. Suppose that G and H have SV sane compatible multiplicity matrices $V = (v_{si}) \in \mathbb{N}_0^{r\times k}$ and $W = (w_{sj}) \in \mathbb{N}_0^{r\times l}$. Then $G \vee H$ is realisable by an orthogonal matrix if either V is a generically realisable SV multiplicity matrix for G, or W is a generically realisable SV multiplicity matrix for H.*

Proof. Let $G = G_1 \cup \cdots \cup G_k$, $H = H_1 \cup \cdots \cup H_l$ be the decomposition of G, H into their connected components. Let us denote the i-th column of V by $\mathbf{v}_i$, and the i-th column of W by $\mathbf{w}_i$. Let $\Sigma = (\sigma_1, \dots \sigma_{r-1}, 1)$ be a singular value list with $0 \leq \sigma_1 < \cdots < \sigma_{r-1} < 1$, and let $\Sigma' = (\sigma_1, \dots, \sigma_{r-1})$. Let $\mathbf{p} = (p_1, \dots, p_{r-1})^\top = \widetilde{V}\mathbf{1}_k = \widetilde{W}\mathbf{1}_l$. Let $D = D_{\Sigma',\mathbf{p}} \neq 0$. Then $D = \bigoplus_{s\in[r-1]} \sigma_s \operatorname{I}_{p_s}$.

Consider the orthogonal matrix

$$X := \begin{pmatrix} D & \bigoplus_{s\in[r-1]}(1-\sigma_s^2)^{\frac{1}{2}} U_s \\ \bigoplus_{s\in[r-1]}(1-\sigma_s^2)^{\frac{1}{2}} V_s^\top & -D \end{pmatrix},$$

where by Proposition 2.3, U_s, V_s are orthogonal matrices for all $s \in [r-1]$. Since $\sigma_2, \dots, \sigma_{r-1} \neq 0$, we have $U_s = V_s$ for $s = 2, \dots, r-1$. We choose to take $U_1 = V_1$ also, regardless of whether σ_1 is zero or not. We additionally take U_s, V_s to be special orthogonal for all $s \in [r]$.

Let $n_1 := e_r^\top V\mathbf{1}_k$ and $n_2 := e_r^\top W\mathbf{1}_l$ be the sum of the entries in the last row of V and W. Let $X' := X \oplus \operatorname{I}_{n_1} \oplus -\operatorname{I}_{n_2}$. Let $D_G = \bigoplus_{t\in[k]} D_{G_t}$, $D_H = \bigoplus_{t\in[l]} D_{H_t}$, where $D_{G_t} := D_{\Sigma,\mathbf{v}_t}$, and $D_{H_t} := D_{\Sigma,\mathbf{w}_t}$. We can permute the rows and columns of X' to obtain the matrix X'', where

$$X'' := \begin{pmatrix} D_G & B(U) \\ B(U)^\top & -D_H \end{pmatrix},$$

with $B(U)$ a $|G| \times |H|$ matrix that depends on our choice of $U = (U_1, \dots, U_{r-1}) \in SO(\mathbf{p})$.

We partition $B(U)$ as a $k \times l$ block matrix $B(U) = (B_{ij}(U))_{i \in [k], j \in [l]}$ corresponding to the direct sum decompositions of D_G, D_H. Then, each $B_{ij}(U)$ is a $|G_i| \times |H_j|$ matrix for all $i \in [k], j \in [l]$. Each pair D_{G_i}, D_{H_i} share a common singular value in Σ' since $\widetilde{V}^\top \widetilde{W} > 0$. Let $Q(i,j) = \{s : s \in [r-1], v_{si}w_{sj} \neq 0\}$. Then $Q(i,j) \neq \varnothing$ for all i, j, and $s \in Q(i,j)$ implies σ_s is a singular value of D_{G_i}, D_{H_j}. By their construction, the rows and columns of $B_{ij}(U)$ can be permuted to obtain

$$B'_{ij}(U) := \begin{pmatrix} \bigoplus_{s \in Q(i,j)} (1-\sigma_s^2)^{\frac{1}{2}} U_s[\mathcal{R}_{si}, \mathcal{C}_{sj}] & 0 \\ 0 & 0 \end{pmatrix} \tag{1}$$

where each $U_s[\mathcal{R}_{si}, \mathcal{C}_{sj}]$ is the appropriate submatrix of U_s with $|\mathcal{R}_{si}| = \operatorname{mult}(\sigma_s, D_{G_i})$, $|\mathcal{C}_{sj}| = \operatorname{mult}(\sigma_s, D_{H_j})$.

Since V, W are SV sane multiplicity matrices for G, H, there exists nowhere-zero orthogonal matrices S_i, T_i, M_j, N_j for $i \in [k]$, $j \in [l]$ so that $S_i D_{G_i} T_i^\top \in \widetilde{S}(G_i)$ and $M_j D_{H_j} N_j^\top \in \widetilde{S}(H_j)$. Let $S = \bigoplus_{i \in [k]} S_i$, $T = \bigoplus_{i \in [k]} T_i$, $M = \bigoplus_{j \in [l]} M_j$, and $N = \bigoplus_{j \in [l]} N_j$. We note that S, T, M, N can be chosen in several ways.

Let $Y = (S \oplus M) X'' (T \oplus N)^\top$. Then,

$$Y = \begin{pmatrix} A_G & SB(U)N^\top \\ MB(U)^\top T^\top & A_H \end{pmatrix}$$

where $A_G \in \widetilde{S}(G)$, $A_H \in \widetilde{S}(H)$. Y is orthogonal, and we have $Y \in \widetilde{S}(G \vee H)$ if we can ensure that $C(U) := SB(U)N^\top$ and $F(U) := (MB(U)^\top T^\top)^\top$ are nowhere-zero. We note that $C(U)$, $F(U)$ inherit the $k \times l$ block-partition of $B(U)$, and write $C(U) = (C_{ij}(U))$ with $C_{ij}(U) = S_i B_{ij}(U) N_j^\top$ and $F(U) = (F_{ij}(U))$ with $F_{ij}(U) = T_i B_{ij}(U) M_j^\top$.

To ensure that $C(U)$, $F(U)$ are nowhere-zero, we look at the choices we have in our construction. Assume that V is a SV generically realisable multiplicity matrix for G. Fix an appropriate M, N. We argue that there exists $U \in SO(\mathbf{p})$ so that $B_{ij}(U)N_j^\top$ and $B_{ij}(U)M_j^\top$ have no zero-columns for all pairs i, j.

To do this, fix any invertible nowhere-zero $|G_i| \times |G_i|$ matrices $\widetilde{S}_i$, $\widetilde{T}_i$ for $i \in [k]$. For $a \in [|G_i|]$, $b \in [|H_j|]$, $i \in [k]$ and $j \in [l]$, consider the linear functionals $\widetilde{L}_{ab}(i,j) : SO(\mathbf{p}) \to \mathbb{R}$, $\widetilde{L}_{ab}(i,j)(U) := (\widetilde{S}_i B_{ij}(U) N_j^\top)_{ab}$ and $\widetilde{M}_{ab}(i,j) : SO(\mathbf{p}) \to \mathbb{R}$, $\widetilde{M}_{ab}(i,j)(U) := (\widetilde{T}_i B_{ij}(U) M_j^\top)_{ab}$. Suppose that for all $U \in SO(\mathbf{p})$, there exists i, j, b such that the bth column of either $B_{ij}(U)N_j^\top$ or $B_{ij}(U)M_j^\top$ is zero. Then $SO(\mathbf{p}) \subseteq \left(\cup_{i,j,b} \cap_a \widetilde{L}_{ab}(i,j)^{-1}(0) \right) \cup \left(\cup_{i,j,b} \cap_a \widetilde{M}_{ab}(i,j)^{-1}(0) \right)$. Since $SO(\mathbf{p})$ is irreducible, WLOG we assume $SO(\mathbf{p}) \subseteq \cup_{i,j,b} \cap_a \widetilde{L}_{ab}(i,j)^{-1}(0)$. But then by the irreducibility of $SO(\mathbf{p})$ there are fixed i_0, j_0, b_0 such that $SO(\mathbf{p}) \subseteq \cap_a \widetilde{L}_{ab_0}(i_0, j_0)^{-1}(0)$. Therefore, for all $U \in SO(\mathbf{p})$ the b_0th column of $\widetilde{S}_{i_0} B_{i_0 j_0}(U) N_{j_0}^\top$ is zero. By invertibility of $\widetilde{S_{i_0}}$ it follows that the b_0th column of $B_{i_0 j_0}(U) N_{j_0}^\top$ is zero. By taking $a = 1$, we have that $\widetilde{L}_{1b_0}(i_0, j_0)(U) = 0$ for all $U \in SO(\mathbf{p})$. Recalling that $B_{ij}(U)$ can be permuted to obtain (1), and using Lemma 2.14, we can write $L_{1b_0}(i_0, j_0)$ as $L_{1b_0}(i_0, j_0) = \sum_{s \in Q(i_0, j_0)} \operatorname{tr}(A_s U_s)$ for $A_s \in \mathbb{R}^{p_s \times p_s}$ with $\operatorname{rank} A_s = 1$ for all $s \in Q(i_0, j_0)$ (which is always non-empty by above). But then by Lemma 2.13, we have $p_s = 1$ for all $s \in Q(i_0, j_0)$. Then $B'_{i_0 j_0}(U)$ in (1) is of the form $B'_{i_0 j_0}(U) = \begin{pmatrix} D' & 0 \\ 0 & 0 \end{pmatrix}$ with $D' := \operatorname{diag}((1-\sigma_s^2)^{\frac{1}{2}} : s \in Q(i_0, j_0))$. Since $\sigma_s \neq 1$ for all $s \in Q(i_0, j_0)$, D' is an invertible diagonal matrix. It follows that the kernel of $B'_{i_0 j_0}(U)$ does not contain a nowhere-zero vector, and the same will be true of $B_{i_0 j_0}(U)$. But $N_{j_0}^\top$ is a nowhere-zero matrix, and so the b_0th column of $B_{i_0 j_0}(U) N_{j_0}^\top$ is non-zero, a contradiction.

Therefore, it is possible to choose $U \in SO(\mathbf{p})$ so that the matrices $B_{ij}(U)N_j^\top$ and $B_{ij}(U)M_j^\top$ contain no zero columns for all $i \in [k]$, $j \in [l]$. By the SV generic realisability of each G_i, we can therefore choose nowhere-zero orthogonal matrices S_i, T_i for all $i \in [k]$ so that $S_i B_{ij}(U) N_j^\top$ and $T_i B_{ij}(U) M_j^\top$ are nowhere-zero for all $(i,j) \in [k] \times [l]$. But then $C(U)$ and $F(U)$ are nowhere-zero, and so $Y \in \widetilde{S}(G \vee H)$, as required. The argument for W being the SV generically realisable multiplicity matrix is symmetric. $\square$

Thus, in using Theorem 2.8 and Theorem 2.15, we arrive at the following:

Corollary 2.16. *If G is SV generically realisable and H is SV sane, then $G \vee H$ is realisable by an orthogonal matrix if and only if G and H have a pair of compatible SV multiplicity matrices.*

3 Applications

In this section, we discuss applications of Corollary 2.16. Clearly, an important question to ask is whether a graph is SV generically realisable. In Section 3.1 we describe relations between SV generic realisability and EV generic realisability as discussed in [2]. Using these, we determine that complete graphs and paths are SV generically realisable, and discuss consequences in each case.

3.1 Singular value and eigenvalue multiplicity matrices.

Consider the diagonal matrix $D_{\Lambda,\mathbf{v}} \in \mathbb{R}^{n\times n}$ with diagonal entries $\Lambda = (\lambda_1, \dots, \lambda_r)$ occurring with multiplicities $\mathbf{v} = (v_1, \dots, v_r)^\top \in \mathbb{N}_0^r$, and $\sum_{i\in[r]} v_i = n$. Let $|\Lambda|$ denote the strictly increasing list of absolute values of entries in Λ. Then there is a corresponding multiplicity list $\mathbf{w}$ for $|\Lambda|$ so that $D_{|\Lambda|,\mathbf{w}} = \Pi^\top(D_{\Lambda,\mathbf{v}}F)\Pi$ for F a diagonal orthogonal matrix with diagonal entries ±1, and Π a permutation matrix.

We can use this idea to obtain SV multiplicity vectors from EV multiplicity vectors, and further show that we can yield SV sane/generically realisable multiplicity matrices from EV sane/generically realisable multiplicity matrices.

Lemma 3.1. *Let G be a connected graph on n vertices, and let $\boldsymbol{w}$ be a multiplicity list. Suppose there exists a singular value list Σ such that $D_{\Sigma,\boldsymbol{w}} = \Pi^\top(D_{\Lambda,\boldsymbol{v}}F)\Pi$ for $F \in \mathbb{R}^{n\times n}$ an orthogonal diagonal matrix, Π a permutation matrix, and Λ an eigenvalue list with $|\Lambda| = \Sigma$. If $\boldsymbol{v}$ is an EV multiplicity vector for G that is realisable in $S(G)$ by using the eigenvalue list Λ, then $\boldsymbol{w}$ is an SV multiplicity vector for G.*

Proof. If $\mathbf{v}$ is an EV multiplicity vector for G realisable with eigenvalue list Λ, then there exists $U \in O(n)$ such that $UD_{\Lambda,\mathbf{v}}U^\top \in S(G)$. Since $D_{\Lambda,\mathbf{v}} = \Pi D_{\Sigma,\mathbf{w}}(F\Pi)^\top$, we have $U\Pi D_{\Sigma,\mathbf{w}}(F\Pi)^\top U^\top = (U\Pi)D_{\Sigma,\mathbf{w}}(UF\Pi)^\top \in S(G)$ with $U\Pi, UF\Pi$ orthogonal, so that $\mathbf{w}$ is realisable by a matrix in $S(G) \subseteq \widetilde{S}(G)$, and so is a SV multiplicity vector. $\square$

Lemma 3.2. *Let G be a connected graph on n vertices, and let $\boldsymbol{w}$ be a SV multiplicity vector for G. Suppose that for any singular value list Σ with $D_{\Sigma,\boldsymbol{w}} \neq 0$ there exists F an orthogonal diagonal matrix and Π a permutation matrix such that $D_{\Sigma,\boldsymbol{w}} = \Pi^\top(D_{\Lambda,\boldsymbol{v}}F)\Pi$ with $|\Lambda| = \Sigma$ and $\boldsymbol{v}$ a multiplicity list. If $\boldsymbol{v}$ is EV sane/EV generically realisable, then $\boldsymbol{w}$ is SV sane/SV generically realisable.*

Proof. If $\mathbf{v}$ is EV sane, then for any eigenvalue list Λ there exists a nowhere-zero orthogonal matrix $U \in O(n)$ such that $UD_{\Lambda,\mathbf{v}}U^\top \in S(G)$. If for any singular value list Σ such that $D_{\Sigma,\mathbf{w}} \neq 0$ we have $D_{\Sigma,\mathbf{w}} = \Pi^\top(D_{\Lambda,\mathbf{v}}F)\Pi$ with $|\Lambda| = \Sigma$, it follows that $U\Pi, UF\Pi$ are nowhere-zero orthogonal matrices, and $(U\Pi)D_{\Sigma,\mathbf{w}}(UF\Pi)^\top \in S(G)$. Therefore $\mathbf{w}$ is SV sane.

Now let $\mathcal{Y} \subseteq \mathbb{R}^n \setminus \{0\}$ be a finite set. Let $\mathcal{Z} := \{\Pi\mathbf{y} : \mathbf{y} \in \mathcal{Y}\} \cup \{F\Pi\mathbf{y} : \mathbf{y} \in \mathcal{Y}\}$. Then $\mathcal{Z}$ is a finite subset of $\mathbb{R}^n$ that does not contain zero (in particular since Π, $F\Pi$ have trivial kernels as orthogonal matrices). If $\mathbf{v}$ is EV generically realisable, then for any eigenvalue list Λ we can find $U \in O(n)$ so that $UD_{\Lambda,\mathbf{v}}U^\top \in S(G)$, and $U\mathbf{z}$ is nowhere-zero for all $\mathbf{z} \in \mathcal{Z}$. Therefore $U\Pi\mathbf{y}$ and $UF\Pi\mathbf{y}$ are nowhere-zero for all $\mathbf{y} \in \mathcal{Y}$ and $(U\Pi)D_{\Sigma,\mathbf{w}}(UF\Pi)^\top \in S(G)$. Thus $\mathbf{w}$ is SV generically realisable. $\square$

3.2 Joins of unions of complete graphs.

We introduce notation for the disjoint unions of complete graphs. For $\mathbf{m} = (m_1, \ldots, m_k) \in \mathbb{N}^k$, let $K_{\mathbf{m}} := \bigcup_{i\in[k]} K_{m_i}$, with K_{m_i} the complete graph on m_i vertices. Let $|K_{\mathbf{m}}| = |\mathbf{m}| = \sum_{i\in[k]} m_i$ be the size of $K_{\mathbf{m}}$. Let kG be the graph that is the disjoint union of k copies of the graph G.

For this section we consider graphs of the form $K_{\mathbf{m}} \vee K_{\mathbf{n}}$ with $\mathbf{m} \in \mathbb{N}^k$, $\mathbf{n} \in \mathbb{N}^l$, and $k \leq l$ for k, $l \in \mathbb{N}$. We first present a necessary condition for $K_{\mathbf{m}} \vee K_{\mathbf{n}}$ to be realisable by an orthogonal matrix. We will prove this more generally for $G \vee H$ where G, H have k, l connected components respectively.

Lemma 3.3. *Let k, $l \in \mathbb{N}$ with $k \leq l$. Consider the graphs G, H with k, l connected components respectively. If there exists an orthogonal matrix in $\widetilde{S}(G \vee H)$, then $l \leq |G|$.*

Proof. Write $G = G_1 \cup \cdots \cup G_k$, $H = H_1 \cup \cdots \cup H_l$. Suppose there exists an orthogonal matrix X in $\widetilde{S}(G \vee H)$, then it is of the form:

$$X := \begin{pmatrix} A = \bigoplus_{i\in[k]} A_i & B \\ C & D = \bigoplus_{j\in[l]} D_j \end{pmatrix},$$

where $A_i \in \widetilde{S}(G_i)$ for all $i \in [k]$ and $D_j \in \widetilde{S}(H_j)$ for all $j \in [l]$. B, C are nowhere-zero $|G| \times |H|$, $|H| \times |G|$ matrices respectively.

Given i, $j \in [l]$, $i \neq j$, then for any row u in $D[H_i]$, and any row v in $D[H_j]$, we have u, v orthogonal. If, for instance, we take the first row d_j of each $D[H_j]$ for all $j \in [l]$, then we have a set of l pairwise orthogonal rows in D.

Let $c_1, \ldots, c_l$ be the l rows of C contained in the rows of X containing the rows d_j of D. These rows of X are pairwise orthogonal, and so the set of vectors $c_1, \ldots, c_l$ must be pairwise orthogonal also. We have $c_1, \ldots, c_l$ non-zero, and as they are orthogonal they must form a linearly independent set. The submatrix corresponding to these $c_1, \ldots c_l$ is a $l \times |G|$ matrix, and as the rows are linearly independent, we have $l \leq |G|$. □

Note that the argument in the proof of Lemma 3.3 can be repeated using the submatrices A, B to yield $k \leq |H|$ also. However, if we assume $k \leq l$, then we have $k \leq l \leq |H|$, so this condition is satisfied automatically.

We will show that the necessary condition of Lemma 3.3 is also sufficient for the realisablility of orthogonal matrices in $K_{\mathbf{m}} \vee K_{\mathbf{n}}$. In [2, Proposition 4.1], it is proven that the complete graph K_n is EV generically realisable for all $n \in \mathbb{N}$. Any EV multiplicity vector for K_n must contain at least two positive entries. We will show additionally that any SV multiplicity vector containing at least one positive entry is SV generically realisable for K_n.

Proposition 3.4. *For $n \in \mathbb{N}$, K_n is SV generically realisable.*

Proof. The result follows trivially for $n = 1$. Otherwise, assume $n \geq 2$. Let $\mathbf{w} \in \mathbb{N}_0^r$ be a SV multiplicity vector for K_n. If $\mathbf{w}$ contains at least two positive coordinates, then $\mathbf{w}$ is EV generically realisable by [2, Proposition 4.1]. Therefore it is SV generically realisable also, as the set of non-negative real singular value lists is contained in the set of real eigenvalue lists.

If $\mathbf{w}$ contains just one non-negative entry, then for any singular value list Σ, we have $D_{\Sigma,\mathbf{w}} = a\,\mathrm{I}_n$ for some $a \in \mathbb{R}_{\geq 0}$. Let $F := -\mathrm{I}_1 \oplus \mathrm{I}_{n-1}$. Then if $a \neq 0$, $D_{\Sigma,\mathbf{w}} = D_{\Lambda,\mathbf{v}} F$ for Λ an eigenvalue list containing $-a$, a and $\mathbf{v}$ a multiplicity list with two positive entries. Since $\mathbf{v}$ is then EV generically realisable for K_n, it follows that $\mathbf{w}$ is SV generically realisable for K_n by Lemma 3.2. □

Lemma 3.5. *Let $\mathbf{m}$, $\mathbf{m}' \in \mathbb{N}^k$, $\mathbf{n}$, $\mathbf{n}' \in \mathbb{N}^l$, with $\mathbf{m} \leq \mathbf{m}'$, $\mathbf{n} \leq \mathbf{n}'$. If $K_{\mathbf{m}} \vee K_{\mathbf{n}}$ is realisable by an orthogonal matrix, then $K_{\mathbf{m}'} \vee K_{\mathbf{n}'}$ is realisable by an orthogonal matrix also.*

Proof. By symmetry, conjugating by permutation matrices, and successively applying the lemma, it is sufficient to prove the statement for $\mathbf{n} = \mathbf{n}'$, and $\mathbf{m} = (m_1, \ldots, m_k)$, $\mathbf{m}' = (m_1', \ldots, m_k')$ with $m_1' = m_1 + 1$, and $m_i' = m_i$ for $i = 2 \ldots k$. Let $X = (x_{ij}) \in \widetilde{S}(K_{\mathbf{m}} \vee K_{\mathbf{n}})$ be an orthogonal matrix.

Let B' be the first row of X less the first entry, and C' the first column of X less the first entry and write:

$$X = \left(\begin{array}{c|c} x_{11} & B' \\ \hline C' & X' \end{array}\right).$$

Let $X' = \mathrm{I}_1 \oplus X$. Let $V_1 = \begin{pmatrix} a & b \\ b & -a \end{pmatrix}$ be a 2×2 symmetric orthogonal matrix with $a, b \neq 0$, and let $V = V_1 \oplus \mathrm{I}_{|\mathbf{m}|+|\mathbf{n}|-1}$. Let $Y = VX'V^\top$. Then Y is orthogonal, and we claim $Y \in \widetilde{S}(K_{\mathbf{m}'} \vee K_{\mathbf{n}'})$. Indeed,

$$Y = VX'V = \left(\begin{array}{cc|ccc} a & b & 0 & \cdots & 0 \\ b & -a & 0 & \cdots & 0 \\ \hline 0 & 0 & & & \\ \vdots & \vdots & & \mathrm{I} & \\ 0 & 0 & & & \end{array}\right) \left(\begin{array}{cc|ccc} 1 & 0 & 0 & \cdots & 0 \\ 0 & x_{11} & & B' & \\ \hline 0 & & & & \\ \vdots & C' & & X' & \\ 0 & & & & \end{array}\right) \left(\begin{array}{cc|ccc} a & b & 0 & \cdots & 0 \\ b & -a & 0 & \cdots & 0 \\ \hline 0 & 0 & & & \\ \vdots & \vdots & & \mathrm{I} & \\ 0 & 0 & & & \end{array}\right)$$

$$= \left(\begin{array}{cc|c} a^2 + b^2 x_{11} & ab(1 - x_{11}) & bB' \\ ab(1 - x_{11}) & a^2 x_{11} + b^2 & -aB' \\ \hline & & \\ bC' & -aC' & X' \end{array}\right).$$

This will be of the required pattern provided that $ab(1 - x_{11})$ is not zero. But $a, b \neq 0$, and $x_{11} \neq 1$, since otherwise all entries of B', C' would be zero (X has orthogonal rows and columns), contradicting the off-diagonal pattern of X. □

Theorem 3.6. *Consider the graph $K_{\boldsymbol{m}} \vee lK_1$ with $\boldsymbol{m} \in \mathbb{N}^k$ and $k \leq l$. If $l \leq |\boldsymbol{m}|$, then $K_{\boldsymbol{m}} \vee lK_1$ is realisable by an orthogonal matrix.*

Proof. We show that $K_{\mathbf{m}}$ and lK_1 have compatible SV multiplicity matrices for $V \in \mathbb{N}_0^{2\times k}$, $W \in \mathbb{N}_0^{2\times l}$. Then by Corollary 2.16, $K_{\mathbf{m}} \vee lK_1$ is realisable by an orthogonal matrix. Let

$$V := \begin{pmatrix} \mathbf{1}_k^\top + \mathbf{a} \\ \mathbf{b} \end{pmatrix}, \quad W := \begin{pmatrix} \mathbf{1}_l^\top \\ \mathbf{0}_l^\top \end{pmatrix},$$

for $\mathbf{a}$, $\mathbf{b} \in \mathbb{N}_0^{1\times k}$ two row-vectors. Then $\widetilde{V}^\top \widetilde{W} > 0$. We want $|\mathbf{a}| = l - k \geq 0$ and $\mathbf{b} = |\mathbf{m}| - l \geq 0$, so that $\widetilde{V}\mathbf{1}_k = \widetilde{W}\mathbf{1}_l$, and that the sum of all entries in V equals $|\mathbf{m}|$.

Write $\mathbf{m} = (m_1, \ldots, m_k)$. Let $\mathbf{b} = (b_i)_{i\in[k]}$ have its coordinates defined successively as $b_i := \min(m_i - 1, |\mathbf{m}| - l - \sum_{j\in[i-1]} b_j)$. We do achieve $|\mathbf{b}| = |\mathbf{m}| - l$, since $\sum_{i\in[k]} m_i - 1 = |\mathbf{m}| - k \geq |\mathbf{m}| - l$. Let $\mathbf{a} = (a_i)_{i\in[k]}$ be defined as $a_i := m_i - 1 - b_i$. Then the i-th column of V contains at least one positive entry and sums to m_i for all $i \in [k]$, and $|\mathbf{a}| = |\mathbf{m}| - k - |\mathbf{b}| = l - k$. □

We are now in a position to establish the converse result of Lemma 3.3 for $K_{\mathbf{m}} \vee K_{\mathbf{n}}$:

Corollary 3.7. *Consider the graph $K_{\boldsymbol{m}} \vee K_{\boldsymbol{n}}$ with $\boldsymbol{m} \in \mathbb{N}^k$, $\boldsymbol{n} \in \mathbb{N}^l$ and $k \leq l$. Then $l \leq |\boldsymbol{m}|$ is a necessary and sufficient condition for $K_{\boldsymbol{m}} \vee K_{\boldsymbol{n}}$ to be realisable by an orthogonal matrix.*

Proof. We have necessity by Lemma 3.3. Conversely, by Theorem 3.6, $K_{\mathbf{m}} \vee lK_1$ is realisable by an orthogonal matrix since $l \leq |\mathbf{m}|$. Therefore by Lemma 3.5, $K_{\mathbf{m}} \vee K_{\mathbf{n}}$ is realisable by an orthogonal matrix also, as $\mathbf{n} \geq \mathbf{1}_l^\top$. □

Remark 3.8. In Proposition 3.4, we actually show the stronger condition that every SV multiplicity vector $\mathbf{w}$ for K_n can be realised by a symmetric matrix, since the process in Lemma 3.2 results in matrices in $S(G)$. Therefore if $K_{\mathbf{m}} \vee K_{\mathbf{n}}$ is realisable by an orthogonal matrix, we can find an orthogonal matrix $X \in \widetilde{S}(K_{\mathbf{m}} \vee K_{\mathbf{n}})$ with $X[K_{\mathbf{m}}]$ and $X[K_{\mathbf{n}}]$ symmetric.

The proof of Theorem 3.6 uses a singular value list of length 2. We can also ask whether we can use singular value lists of length greater than two. The following two results briefly explore this.

Lemma 3.9. *Consider the graph $K_{\boldsymbol{m}} \vee lK_1$, with $\boldsymbol{m} \in \mathbb{N}^k$, and $k \le l$. Let $\tilde{m} = \min\{m_i : i \in [k]\}$. If $X \in \widetilde{S}(K_{\boldsymbol{m}} \vee lK_1)$ is orthogonal, then $X[lK_1]$ has at most $\tilde{m}$ distinct singular values.*

Proof. Let $A = X[K_{\mathbf{m}}]$, $D = X[lK_1]$. Then $A = \bigoplus_{i\in[m]} A_i$ for $A_i \in \widetilde{S}(K_{m_i})$. Let A_j be the $\tilde{m} \times \tilde{m}$ submatrix of A corresponding to the smallest connected component of $K_{\mathbf{m}}$. Then A_j has at most $\tilde{m}$ distinct non-one singular values. By Theorem 2.8, each singular value of D must be a singular value of A_j, by the compatibility of $K_{\mathbf{m}}$ and lK_1. Thus up to sign, D has at most $\tilde{m}$ distinct diagonal entries. □

Lemma 3.10. *Let $k \le l$ and consider the graph $kK_2 \vee lK_1$. If $l < 2k$, then any $X \in \widetilde{S}(kK_2 \vee lK_1)$ orthogonal has $X[kK_2]$, $X[lK_1]$ with one singular value distinct from 1.*

Proof. By Lemma 3.9, $X[kK_2]$, $X[lK_1]$ can have at most 2 singular values distinct from one. We show that there does not exist compatible multiplicity matrices $V \in \mathbb{N}_0^{3\times k}$, $W \in \mathbb{N}_0^{3\times l}$ for $l < 2k$.

Indeed suppose $X \in kK_2 \vee lK_1$ is orthogonal with $X[kK_2]$, $X[lK_1]$ having two distinct non-one singular values $\sigma_1, \sigma_2 \in \mathbb{R}_{\ge 0}$. Let V, W be the resulting compatible SV multiplicity matrices. The singular values of $X[lk_1]$ will be all less than one by the non-zero pattern of the off-diagonal blocks of X. Since $|lK_1| = l < 2k = |kK_2|$, $X[kK_2]$ will have at least one singular value equal to 1 (as $\widetilde{V}\mathbf{1}_k = \widetilde{W}\mathbf{1}_l$). We have $X[kK_2] = \bigoplus_{i\in[k]} A_i$ for $A_i \in \widetilde{S}(K_2)$ and $X[lK_1] = \bigoplus_{j\in[l]} D_j$ for each D_j a 1×1 matrix. Let A_s be the matrix with one as a singular value. Then A_s has just one singular value less than one, say σ_1. But there exists some some D_j with σ_2 as its singular value, so that $(\widetilde{V}^\top \widetilde{W})_{sj} = 0$, a contradiction. Therefore $X[lK_1]$ has just one distinct singular value σ_1. □

We finish this sub-section with a computed example:

Example 3.11. Consider $G = 3K_2 \vee 4K_1$. G is not realisable by an orthogonal symmetric matrix (see [2, Example 4.23]), but it can be realised by an orthogonal matrix by Theorem 3.6, since $l = 4 \le |\mathbf{m}| = 6$. By Lemma 3.10, since $l < 2k$, any $X \in \widetilde{S}(G)$ orthogonal has $X[3K_2]$, $X[4K_1]$ with at most one distinct non-one singular value. Recall the notation used in the proof of Theorem 2.15. Let $D = \frac{1}{\sqrt{2}} \mathrm{I}_4$, and let $U_1 = \frac{1}{2\sqrt{2}} H_4$, where H_4 is the 4×4 Hadamard matrix. Let $S_1 = S_2 = S_3 = T_1 = T_2 = \begin{pmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}$, and let $T_3 = \begin{pmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{\sqrt{3}}{2} \end{pmatrix}$. We take $M = N = \mathrm{I}_4$. Then,

$$X = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ 0 & 1 & 0 & 0 & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\ 0 & 0 & 1 & 0 & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 0 & 1 & \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & -1 & 0 & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & 0 & -1 & 0 & 0 \\ \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & 0 & 0 & -1 & 0 \\ \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & 0 & 0 & 0 & -1 \end{pmatrix}, \quad X'' = \frac{1}{\sqrt{2}} \begin{pmatrix} \sqrt{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & \sqrt{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\ 0 & 0 & 0 & 0 & 1 & 0 & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 0 & 0 & 0 & 1 & \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & -1 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & 0 & -1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & 0 & 0 & -1 & 0 \\ 0 & \frac{1}{2} & 0 & -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & 0 & 0 & 0 & -1 \end{pmatrix},$$

$$Y \approx \frac{1}{\sqrt{2}} \begin{pmatrix} 1.1 & 0.179 & 0 & 0 & 0 & 0 & 0.433 & 0.433 & 0.433 & 0.433 \\ 0.179 & 1.31 & 0 & 0 & 0 & 0 & -0.25 & -0.25 & -0.25 & -0.25 \\ 0 & 0 & 1.1 & 0.179 & 0 & 0 & 0.433 & 0.433 & -0.433 & -0.433 \\ 0 & 0 & 0.179 & 1.31 & 0 & 0 & -0.25 & -0.25 & 0.25 & 0.25 \\ 0 & 0 & 0 & 0 & 0.866 & 0.5 & 0.683 & -0.683 & 0.183 & -0.183 \\ 0 & 0 & 0 & 0 & 0.5 & -0.866 & 0.183 & -0.183 & -0.683 & 0.683 \\ 0.433 & -0.25 & 0.433 & -0.250 & 0.683 & 0.183 & -1 & 0 & 0 & 0 \\ 0.433 & -0.25 & 0.433 & -0.250 & -0.683 & -0.183 & 0 & -1 & 0 & 0 \\ 0.433 & -0.25 & -0.433 & 0.250 & -0.183 & 0.683 & 0 & 0 & -1 & 0 \\ 0.433 & -0.25 & -0.433 & 0.250 & 0.183 & -0.683 & 0 & 0 & 0 & -1 \end{pmatrix} \in \widetilde{S}(G),$$

and Y is indeed orthogonal.

3.3 SV generic realisability of paths.

We now turn our focus to paths. Let P_n denote the path on n vertices. The SV multiplicity vectors for P_n are much more constrained than those for K_n, as Proposition 3.12 shows, but we will yield that P_n is SV generically realisable also.

Proposition 3.12. *Let $r,\ n \in \mathbb{N}$. Let $V = (v_1, \ldots, v_r)^\top \in \mathbb{N}_0^r$ be a SV multiplicity vector for P_n. Then the coordinates of V are in the set $\{0, 1, 2\}$.*

Proof. Equivalently, we show that any $A \in \widetilde{S}(P_n)$ has singular values of multiplicity at most 2.

For $n = 1, 2$, the result follows trivially.

Otherwise, assume $n \geq 3$. Let $X = (x_{ij}) \in \widetilde{S}(P_n)$. Then the super-diagonal and sub-diagonal of X are nowhere-zero. Let $\lambda \in \mathbb{R}_{\geq 0}$ be an eigenvalue of $XX^\top$ (or equivalently $X^\top X$) and take the matrix $C = (c_{ij}) := XX^\top - \lambda\,\mathrm{I}_n$. The nullity of C corresponds to the multiplicity of λ as an eigenvalue of $XX^\top$. We show that C has rank at least $n - 2$.

Indeed, for $i = 3, \ldots, n$, we have $c_{i,i-2} \neq 0$ since $c_{i,i-2} = \sum_{k\in[n]} x_{i,k}x_{i-2,k} = x_{i,i-1}x_{i-2,i-1}$ with $x_{i,i-1}$, $x_{i-2,i-1}$ both non-zero. It is clear that $c_{ij} = 0$ if $|i-j| > 2$. Therefore, if we take the submatrix $\overline{C}$ of C obtained by deleting the first two rows of C and the last two columns, we have $\overline{C}$ an upper-triangular matrix with a nowhere-zero diagonal. Therefore this submatrix will have rank $n - 2$, so that the rank of C is greater than $n - 2$.

Thus the eigenvalues λ occur with multiplicity at most two. These are all non-negative, and the square roots $\sqrt{\lambda}$ are the singular values of X, and so occur with multiplicity at most two also. Therefore, if V is realisable by a matrix in $\widetilde{S}(P_n)$, the coordinates of V are all at most 2. □

Let G be a connected graph on n vertices. Any $\{0, 1\}$ EV multiplicity vector $\mathbf{v}$ is generically realisable for G as shown in [5, Theorem 2.5]. We apply this result to show that P_n is SV generically realisable.

Proposition 3.13. *Let G be a connected graph on n vertices such that any matrix $X \in \widetilde{S}(G)$ has rank at least $n-1$. Let $\boldsymbol{w} = (w_1, \ldots, w_r) \in \mathbb{N}_0^r$ be a multiplicity list with the entries of $\boldsymbol{w}$ in $\{0, 1, 2\}$ and $\sum_{i\in[r]} w_i = n$. Then $\boldsymbol{w}$ is SV generically realisable for G.*

Proof. Since any matrix in $\widetilde{S}(G)$ has rank at least $n - 1$, the multiplicity of zero as a singular value for any matrix in $\widetilde{S(G)}$ is at most one. Consider $D_{\Sigma,\mathbf{w}} \neq 0$ for $\Sigma \in \mathbb{R}^r_{\geq 0}$ a singular value list such that $\mathrm{mult}(0, D_{\Sigma,\mathbf{w}}) < 2$. There is an orthogonal diagonal matrix F and permutation matrix Π such that $D_{\Sigma,\mathbf{w}} = \Pi^\top (D_{\Lambda,\mathbf{v}} F)\Pi$ where $|\Lambda| = \Sigma$ and $\mathbf{v}$ is a $\{0, 1\}$ multiplicity list. This corresponds to the singular values σ_s in $D_{\Sigma,\mathbf{w}}$ with multiplicities two occurring as $-\sigma_s$, σ_s with multiplicity one in $D_{\Lambda,\mathbf{v}}$. As $\mathbf{v}$ is EV generically realisable for G by [5, Theorem 2.5], we have that $\mathbf{w}$ is SV generically realisable by Lemma 3.2. □

Corollary 3.14. *The path P_n is SV generically realisable for all $n \in \mathbb{N}$.*

Proof. Any SV multiplicity vector for P_n has entries in $\{0, 1, 2\}$ by Proposition 3.12. Also, a matrix in $\widetilde{S}(P_n)$ has rank at least $n-1$, as a consequence of the nowhere-zero super and sub-diagonals. Then any such $\{0, 1, 2\}$ singular value list is SV generically realisable for P_n by Proposition 3.13. Thus P_n is SV generically realisable. □

By Proposition 3.4 and Corollary 3.14, K_n and P_n are generically realisable. Therefore by Corollary 2.16, we yield the following.

Corollary 3.15. *Let G, H be two graphs whose connected components are all complete graphs or paths. Then $G \vee H$ is realisable by an orthogonal matrix if and only if G and H have compatible SV multiplicity matrices.*

Determining precisely the criteria for graphs of the above form in Corollary 3.15 to have compatible SV multiplicity matrices is a more complicated problem than the clean result we were able to derive in Corollary 3.7 for the joins of unions of complete graphs. We finish with some examples of instances when orthogonal matrices can be realised.

Example 3.16. The following is an orthogonal matrix in $\widetilde{S}(P_3 \vee 2K_3)$:

$$X = \begin{pmatrix} 0.5 & -0.5 & 0 & -0.24 & 0.126 & -0.219 & 0.0491 & 0.562 & 0.246 \\ 0.5 & 0.5 & -0.5 & -0.0557 & 0.0292 & -0.0508 & -0.399 & 0.103 & -0.271 \\ 0 & 0.5 & 0.5 & 0.24 & -0.126 & 0.219 & -0.0433 & 0.586 & 0.183 \\ -0.064 & 0.0919 & 0.064 & -0.676 & -0.725 & -0.0306 & 0 & 0 & 0 \\ -0.239 & 0.343 & 0.239 & -0.398 & 0.483 & -0.614 & 0 & 0 & 0 \\ 0.0188 & -0.027 & -0.0188 & 0.516 & -0.457 & -0.724 & 0 & 0 & 0 \\ 0.498 & -0.0617 & 0.587 & 0 & 0 & 0 & -0.378 & -0.503 & 0.087 \\ -0.436 & -0.264 & -0.0562 & 0 & 0 & 0 & -0.833 & 0.179 & 0.107 \\ -0.0175 & -0.222 & 0.302 & 0 & 0 & 0 & -0.0103 & 0.213 & -0.902 \end{pmatrix}.$$

With respect to the singular value list $\Sigma = \{\frac{1}{2}, \frac{\sqrt{3}}{2}, 1\}$, we have the following compatible multiplicity matrices V, W for P_3, $2K_3$ respectively:

$$V = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}, \quad W = \begin{pmatrix} 0 & 1 \\ 1 & 1 \\ 2 & 1 \end{pmatrix}.$$

Example 3.17. Consider $n, k \in \mathbb{N}$ such that $2k \leq n \leq 2k+2$. If $m_1, m_2 \geq k$, then $P_n \vee (K_{m_1} \cup K_{m_2})$ is realisable by an orthogonal matrix. To show this, consider the compatible SV multiplicity matrices V, W for P_n, $K_{m_1} \cup K_{m_2}$ respectively:

$$V = \begin{pmatrix} 2 \cdot \mathbf{1}_k^\top \\ v_{k+1} \end{pmatrix} \in \mathbb{N}_0^{k+1,1}, \quad W = \begin{pmatrix} \mathbf{1}_k^\top & \mathbf{1}_k^\top \\ w_{k+1,1} & w_{k+1,2} \end{pmatrix} \in \mathbb{N}_0^{k+1,2}.$$

Then $v_{k+1} = n - 2k \leq 2$ and $w_{k+1,1} = m_1 - k \geq 0$ and $w_{k+1,2} = m_2 - k \geq 0$. Thus V, W are SV multiplicity matrices for P_n and $K_{m_1} \cup K_{m_2}$ and are compatible. By Corollary 3.15, there exists an orthogonal matrix in $\widetilde{S}(P_n \vee (K_{m_1} \cup K_{m_2}))$.

4 Possible future directions

We finish with a discussion of questions and possible directions that have not been explored in this report.

Naturally, the theory presented in Section 2 could be extended to other classes of graphs, such as trees and cycles.

We find that the necessary condition in Lemma 3.3 for $G \vee H$ to be realisable by an orthogonal matrix is also sufficient if G and H are disjoint unions of complete graphs. One could ask whether there are any other types of graphs G, H for which the condition of Lemma 3.3 is necessary and sufficient also.

In this report we have relied on obtaining SV multiplicity vectors from EV multiplicity vectors, as per the process described in Lemma 3.1. An option would be to explore instances where we have SV multiplicity vectors that cannot be obtained from EV multiplicity vectors. Or, are there graphs that are SV generically realisable, but not EV generically realisable?

Additionally, one could consider looking at directed graphs. Given an undirected graph G, the matrices in $\widetilde{S}(G)$ satisfy a symmetric zero non-zero pattern. That is, $X = (x_{ij}) \in \widetilde{S}(G)$ has for $i \neq j$ that $x_{ij} \neq 0 \iff x_{ji} \neq 0$. In the context of the study of $q(G)$ in the symmetric case as part of the IEP-G, this is a natural choice. With respect to the context of orthogonal matrices and SVD, another possible option is to instead consider directed graphs, and matrices $Y = (y_{ij})$ satisfying $y_{ij} \neq 0$ if and only if there is an edge *from* vertex i *to* vertex j in G, for $i \neq j$. A possible direction of interest could be to consider directed paths, which should have quite constrained SV multiplicity vectors, similar to their undirected counterparts.

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