Patterns of orthogonal matrices.

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Introduction

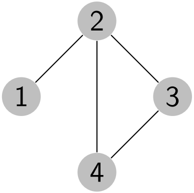
- Patterns in orthogonal matrices have been widely studied. Here we are looking at zero-nonzero patterns on the off-diagonals of matrices.
- The symmetric case of this problem has been studied by Levene, Oblak, and Šmigoc in [LOŠ, 2020], [LOŠ, 2021].
- The symmetric case has applications in the Inverse Eigenvalue Problem for Graphs.

Matrices with patterns determined by a graph G

Definition

Given a simple graph G with labelled vertex set $V = \{1, \ldots, n\}$ and edge set $E \subseteq V \times V$, we define the set of matrices

$$\widetilde{S}(G) = \{B = (b_{ij}) \in \mathbb{R}^{n\times n} : \text{for } i \neq j, b_{ij} \neq 0 \iff \{i,j\} \in E\}.$$



$$B = \begin{pmatrix} \circledast & * & 0 & 0 \\ * & \circledast & * & * \\ 0 & * & \circledast & * \\ 0 & * & * & \circledast \end{pmatrix}$$

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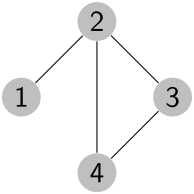

$$B = \begin{pmatrix} 3 & 1 & 0 & 0 \\ 1 & 0 & 0.5 & 7 \\ 0 & 0.5 & \sqrt{2} & -1 \\ 0 & 7 & -1 & 1 \end{pmatrix}$$

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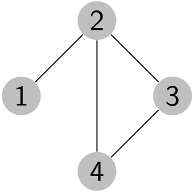

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The subset $S(G) \subseteq \widetilde{S}(G)$ of symmetric matrices and the eigenvalues of such matrices have been studied in detail (see [BA, 2013], [LOŠ, 2020], etc.).

Minimum number of distinct eigenvalues

Definition

For a matrix $A \in \mathbb{R}^{n\times n}$, let $q(A)$ denote the number of distinct eigenvalues of A. Given a graph G and the corresponding set of symmetric matrices $S(G)$, we let

$$q(G) = \min\{q(A) : A \in S(G)\}.$$

Example

$q(G) = 1$ if and only if the graph G has no edges. [BA, 2013]

Orthogonal Matrices and two distinct eigenvalues

Lemma

Let G be a graph with a non-empty edge set. Then $q(G) = 2$ if and only if there exists an orthogonal symmetric matrix in $S(G)$. [BA, 2013]

Example

Let K_n be the complete graph on n vertices. Then $q(K_n) = 2$. [BA, 2013]

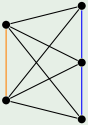
Joins of Unions of Complete Graphs

Definition (Join of Graphs)

Given two graphs G, H, the *join* $G \vee H$ is the disjoint graph union $G \cup H$ with all possible additional edges joining every vertex of G to every vertex of H.

Example

$$P_2 \vee P_3 =$$



Joins of Unions of Complete Graphs

Given $\boldsymbol{m} = (m_1, \ldots, m_k) \in \mathbb{N}^k$, let $K_{\boldsymbol{m}} = K_{m_1} \cup \cdots \cup K_{m_k}$ be the disjoint union of k complete graphs. We look at matrices in $\widetilde{S}(K_{\boldsymbol{m}} \vee K_{\boldsymbol{n}})$ with $\boldsymbol{m} \in \mathbb{N}^k$, $\boldsymbol{n} \in \mathbb{N}^l$.

Example

Let $\boldsymbol{m} = (1, 2)$, $\boldsymbol{n} = (1, 1, 2, 2)$. Then $K_{\boldsymbol{m}} \vee K_{\boldsymbol{n}} = (K_1 \cup K_2) \vee (2K_1 \cup 2K_2)$, and matrices in $\widetilde{S}(K_{\boldsymbol{m}} \vee K_{\boldsymbol{n}})$ are of the form:

$$\left(\begin{array}{ccc|cccccc}
\circledast & 0 & 0 & * & * & * & * & * & * \\
0 & \circledast & * & * & * & * & * & * & * \\
0 & * & \circledast & * & * & * & * & * & * \\
\hline
* & * & * & \circledast & 0 & 0 & 0 & 0 & 0 \\
* & * & * & 0 & \circledast & 0 & 0 & 0 & 0 \\
* & * & * & 0 & 0 & \circledast & * & 0 & 0 \\
* & * & * & 0 & 0 & * & \circledast & 0 & 0 \\
* & * & * & 0 & 0 & 0 & 0 & \circledast & * \\
* & * & * & 0 & 0 & 0 & 0 & * & \circledast
\end{array}\right)$$

Orthogonal matrices and joins of graphs

Lemma

Let $k, l \in \mathbb{N}$ with $k \leq l$. Consider the graphs $K_{\boldsymbol{m}}$, $K_{\boldsymbol{n}}$ with $\boldsymbol{m} \in \mathbb{N}^k$, $\boldsymbol{n} \in \mathbb{N}^l$. If there exists an orthogonal matrix in $\widetilde{S}(K_{\boldsymbol{m}} \vee K_{\boldsymbol{n}})$, then $l \leq |K_{\boldsymbol{m}}|$.

Example

Let $\boldsymbol{m} = (1, 2)$, $\boldsymbol{n} = (1, 1, 2, 2)$. Then $K_{\boldsymbol{m}} \vee K_{\boldsymbol{n}}$ is *not* realisable by an orthogonal matrix.

$$\widetilde{S}(K_{\boldsymbol{m}} \vee K_{\boldsymbol{n}}) = \left\{ \left(\begin{array}{ccc|cccccc} \circledast & 0 & 0 & * & * & * & * & * & * \\ 0 & \circledast & * & * & * & * & * & * & * \\ 0 & * & \circledast & * & * & * & * & * & * \\ \hline * & * & * & \circledast & 0 & 0 & 0 & 0 & 0 \\ * & * & * & 0 & \circledast & 0 & 0 & 0 & 0 \\ * & * & * & 0 & 0 & \circledast & * & 0 & 0 \\ * & * & * & 0 & 0 & * & \circledast & 0 & 0 \\ * & * & * & 0 & 0 & 0 & 0 & \circledast & * \\ * & * & * & 0 & 0 & 0 & 0 & * & \circledast \end{array} \right) \right\}$$

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Orthogonal matrices and joins of graphs

Theorem

Consider the graph $K_{\boldsymbol{m}} \vee K_{\boldsymbol{n}}$ with $\boldsymbol{m} \in \mathbb{N}^k$, $\boldsymbol{n} \in \mathbb{N}^l$ and $k \leq l$. Then $l \leq |K_{\boldsymbol{m}}|$ is a necessary and sufficient condition for $K_{\boldsymbol{m}} \vee K_{\boldsymbol{n}}$ to be realisable by an orthogonal matrix.

Orthogonal matrices and joins of graphs

Example

Consider $G = 2K_2 \vee 3K_1$. G is not realisable by an orthogonal symmetric matrix (see [LOŠ, 2020, Example 4.23]), but it can be realised by an orthogonal matrix:

$$Y \approx \left(\begin{array}{cccc|ccc} 0.414 & 0.573 & 0 & 0 & 0.296 & 0.598 & 0.231 \\ 0.573 & -0.413 & 0 & 0 & 0.032 & -0.267 & 0.653 \\ 0 & 0 & 0.352 & 0.692 & 0.57 & -0.234 & -0.124 \\ 0 & 0 & 0.692 & -0.645 & 0.292 & -0.12 & -0.063 \\ \hline 0.199 & 0.221 & 0.570 & 0.292 & -0.707 & 0 & 0 \\ 0.133 & 0.642 & -0.234 & -0.12 & 0 & -0.707 & 0 \\ 0.665 & -0.195 & -0.124 & -0.063 & 0 & 0 & -0.707 \end{array}\right) \in \widetilde{S}(G).$$

Compatible Multiplicity Matrices: Example

Example

Consider $G = 2K_2 \vee 3K_1$. We have $Y \in \widetilde{S}(G)$ orthogonal:

$$Y \approx \left(\begin{array}{cccc|ccc} 0.414 & 0.573 & 0 & 0 & 0.296 & 0.598 & 0.231 \\ 0.573 & -0.413 & 0 & 0 & 0.032 & -0.267 & 0.653 \\ 0 & 0 & 0.352 & 0.692 & 0.57 & -0.234 & -0.124 \\ 0 & 0 & 0.692 & -0.645 & 0.292 & -0.12 & -0.063 \\ \hline 0.199 & 0.221 & 0.570 & 0.292 & -0.707 & 0 & 0 \\ 0.133 & 0.642 & -0.234 & -0.12 & 0 & -0.707 & 0 \\ 0.665 & -0.195 & -0.124 & -0.063 & 0 & 0 & -0.707 \end{array}\right).$$

We can perform a SVD on $Y[2K_2]$ and write $Y[2K_2] = UDV^\top$ with U, V orthogonal and $D = \left(\begin{array}{cc|cc} 0.707 & 0 & 0 & 0 \\ 0 & 0.707 & 0 & 0 \\ \hline 0 & 0 & 0.707 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right)$. With respect to the singular value list $\Sigma = \{0.707, 1\}$, we define *singular value multiplicity matrices* V, W for $2K_2$, $3K_1$ respectively. These encode the multiplicities of the singular values in the connected components of the graphs. We have

$$V = \begin{pmatrix} 2 & 1 \\ 0 & 1 \end{pmatrix}, \quad W = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

Compatible Multiplicity Matrices and Singular Value Decomposition

In [LOŠ, 2020], a machinery of *compatible eigenvalue multiplicity matrices* is developed to give a necessary condition for $G \vee H$ to be realisable by a symmetric orthogonal matrix. Under additional conditions, these compatibility relations become a sufficient condition for the existence of an orthogonal symmetric matrix $X \in S(G \vee H)$ also.

For the non-symmetric case, the notion of compatible eigenvalue multiplicity matrices extends naturally to that of *compatible singular value multiplicity matrices*. In a similar capacity we are able to obtain a necessary and sufficient condition for the join of two graphs to be realisable by an orthogonal matrix in certain cases.

Key References

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