

GENERALIZED FISHBURN NUMBERS AND TORUS KNOTS

COLIN BIJAOU, HANS U. BODEN, BECKHAM MYERS, ROBERT OSBURN, WILLIAM RUSHWORTH,
AARON TRONSGARD, AND SHAOYANG ZHOU

ABSTRACT. Andrews and Sellers recently initiated the study of arithmetic properties of Fishburn numbers. In this paper, we prove prime power congruences for generalized Fishburn numbers. These numbers are the coefficients in the $1 - q$ expansion of the Kontsevich-Zagier series $\mathcal{F}_t(q)$ for the torus knots $T(3, 2^t)$, $t \geq 2$. The proof uses a strong divisibility result of Ahlgren, Kim and Lovejoy and a new “strange identity” for $\mathcal{F}_t(q)$.

1. INTRODUCTION

The Fishburn numbers $\xi(n)$ are the coefficients in the formal power series expansion

$$F(1 - q) =: \sum_{n \geq 0} \xi(n)q^n = 1 + q + 2q^2 + 5q^3 + 15q^4 + 53q^5 + \dots \quad (1.1)$$

where $F(q) := \sum_{n \geq 0} (q)_n$ is the Kontsevich-Zagier “strange” series [28] and

$$(a_1, a_2, \dots, a_j)_n = (a_1, a_2, \dots, a_j; q)_n := \prod_{k=1}^n (1 - a_1 q^{k-1})(1 - a_2 q^{k-1}) \dots (1 - a_j q^{k-1})$$

is the standard q -hypergeometric notation, valid for $n \in \mathbb{N} \cup \{\infty\}$. Here, the moniker “strange” is used as $F(q)$ does not converge on any open subset of \mathbb{C} , but is well-defined when q is a root of unity (where it is finite) and when q is replaced by $1 - q$ as in (1.1). The Fishburn numbers are of interest for their numerous combinatorial variants (see A022493 in [23]), asymptotics [24, 28] and arithmetic properties [1, 3, 7, 8, 25]. In their marvelous paper, Andrews and Sellers [3] proved congruences for $\xi(n)$ modulo primes which were then extended to prime powers [1, 25]. For example, we have

$$\xi(5^r n - 1) \equiv \xi(5^r n - 2) \equiv 0 \pmod{5^r}, \quad (1.2)$$

$$\xi(7^r n - 1) \equiv 0 \pmod{7^r} \quad (1.3)$$

and

$$\xi(11^r n - 1) \equiv \xi(11^r n - 2) \equiv \xi(11^r n - 3) \equiv 0 \pmod{11^r} \quad (1.4)$$

for all natural numbers r and n . Our interest in this paper lies in the knot theoretic interpretation of $F(q)$ as it leads to a natural generalization of the coefficients $\xi(n)$.

Let K be a knot and $J_N(K; q)$ be the usual colored Jones polynomial, normalized to be 1 for the unknot. As a knot invariant, the colored Jones polynomial plays the lead role in many open

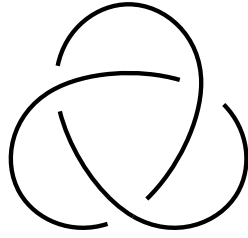
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problems in quantum topology. The sequence $\{J_N(K; q)\}_{N \in \mathbb{N}}$ appears to encode many subtle geometric and topological properties of the knot K at a remarkably deep level. For example, the Volume Conjecture [15, 21, 22] relates the value at $\zeta_N = e^{2\pi i/N}$ of the N th colored Jones polynomial (or, equivalently, the N th Kashaev invariant) of a knot to its hyperbolic volume, the Strong Slope Conjecture [6, 14] says that the maximal and minimal degrees of $J_N(K; q)$ in q contain information about essential surfaces in knot exteriors and the AJ Conjecture [5] connects the recurrence relation for $J_N(K; q)$ to the A -polynomial of the knot [4], a plane curve describing the $SL(2, \mathbb{C})$ character variety of the knot complement. Explicit formulas for $J_N(K; q)$ in terms of q -hypergeometric series are also of importance and have been proven for various families of knots [9, 11, 12, 17, 18, 19, 20, 26, 27]. For example, if $T(3, 2) = T(2, 3)$ is the right-handed torus knot as in Figure 1, then [9, 17]

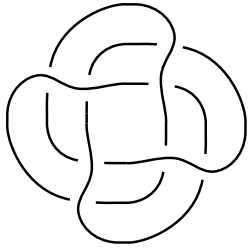
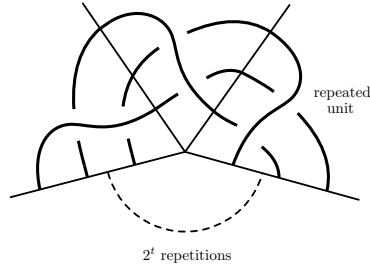
$$J_N(T(3, 2); q) = q^{1-N} \sum_{n \geq 0} q^{-nN} (q^{1-N})_n.$$

FIGURE 1. $T(3, 2)$

Observe that the Kontsevich-Zagier series $F(q)$ matches the colored Jones polynomial for $T(3, 2)$ at roots of unity, that is, for $q = \zeta_N$

$$\zeta_N F(\zeta_N) = J_N(T(3, 2); \zeta_N).$$

Consider the family of torus knots $T(3, 2^t)$ for $t \geq 2$ as in Figures 2 and 3.

FIGURE 2. $T(3, 4)$ FIGURE 3. $T(3, 2^t)$

In this case, a q -hypergeometric expression for the colored Jones polynomial has been computed, namely (see page 41, Théorème 3.2 in [16])

$$\begin{aligned}
J_N(T(3, 2^t); q) &= (-1)^{h''(t)} q^{2^t-1-h'(t)-N} \sum_{n \geq 0} (q^{1-N})_n q^{-Nnm(t)} \\
&\times \sum_{3 \sum_{\ell=1}^{m(t)-1} j_\ell \equiv 1 \pmod{m(t)}} (-q^{-N})^{\sum_{\ell=1}^{m(t)-1} j_\ell} q^{\frac{-a(t)+\sum_{\ell=1}^{m(t)-1} j_\ell \ell}{m(t)} + \sum_{\ell=1}^{m(t)-1} \binom{j_\ell}{2}} \\
&\times \sum_{k=0}^{m(t)-1} q^{-kN} \prod_{\ell=1}^{m(t)-1} \begin{bmatrix} n + I(\ell \leq k) \\ j_\ell \end{bmatrix}
\end{aligned} \tag{1.5}$$

where

$$h''(t) = \begin{cases} \frac{2^t-1}{3} & \text{if } t \text{ is even,} \\ \frac{2^t-2}{3} & \text{if } t \text{ is odd,} \end{cases} \quad h'(t) = \begin{cases} \frac{2^t-4}{3} & \text{if } t \text{ is even,} \\ \frac{2^t-5}{3} & \text{if } t \text{ is odd,} \end{cases} \quad a(t) = \begin{cases} \frac{2^{t-1}+1}{3} & \text{if } t \text{ is even,} \\ \frac{2^{t+1}}{3} & \text{if } t \text{ is odd,} \end{cases}$$

$m(t) = 2^{t-1}$, $I(*)$ is the characteristic function and

$$\begin{bmatrix} n \\ k \end{bmatrix} = \begin{bmatrix} n \\ k \end{bmatrix}_q := \frac{(q)_n}{(q)_{n-k}(q)_k}$$

is the q -binomial coefficient. We note that the $t = 2$ case of (1.5) recovers equation (16) in [11]. We now define the *Kontsevich-Zagier series for torus knots* as

$$\begin{aligned}
\mathcal{F}_t(q) &= (-1)^{h''(t)} q^{-h'(t)} \sum_{n \geq 0} (q)_n \sum_{3 \sum_{\ell=1}^{m(t)-1} j_\ell \equiv 1 \pmod{m(t)}} q^{\frac{-a(t)+\sum_{\ell=1}^{m(t)-1} j_\ell \ell}{m(t)} + \sum_{\ell=1}^{m(t)-1} \binom{j_\ell}{2}} \\
&\times \sum_{k=0}^{m(t)-1} \prod_{\ell=1}^{m(t)-1} \begin{bmatrix} n + I(\ell \leq k) \\ j_\ell \end{bmatrix}.
\end{aligned} \tag{1.6}$$

As with the original Kontsevich-Zagier series, $\mathcal{F}_t(q)$ converges in a similar fashion and thus we may write

$$\mathcal{F}_t(1-q) =: \sum_{n \geq 0} \xi_t(n) q^n.$$

For example,

$$\mathcal{F}_2(1-q) = 1 + 3q + 11q^2 + 50q^3 + 280q^4 + 1890q^5 + \dots$$

and

$$\mathcal{F}_3(1-q) = 1 + 7q + 49q^2 + 420q^3 + 4515q^4 + 59367q^5 + \dots$$

In addition, we have

$$\zeta_N^{2^t-1} \mathcal{F}_t(\zeta_N) = J_N(T(3, 2^t); \zeta_N).$$

The purpose of this paper is to illustrate how a recent result of Ahlgren, Kim and Lovejoy [2] combined with a new “strange identity” for $\mathcal{F}_t(q)$ allow one to prove prime power congruences

akin to (1.2)–(1.4) for the *generalized Fishburn numbers* $\xi_t(n)$. For natural numbers s and $t \geq 2$ and the periodic function

$$\chi_t(n) = \chi_{3 \cdot 2^{t+1}}(n) := \begin{cases} 1 & \text{if } n \equiv 2^{t+1} - 3, 3 + 2^{t+2} \pmod{3 \cdot 2^{t+1}}, \\ -1 & \text{if } n \equiv 2^{t+1} + 3, 2^{t+2} - 3 \pmod{3 \cdot 2^{t+1}}, \\ 0 & \text{otherwise,} \end{cases} \quad (1.7)$$

we define the set

$$S_{t,\chi_t}(s) = \left\{ 0 \leq j \leq s-1 : j \equiv \frac{n^2 - (2^{t+1} - 3)^2}{3 \cdot 2^{t+2}} \pmod{s} \text{ where } \chi_t(n) \neq 0 \right\}.$$

Our main result is now the following.

Theorem 1.1. *If $p \geq 5$ is a prime and $j \in \{1, 2, \dots, p-1 - \max S_{t,\chi_t}(p)\}$, then*

$$\xi_t(p^r n - j) \equiv 0 \pmod{p^r}$$

for all natural numbers r , n and $t \geq 2$.

One can check that $S_{2,\chi_2}(5) = \{0, 2, 3\}$, $S_{2,\chi_2}(17) = \{0, 2, 3, 4, 7, 8, 9, 11, 14\}$, $S_{3,\chi_3}(7) = \{0, 2, 3, 4\}$ and $S_{3,\chi_3}(13) = \{0, 2, 5, 6, 7, 8, 11\}$. Thus, by Theorem 1.1, we have

$$\begin{aligned} \xi_2(5^r n - 1) &\equiv 0 \pmod{5^r}, \\ \xi_2(17^r - 1) &\equiv \xi_2(17^r - 2) \equiv 0 \pmod{17^r}, \\ \xi_3(7^r n - 1) &\equiv \xi_3(7^r n - 2) \equiv 0 \pmod{7^r} \end{aligned}$$

and

$$\xi_3(13^r n - 1) \equiv 0 \pmod{13^r}$$

for all natural numbers r and n .

The paper is organized as follows. In Section 2, we recall the main result from [2] and then record some preliminaries, including the new “strange identity”. In Section 3, we prove Theorem 1.1. Finally, in Section 4, we discuss some possibilities for future work.

2. PRELIMINARIES

Our first step is to recall the setup from [2]. Let \mathcal{F} be a function of the form

$$\mathcal{F}(q) = \sum_{n \geq 0} (q)_n f_n(q) \quad (2.1)$$

where $f_n(q) \in \mathbb{Z}[q]$. For positive integers s and N , consider the partial sum

$$\mathcal{F}(q; N) := \sum_{n=0}^N (q)_n f_n(q)$$

and its s -dissection

$$\mathcal{F}(q; N) = \sum_{i=0}^{s-1} q^i A_{\mathcal{F},s}(N, i, q^s)$$

where $A_{\mathcal{F},s}(N, i, q) \in \mathbb{Z}[q]$. Now, consider the partial theta functions

$$P_{a,b,\chi}^{(\nu)}(q) := \sum_{n \geq 0} n^\nu \chi(n) q^{\frac{n^2-a}{b}} \quad (2.2)$$

where $\nu \in \{0, 1\}$, $a \geq 0$, $b > 0$ are integers and $\chi : \mathbb{Z} \rightarrow \mathbb{C}$ is a function satisfying the following two conditions:

$$\chi(n) \neq 0 \text{ only if } \frac{n^2 - a}{b} \in \mathbb{Z} \quad (2.3)$$

and for each root of unity ζ ,

$$\text{the function } n \rightarrow \zeta^{\frac{n^2-a}{b}} \chi(n) \text{ is periodic and has mean value zero.} \quad (2.4)$$

Finally, define the set $S_{a,b,\chi}(s)$ by

$$S_{a,b,\chi}(s) = \left\{ 0 \leq j \leq s-1 : j \equiv \frac{n^2 - a}{b} \pmod{s} \text{ where } \chi(n) \neq 0 \right\}.$$

We can now state the main result in [2].

Theorem 2.1. *Suppose that \mathcal{F} and $P_{a,b,\chi}^{(\nu)}$ are functions as in (2.1) and (2.2) and for each root of unity ζ , we have the asymptotic expansion*

$$P_{a,b,\chi}^{(\nu)}(\zeta e^{-t}) \sim \mathcal{F}(\zeta e^{-t})$$

as $t \rightarrow 0^+$. Suppose that s and N are positive integers and $i \notin S_{a,b,\chi}(s)$. Then

$$(q)_{\lambda(N,s)} \mid A_{\mathcal{F},s}(N, i, q)$$

where $\lambda(N, s) := \lfloor \frac{N+1}{s} \rfloor$.

We now turn to our situation and record the following result. Throughout, we assume that $t \geq 2$.

Proposition 2.2. *The periodic function χ_t (as defined by (1.7)) satisfies (2.3) and (2.4).*

Proof. A straightforward calculation using (1.7) confirms (2.3) with $a = (2^{t+1} - 3)^2$ and $b = 3 \cdot 2^{t+2}$. Suppose ζ is a root of unity of order M and define

$$\psi_t(n) = \zeta^{\frac{n^2 - (2^{t+1} - 3)^2}{3 \cdot 2^{t+2}}} \chi_t.$$

Note that ψ_t has period $M(3 \cdot 2^{t+1})$. We now claim that

$$\sum_{n=1}^{M(3 \cdot 2^{t+1})} \psi_t(n) = 0. \quad (2.5)$$

Suppose M is even. By (1.7), we obtain

$$\chi_t(n + 3 \cdot 2^t M) = \chi_t(n) \quad (2.6)$$

for all n . We also have

$$\zeta^{\frac{(n+3 \cdot 2^t M)^2 - (2^{t+1} - 3)^2}{3 \cdot 2^{t+2}}} = \zeta^{\frac{M(n+3 \cdot 2^{t-1} M^2)}{2}} \zeta^{\frac{n^2 - (2^{t+1} - 3)^2}{3 \cdot 2^{t+2}}}. \quad (2.7)$$

As ψ_t is supported on odd integers, we can assume that n is odd. Then $M(n + 3 \cdot 2^{t-1} M^2)$ is an odd multiple of M and so (2.7) equals $-\zeta^{\frac{n^2 - (2^{t+1} - 3)^2}{3 \cdot 2^{t+2}}}$. This fact and (2.6) imply

$$\sum_{n=1}^{M(3 \cdot 2^{t+1})} \psi_t(n) = \sum_{n=1}^{M(3 \cdot 2^t)} (\psi_t(n) + \psi_t(n + 3 \cdot 2^t M)) = 0.$$

Now, suppose M is odd. We break up (2.5) into four sums, one for each congruence class modulo $3 \cdot 2^{t+1}$ in (1.7). Specifically,

$$\begin{aligned} \sum_{n=1}^{M(3 \cdot 2^{t+1})} \psi_t(n) &= \sum_{m=0}^{M-1} \zeta^{3 \cdot 2^t m^2 + (2^{t+2} + 3)m + 2^t + 2 + 1} - \sum_{m=0}^{M-1} \zeta^{3 \cdot 2^t m^2 + (2^{t+2} - 3)m + 2^t - 1} \\ &\quad + \sum_{m=0}^{M-1} \zeta^{3 \cdot 2^t m^2 + (2^{t+1} - 3)m} - \sum_{m=0}^{M-1} \zeta^{3 \cdot 2^t m^2 + (2^{t+1} + 3)m + 2}. \end{aligned} \quad (2.8)$$

Let $i_{t,M} = 2^{-t}$ be the unique solution to $2^t i \equiv 1 \pmod{M}$. Performing the shift $m \rightarrow m + i_{t,M}$ (respectively, $m \rightarrow m - i_{t,M}$) to the second (respectively, fourth) sum in (2.8) followed by a routine simplification implies (2.5). The result now follows. \square

For χ_t as in (1.7), consider the series

$$H_t(x) = H_t(x, q) = \sum_{n \geq 0} \chi_t(n) q^{\frac{n^2 - (2^{t+1} - 3)^2}{3 \cdot 2^{t+2}}} x^{\frac{n - (2^{t+1} - 3)}{2}}.$$

Proposition 2.3. *Let $h(t) = 2^t - 2$. We have*

$$\begin{aligned} H_t(x) &= (-1)^{h''(t)} q^{-h'(t)} x^{-h(t)} \sum_{n \geq 0} (x)_{n+1} x^{nm(t)} \\ &\quad \times \sum_{\substack{3 \sum_{\ell=1}^{m(t)-1} j_\ell \equiv 1 \\ \pmod{m(t)}}} (-x)^{\sum_{\ell=1}^{m(t)-1} j_\ell} q^{\frac{-a(t) + \sum_{\ell=1}^{m(t)-1} j_\ell \ell}{m(t)} + \sum_{\ell=1}^{m(t)-1} \binom{j_\ell}{2}} \\ &\quad \times \sum_{k=0}^{m(t)-1} x^k \prod_{\ell=1}^{m(t)-1} \left[\begin{matrix} n + I(\ell \leq k) \\ j_\ell \end{matrix} \right]. \end{aligned} \quad (2.9)$$

Proof. Let $f_t(x)$ denote the right-hand side of (2.9). Then $f_t(x)$ satisfies the difference equation (see (3.3.11) in [16])

$$f_t(x) = 1 - q^2 x^3 - q^{2^t - 1} x^{2^t} + q^{3+2^t} x^{3+2^t} + q^{5 \cdot 2^t - 3} x^{3 \cdot 2^t} f_t(q^2 x). \quad (2.10)$$

One can directly verify that $H_t(x)$ also satisfies (2.10) using (1.7). \square

Recall that the Kontsevich-Zagier series $F(q)$ satisfies the “strange identity”

$$F(q) \text{ “} = \text{”} - \frac{1}{2} \sum_{n \geq 1} n \left(\frac{12}{n} \right) q^{\frac{n^2-1}{24}}$$

where “ = ” means that the two sides agree to all orders at every root of unity (for further details, see Sections 2 and 5 in [28]) and $\left(\frac{12}{*} \right)$ is the quadratic character of conductor 12. We now prove the following new strange identity for the Kontsevich-Zagier series for torus knots.

Proposition 2.4. *We have*

$$\mathcal{F}_t(q) \text{ “} = \text{”} - \frac{1}{2} \sum_{n \geq 0} n \chi_t(n) q^{\frac{n^2 - (2^{t+1} - 3)^2}{3 \cdot 2^{t+2}}}. \quad (2.11)$$

Proof. For ease of notation, let $'$ denote the condition

$$3 \sum_{\ell=1}^{m(t)-1} j_\ell \ell \equiv 1 \pmod{m(t)}$$

occurring in the second sum in (1.5) on the j_ℓ 's,

$$v = v(j_1, \dots, j_{m(t)-1}) := \frac{-a(t) + \sum_{\ell=1}^{m(t)-1} j_\ell \ell}{m(t)} + \sum_{\ell=1}^{m(t)-1} \binom{j_\ell}{2}$$

and \bar{c} the reduction of an integer c modulo $m(t)$. The following identity¹ implies (2.11) upon setting $q = \zeta e^{-t}$, then letting $t \rightarrow 0^+$:

$$\begin{aligned} & \frac{1}{2} \sum_{n \geq 0} n \chi_t(n) q^{\frac{n^2 - (2^{t+1} - 3)^2}{3 \cdot 2^{t+2}}} - \frac{2^{t+1} - 3}{2} (q^{2^{t+1}}, q^{2^t-1}, q^{2^t+1}; q^{2^{t+1}})_\infty (q^2, q^{2^{t+2}-2}; q^{2^{t+2}})_\infty \\ &= (-1)^{h''(t)+1} q^{-h'(t)} \sum_{n \geq 0} \left[(q)_n - (q)_\infty \right] \\ & \quad \times \sum'_{j_1, \dots, j_{m(t)-1}} (-1)^{\sum_{\ell=1}^{m(t)-1} j_\ell} q^v \sum_{k=0}^{m(t)-1} \prod_{\ell=1}^{m(t)-1} \left[\begin{matrix} n + I(\ell \leq k) \\ j_\ell \end{matrix} \right] \\ &+ (-1)^{h''(t)+1} q^{-h'(t)} (q)_\infty \left(\sum_{i=1}^{\infty} \frac{q^i}{1 - q^i} \right) \sum_{n \geq 0} b_{n,t}(q) + (-1)^{h''(t)} q^{-h'(t)} (q)_\infty \sum_{n \geq 0} (n - h(t)) b_{n,t}(q) \end{aligned} \quad (2.12)$$

where

$$b_{n,t}(q) = a_{n,t}(q) - a_{n-1,t}(q) \quad (2.13)$$

and

¹The $t = 2$ case of (2.12) gives an alternative (and corrected) version of Proposition 5 in [13].

$$a_{n,t}(q) = \sum'_{j_1, \dots, j_{m(t)-1}} (-1)^{\sum_{\ell=1}^{m(t)-1} j_\ell} q^v \prod_{\ell=1}^{m(t)-1} \left[\frac{n - \sum_{\ell=1}^{m(t)-1} j_\ell - \overline{(n - \sum_{\ell=1}^{m(t)-1} j_\ell)}}{m(t)} + I(\ell \leq \overline{(n - \sum_{\ell=1}^{m(t)-1} j_\ell)}) \right]. \quad (2.14)$$

To prove (2.12), we begin by rewriting (2.9) as

$$\begin{aligned} H_t(x) &= (-1)^{h''(t)} q^{-h'(t)} (1-x) \sum_{n \geq 0} \left[(qx)_n - (qx)_\infty \right] x^{nm(t)} \\ &\times \sum'_{j_1, \dots, j_{m(t)-1}} (-1)^{\sum_{\ell=1}^{m(t)-1} j_\ell} x^{-h(t) + \sum_{\ell=1}^{m(t)-1} j_\ell} q^v \\ &\times \sum_{k=0}^{m(t)-1} x^k \prod_{\ell=1}^{m(t)-1} \left[n + I(\ell \leq k) \right] + (qx)_\infty (-1)^{h''(t)} q^{-h'(t)} x^{-h(t)} (1-x) M_t(x, q) \end{aligned} \quad (2.15)$$

where

$$M_t(x, q) := \sum_{n \geq 0} x^{nm(t)} \sum'_{j_1, \dots, j_{m(t)-1}} (-x)^{\sum_{\ell=1}^{m(t)-1} j_\ell} q^v \sum_{k=0}^{m(t)-1} x^k \prod_{\ell=1}^{m(t)-1} \left[n + I(\ell \leq k) \right]. \quad (2.16)$$

We now claim that

$$(1-x)M_t(x, q) = \sum_{n \geq 0} b_{n,t}(q) x^n \quad (2.17)$$

where $b_{n,t}(q)$ is given by (2.13) and (2.14). To see (2.17), we first write

$$\sum_{\ell=1}^{m(t)-1} j_\ell = m(t) \left\lfloor \frac{\sum_{\ell=1}^{m(t)-1} j_\ell}{m(t)} \right\rfloor + \overline{\sum_{\ell=1}^{m(t)-1} j_\ell}. \quad (2.18)$$

We now have that $M_t(x, q)$ equals

$$\begin{aligned} &\sum'_{j_1, \dots, j_{m(t)-1}} \sum_{k=0}^{m(t)-1} \sum_{n \geq 0} (-1)^{\sum_{\ell=1}^{m(t)-1} j_\ell} x^{m(t) \left(n + \left\lfloor \frac{\sum_{\ell=1}^{m(t)-1} j_\ell}{m(t)} \right\rfloor \right) + k + \overline{\sum_{\ell=1}^{m(t)-1} j_\ell}} q^v \prod_{\ell=1}^{m(t)-1} \left[n + I(\ell \leq k) \right] \\ &\quad \text{(using (2.16) and (2.18))} \\ &= \sum'_{j_1, \dots, j_{m(t)-1}} \sum_{k=0}^{m(t)-1} \sum_{n \geq 0} (-1)^{\sum_{\ell=1}^{m(t)-1} j_\ell} x^{m(t)n + k + \overline{\sum_{\ell=1}^{m(t)-1} j_\ell}} q^v \prod_{\ell=1}^{m(t)-1} \left[n - \left\lfloor \frac{\sum_{\ell=1}^{m(t)-1} j_\ell}{m(t)} \right\rfloor + I(\ell \leq k) \right] \\ &\quad \left(\text{letting } n \rightarrow n - \left\lfloor \frac{\sum_{\ell=1}^{m(t)-1} j_\ell}{m(t)} \right\rfloor \right) \end{aligned}$$

$$\begin{aligned}
&= \sum_{k=0}^{m(t)-1} \sum'_{j_1, \dots, j_{m(t)-1}} \sum_{n \equiv k + \overline{\sum_{\ell=1}^{m(t)-1} j_\ell} \pmod{m(t)}} (-1)^{\sum_{\ell=1}^{m(t)-1} j_\ell} x^n q^v \prod_{\ell=1}^{m(t)-1} \left[\frac{n - k - \sum_{\ell=1}^{m(t)-1} j_\ell}{m(t)} + I(\ell \leq k) \right] \\
&\quad \left(\text{letting } n \rightarrow \frac{n - k - \sum_{\ell=1}^{m(t)-1} j_\ell}{m(t)} \text{ and using (2.18)} \right) \\
&= \sum_{n \geq 0} \sum'_{j_1, \dots, j_{m(t)-1}} x^n (-1)^{\sum_{\ell=1}^{m(t)-1} j_\ell} q^v \prod_{\ell=1}^{m(t)-1} \left[\frac{n - \sum_{\ell=1}^{m(t)-1} j_\ell - \overline{(n - \sum_{\ell=1}^{m(t)-1} j_\ell)}}{m(t)} + I(\ell \leq \overline{n - \sum_{\ell=1}^{m(t)-1} j_\ell}) \right] \\
&= \sum_{n \geq 0} a_{n,t}(q) x^n.
\end{aligned}$$

Thus, we obtain (2.17) via (2.13). We now substitute (2.17) into (2.15), then take the derivative of both sides of (2.15) with respect to x , set $x = 1$ and apply Watson's quintuple product identity

$$\sum_{k \in \mathbb{Z}} q^{\frac{k(3k-1)}{2}} x^{3k} (1 - xq^k) = (q, x, qx^{-1})_\infty (qx^2, qx^{-2}; q^2)_\infty \quad (2.19)$$

to the resulting left-hand side with $q \rightarrow q^{2^{t+1}}$ and $x = q^{2^t - 1}$ to obtain (2.12). This proves the result. \square

3. PROOF OF THEOREM 1.1

Proof of Theorem 1.1. Let p be a prime ≥ 5 and consider the truncation $\mathcal{F}_t(q, N)$ of (1.6) and its p -dissection

$$\mathcal{F}_t(q, N) = \sum_{i=0}^{p-1} q^i A_{\mathcal{F}_t, p}(N, i, q^p).$$

We have

$$\mathcal{F}_t(1 - q, N) = \sum_{i \in S_{t, \chi_t}(p)} (1 - q)^i A_{\mathcal{F}_t, p}(N, i, (1 - q)^p) + \sum_{i \notin S_{t, \chi_t}(p)} (1 - q)^i A_{\mathcal{F}_t, p}(N, i, (1 - q)^p).$$

By Theorem 2.1, Propositions 2.2 and 2.4 and the fact that $(1 - q^k)^n \mid (q)_m$ for $m \geq kn$, we can write

$$\mathcal{F}_t(1 - q, N) = \sum_{i \in S_{t, \chi_t}(p)} (1 - q)^i A_{\mathcal{F}_t, p}(N, i, (1 - q)^p) + (1 - (1 - q)^p)^n \sum_{i \notin S_{t, \chi_t}(p)} (1 - q)^i g_i(q)$$

for n such $N \geq np^2 - 1$ and $g_i(q) \in \mathbb{Z}[q]$. By Lemma 3.3 in [25],

$$(1 - (1 - q)^p)^n \equiv O(q^{pn - (p-1)(r-1)}) \pmod{p^r}$$

and so

$$\mathcal{F}_t(1 - q, N) = \sum_{i \in S_{t, \chi_t}(p)} (1 - q)^i A_{\mathcal{F}_t, p}(N, i, (1 - q)^p) + O(q^{pn - (p-1)(r-1)}) \pmod{p^r}. \quad (3.1)$$

The sum in (3.1) is an integral linear combination of terms of the form $(1 - q)^{i+lp}$ for $i \in S_{t, \chi_t}(p)$ and natural numbers l . Thus, it suffices to show that

$$\binom{i + lp}{p^r n - j} \equiv 0 \pmod{p^r} \quad (3.2)$$

for all $j \in \{1, 2, \dots, p-1 - \max S_{t, \chi_t}(p)\}$. The condition on j implies that $j < p - i$ and so (3.2) follows from Lemma 3.4 in [25]. The result now follows after letting $n \rightarrow \infty$ in (3.1). \square

4. FURTHER COMMENTS

There are several avenues for further study. First, one could investigate combinatorial descriptions and asymptotic properties for the numbers $\xi_t(n)$. Second, the colored Jones polynomial of a knot K satisfies a *cyclotomic expansion* of the form

$$J_N(K; q) = \sum_{n \geq 0} (q^{1+N})_n (q^{1-N})_n C_n(K; q),$$

where the *cyclotomic coefficients* $C_n(K; q)$ are Laurent polynomials independent of N [10]. It would be highly desirable to find the cyclotomic expansion for the torus knots $T(3, 2^t)$, $t \geq 2$. For $t = 1$, the cyclotomic expansion has been found by Masbaum [20]. However, it is unclear if his techniques are sufficient when $t \geq 2$. They require finding a link whose components are unknotted, from which $T(3, 2)$ can be recovered by introducing twists into a single region of one of the components. Employing this process for $t \geq 2$ appears to require extending it to allow for multiple twist regions. Third, in relation to Proposition 2.4, a strange identity for the Kontsevich-Zagier series associated to the torus knots $T(2, 2t + 1)$ for $t \geq 1$ has also been computed (see (15) in [12]). Do strange-type identities exist for q -series associated to satellite or hyperbolic knots? Finally, as the Kontsevich-Zagier series $F(q)$ is a foundational example of a quantum modular form [29], it is natural to wonder if the same is true for $\mathcal{F}_t(q)$. This is the subject of forthcoming work.

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DEPARTMENT OF MATHEMATICS & STATISTICS, MCMASTER UNIVERSITY, HAMILTON HALL, 1280 MAIN STREET WEST, HAMILTON, ONTARIO, CANADA L8S 4K1

E-mail address: `bijaouc@mcmaster.ca`, `boden@mcmaster.ca`, `will.rushworth@math.mcmaster.ca`

DEPARTMENT OF MATHEMATICS, HARVARD UNIVERSITY, SCIENCE CENTER ROOM 325, 1 OXFORD STREET, CAMBRIDGE, MA 02138, USA

E-mail address: `bmyers@college.harvard.edu`

SCHOOL OF MATHEMATICS AND STATISTICS, UNIVERSITY COLLEGE DUBLIN, BELFIELD, DUBLIN 4, IRELAND

E-mail address: `robert.osburn@ucd.ie`

DEPARTMENT OF MATHEMATICAL AND STATISTICAL SCIENCES, UNIVERSITY OF ALBERTA, EDMONTON, ALBERTA, T6G 2G1, CANADA

E-mail address: `tronsgar@ualberta.ca`

DEPARTMENT OF MATHEMATICS, VANDERBILT UNIVERSITY, NASHVILLE, TN 37240, USA

E-mail address: `shaoyang.zhou@vanderbilt.edu`