# The Nonnegative Inverse Eigenvalue Problem and Johnson's Conjecture


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**Abstract**

The nonnegative inverse eigenvalue problem (NIEP) is a question in matrix theory which concerns the characterisation of spectra of entrywise nonnegative matrices. It is quite a high level question, and gives rise to a panoply of mathematical work regarding lower level details of it. In this report I will first provide to the reader an introduction to the NIEP, and then provide some existing results for specific cases of the problem. Finally, I will discuss Johnson's conjecture, which concerns the derivative of a characteristic polynomial of an $n \times n$ nonnegative matrix (a "realisable" polynomial), wherein I will provide some original work, which serves as progress towards a proof of this conjecture for the $n = 5$ case.

# 1 Introduction: The Spectrum of a Nonnegative Matrix

We denote by $M_n(\mathbb{R}^+)$ the set of all $n \times n$ matrices with nonnegative real entries. The spectrum of a matrix $A \in M_n(\mathbb{R}^+)$, denoted here as $\sigma(A) = (\lambda_1, \lambda_2, ..., \lambda_n)$ (repetitions included), or simply $\sigma$ when there is no ambiguity, is the set of eigenvalues of $A$. When such an $A$ exists for $\sigma$ we say that $A$ "realises" $\sigma$. The type of matrix of concern here is entrywise nonnegative, referred to from here on as nonnegative, and we use $A \geq 0$ as its notation. The NIEP asks the following:

Given a list of complex numbers $\sigma$, what are the necessary and sufficient conditions for it to be the spectrum of a nonnegative matrix?

**Example 1.** *Let* $A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$*, then* $\sigma = (1, -1)$*. Alternatively, we could start with letting* $\sigma = (2, 0)$*, and subsequently find a realising matrix* $A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$*. (It is this latter order that we are concerned with in the NIEP.)*

## 1.1 Necessary Conditions

This section addresses the criteria a set of values must satisfy in order for it to be realisable. Note that the satisfaction of these criteria does not imply realisability. Before listing the necessary conditions, some preliminary results must be stated.

By our theory of nonnegative matrices, we know that for any nonnegative matrix $A$, its spectral radius $\rho(A)$ is an element of its spectrum $\sigma$.

**Definition 1.** *A matrix $A$ is irreducible when it is not similar via permutation to a block upper diagonal matrix. That is to say, when there is no permutation matrix $P$, and block upper diagonal matrix $B$, with at least two blocks of size $\geq 1$, such that $P^T AP = B$.*

**Theorem 1.** *(Perron-Frobenius Theorem) If $A$ is an irreducible, nonnegative square matrix, then the following properties hold:*

- *$\rho(A) \in \sigma(A)$, where $\rho(A) = max\{|\lambda| : \lambda \in \sigma(A)\}$,*
- *alg $mult_A(\rho(A)) = 1$,*
- *There exists an entrywise positive eigenvector $v > 0$ of $A$ such that $Av = \rho(A)v$. ($v$ is called the Perron eigenvector when the sum of its entries equals 1),*
- *The only nonnegative eigenvectors of $A$ are positive multiples of its Perron eigenvector.*

**Definition 2.** *Let $A$ be a nonnegative $n \times n$ matrix, and $\sigma(A) = (\lambda_1, \lambda_2, ..., \lambda_n)$, then the $k^{th}$ moment of $\sigma$ is defined to be*

$$s_k := \lambda_1^k + \lambda_2^k + ... + \lambda_n^k. \tag{1}$$

Note that the $k^{th}$ moment of $\sigma(A)$ is equal to $tr(A^k)$. Thus, one of the necessary conditions follows naturally, which is that if $A$ is a nonnegative matrix, and $s_k$ is its $k^{th}$ moment, then

$$s_k = tr(A^k) \geq 0 \tag{2}$$

as all entries of $A^k$ must be nonnegative.
Another necessary condition was obtained by Loewy and London in 1978, and independently by Johnson in 1981, to give what is called the J-LL inequalities, which state that given a nonnegative $n \times n$ matrix $A$, and for any $m, k \in \mathbb{N}$, the following inequality holds:

$$n^{m-1} s_{km} \geq s_k^m. \tag{3}$$

A proof of this is given by Cronin, and goes as follows.
Let $A$ be an $n \times n$ nonnegative matrix, and let $A^k = D + C$ where $D$ is a diagonal matrix, $C$ has trace zero, and both are nonnegative. Then

$$\begin{aligned} s_{km} &= tr((A^k)^m) = tr((D+C)^m) \\ &\geq tr(D^m) \\ &\geq \frac{1}{n^{m-1}} tr(D)^m \\ &= \frac{1}{n^{m-1}} s_k^m. \end{aligned}$$

Another necessary condition comes from the fact that the characteristic polynomial of a real-valued matrix has real coefficients. The upshot of this is that the set of roots of the polynomial is closed under complex conjugation, thus the spectrum $\sigma$ of $A$ is closed under conjugation. Now we will go through some simple examples of the application of these necessary conditions.

**Example 2.** *Let $\sigma = (5, 3+2i, 3-2i, -6)$. Here $\sigma$ is not realisable, as the spectral radius is 6, but $6 \notin \sigma$.*

**Example 3.** *Let $\sigma = (3, 3, -2, -2, -2)$. Here $\sigma$ is not realisable, as we observe that it does not have an algebraically simple Perron eigenvalue, thus any nonnegative realisation of it must be reducible (by the Perron-Frobenius Theorem). Thus, a block diagonal matricial realisation must be possible, meaning either $\sigma_1 = (3, -2, -2, )$ or $\sigma_2 = (3, -2, -2, -2, )$ must be realisable. Neither of those cases are realisable, as the sum of the values in both cases is negative meaning any 'realizing' matrix would have negative trace.*

**Example 4.** *Let $\sigma = (\sqrt{2}, i, -i)$, then by using the J-LL inequality (3) for $n = 3, m = 2, k = 1$ , we get $3s_2 \geq s_1^2$, which implies that $0 \geq 2$, meaning that $\sigma$ cannot be realisable.*

As is made clear from the above examples, the necessary conditions pertaining to the NIEP are useful for determining when a set of numbers is not realisable. There are likely many other necessary conditions that have not been discovered yet which could prove useful in attaining results that are not reachable given our current arsenal. One slightly more complicated result that has proven to be quite useful is given by Cronin and is as follows:

**Theorem 2.** *For any $n \in \mathbb{N}$ and nonnegative $n \times n$ matrix $A$,*

$$n^2 s_3 - 3n s_1 s_2 + 2s_1^3 + \frac{n-2}{\sqrt{n-1}}(n s_2 - s_1^2)^{3/2} \geq 0. \tag{4}$$

**Example 5.** *Let $\sigma = (17, -9, 7+9i, 7-9i)$, then by (4), $\sigma$ is not realisable.*

## 1.2 Sufficient Conditions

This section addresses the conditions which if satisfied, guarantee the realisability of a given list of numbers. In order to convey this topic, I will go through some examples of particular cases.

First we will consider the $2\times2$ case. Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ be a nonnegative matrix. The characteristic polynomial of $A$ is

$$p(x) = det(xI - A) = x^2 - (a+d)x + (ad - bc) \tag{5}$$

and to find the spectrum of $A$ we find the roots of $p(x)$, which are of the form

$$x = \frac{a + b \pm \sqrt{(a+d)^2 - 4(ad - bc)}}{2} = \frac{a + b \pm \sqrt{(a-d)^2 + 4bc}}{2} \in \mathbb{R} \tag{6}$$

thus a necessary condition for a list to be the spectrum of a $2 \times 2$ matrix is that it is real. Now, let $\sigma = (\alpha, \beta)$ satisfy all of the necessary conditions seen thus far ($\alpha, \beta \in \mathbb{R}, \alpha + \beta \geq 0$), and without loss of generality let $\alpha \geq |\beta|$. If $\beta \geq 0$, then $\begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}$ is a realising matrix of $\sigma$. If $\beta < 0$ then

$$(x - \alpha)(x - \beta) = x^2 - (\alpha + \beta)x + \alpha\beta = x^2 + p_1 x + p_2 \tag{7}$$

is the characteristic polynomial of a realising matrix of $\sigma$, and since $p_1$ and $p_2$ are not positive, the companion matrix $\begin{pmatrix} 0 & 1 \\ -p_2 & -p_1 \end{pmatrix}$ is a nonnegative realisation of $\sigma$. Thus for the $2 \times 2$ case, $\sigma$ is realisable if and only if it is real, and has a nonnegative summation.

Now we will observe the $3 \times 3$ case, letting $\sigma = (\lambda_1, \lambda_2, \lambda_3)$. First we will look at when $\sigma$ consists of three real values. Given $\sigma \in \mathbb{R}$, without loss of generality, let $\lambda_1 \geq \lambda_2 \geq \lambda_3$, and $\lambda_1$ be equal to the spectral radius.

Case 1: If $\lambda_3 \geq 0$, then $\begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}$ is a nonnegative realisation of $\sigma$.

Case 2: If $\lambda_2 \geq 0 > \lambda_3$, then $\begin{pmatrix} \lambda_2 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -\lambda_1\lambda_3 & \lambda_1 + \lambda_3 \end{pmatrix}$ is a nonnegative realisation of $\sigma$.

Case 3: If $0 > \lambda_2 > \lambda_3$, then we have that

$$(x - \lambda_1)(x - \lambda_2)(x - \lambda_3) = x^3 - (\lambda_1 + \lambda_2 + \lambda_3)x^2 + (\lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_2\lambda_3)x - \lambda_1\lambda_2\lambda_3 \tag{8}$$

is the characteristic polynomial of a realising matrix of $\sigma$, and all of the non leading coefficients are clearly not positive, the companion matrix

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \lambda_1\lambda_2\lambda_3 & -(\lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_2\lambda_3) & \lambda_1 + \lambda_2 + \lambda_3 \end{pmatrix}$$

is a nonnegative realisation of $\sigma$.

Thus, a 3-set of real numbers is realisable if and only if it has a nonnegative summation.

Now to visit the case where the 3-set values are allowed to be non-real complex numbers, Loewy and London have provided a complete solution to the NIEP for $n = 3$.

**Theorem 3.** *Let $\sigma = (\lambda_1, \lambda_2, \lambda_3)$ be a list of complex numbers. Then $\sigma$ is realisable if and only if:*

- $max\{|\lambda_i| : \lambda_i \in \sigma\} \in \sigma$
- $\sigma = \bar{\sigma}$
- $s_1 \geq 0$
- $3s_2 \geq s_1^2$

In order to show a constructive solution to the $n = 3$ case of the NIEP, we will first take a look at a class of matrix called circulant matrices.

**Definition 3.** *An $n \times n$ circulant matrix is one of the form* $\begin{pmatrix} x_1 & x_2 & x_3 & \dots & x_n \\ x_n & x_1 & x_2 & \dots & x_{n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ x_2 & x_3 & x_4 & \dots & x_1 \end{pmatrix}$.

The normalized eigenvectors of a circulant matrix are of the form $v_j = \frac{1}{\sqrt{n}}(1, \omega^j, \omega^{2j}, ..., \omega^{(n-1)j})$, with $\omega = \exp(\frac{2\pi i}{n})$, and corresponding eigenvalue $\lambda_j = x_1 + x_2\omega^j + ... + x_n\omega^{(n-1)j}$, for $j = 1, ..., n$. By this property we can proceed with the following:

If a list $\sigma = (\lambda_1, \lambda_2, \bar{\lambda}_2)$ satisfies the properties of Theorem 3, then we can consider

$$\hat{\sigma} = \frac{\sigma}{|\lambda_2|} = (\lambda, e^{i\theta}, e^{-i\theta}),$$

which has the nonnegative (circulant) realisation

$$\hat{A} = \frac{1}{3}\begin{pmatrix} \lambda + \cos 2\theta & \lambda - 2\cos(\frac{\pi}{3} + \theta) & \lambda - 2\cos(\frac{\pi}{3} - \theta) \\ \lambda - 2\cos(\frac{\pi}{3} - \theta) & \lambda + \cos 2\theta & \lambda - 2\cos(\frac{\pi}{3} + \theta) \\ \lambda - 2\cos(\frac{\pi}{3} + \theta) & \lambda - 2\cos(\frac{\pi}{3} - \theta) & \lambda + \cos 2\theta \end{pmatrix},$$

giving $A = |\lambda_2|\hat{A}$ as a nonnegative realisation of $\sigma$.

## 2 Johnson's Conjecture

In this section focusing on Johnson's Conjecture we first will look at its existing results, which are provided by Cronin in his 2012 thesis, however I will provide my own proofs of these results. I will then provide work towards a proof of Johnson's Conjecture for the $n = 5$ case. First we will look at a result which is very useful for relating the coefficients of a polynomial to its roots.

**Theorem 4.** *(Newton's Identities) Let $(\lambda_1, \dots, \lambda_n)$ be a list of variables, and denote by $s_k$ the $k^{th}$ power sum $\lambda_1^k + \dots + \lambda_n^k$. Letting $e_k$ denote the elementary symmetric polynomial such that*

$$\begin{aligned} e_0 &= 1 \\ e_1 &= \sum_{i=1}^{n} \lambda_i \\ e_2 &= \sum_{1 \leq i \leq j \leq n} \lambda_i \lambda_j \\ e_n &= \lambda_1 ... \lambda_n \\ e_k &= 0, \quad for \;\; k > n, \end{aligned}$$

*we have the relation*

$$ke_k = \sum_{i=1}^{k} (-1)^{i-1} e_{k-i} s_i.$$

The statement of Johnson's Conjecture is as follows.

**Conjecture 1.** *If $f(x) = x^n + p_1x^{n-1} + ... + p_n$ is the characteristic polynomial of an $n \times n$ nonnegative matrix, then $g(x) := \frac{f'(x)}{n}$ is the characteristic polynmial of a nonnegative $(n-1) \times (n-1)$ matrix.*

It is not unanimously believed that Johnson's Conjecture holds for all $n$, but of course no one can be certain yet. The cases of $n \leq 4$ have been proven by Cronin, but we will here go through some proofs of my own of those same results.

First we will consider the simplest case where $n = 2$. Here we have

$$\begin{aligned} f(x) &= x^2 - (\lambda_1 + \lambda_2)x + \lambda_1\lambda_2 \\ \implies g(x) &= \frac{f'(x)}{2} = x - \frac{\lambda_1 + \lambda_2}{2}, \end{aligned}$$

and by an aforementioned result on the realisability of polynomials of degree 2, we have that $\frac{\lambda_1+\lambda_2}{2} \geq 0$, meaning $g(x)$ is realisable.

The case for $n = 3$ is similarly simple to prove. Letting

$$f(x) = x^3 - (\lambda_1 + \lambda_2 + \lambda_3)x^2 + (\lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_2\lambda_3)x - (\lambda_1\lambda_2\lambda_3)$$

we have

$$g(x) = \frac{f'(x)}{3} = x^2 - \frac{2(\lambda_1 + \lambda_2 + \lambda_3)}{3}x + \frac{\lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_2\lambda_3}{3}.$$

Furthermore, $\frac{2(\lambda_1+\lambda_2+\lambda_3)}{3} \geq 0$, and

$$(\lambda_1 + \lambda_2 + \lambda_3)^2 - \lambda_1\lambda_2 - \lambda_1\lambda_3 - \lambda_2\lambda_3 = \lambda_1^2 + \lambda_2^2 + \lambda_3^2 + \lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_2\lambda_3 \geq 0. \tag{9}$$

The last inequality follows by considering that $\lambda_1\lambda_2+\lambda_1\lambda_3+\lambda_2\lambda_3 \in \mathbb{R}$ and if it is negative, the first expression in (9) is clearly positive, and if it is nonnegative, the second expression in (9) is clearly nonnegative by the realisability of $f(x)$. We thus have that the roots of $g(x)$ are real and have a nonnegative summation. Thus $g(x)$ is realisable by an aforementioned result. To prove higher degree results, we will first look at the following results.

**Theorem 5.** *(Torre-Mayo) Let $p(x) = x^n + k_1x^{n-1} + k_2x^{n-2} + ... + k_n$ be the characteristic polynomial, of degree $\geq 3$, of a nonnegative matrix $A$. Then:*

- $k_1 \leq 0$
- $k_2 \leq \frac{n-1}{2n}k_1^2$
- $k_3 \leq \begin{cases} \frac{n-2}{n}(k_1k_2 + \frac{n-1}{3n}((k_1^2 - \frac{2nk_2}{n-1})^{3/2} - k_1^3)), & \text{if } \frac{(n-4)(n-1)}{2(n-2)^2}k_1^2 < k_2 \\ k_1k_2 - \frac{(n-3)(n-1)}{3(n-2)^2}k_1^3, & \text{otherwise.} \end{cases}$

The following corollary is an original result regarding the derivative of a realisable polynomial, which I hope to make use of for a proof of the $n = 5$ case of Johnson's Conjecture.

**Corollary 1.** *(Fulcher, 2021) Let $f(x) = x^n + p_1x^{n-1} + p_2x^{n-2} + ... + p_n$ be a realisable polynomial of degree $\geq 4$. Then $g(x) = \frac{f'(x)}{n}$ satisfies the criteria listed in theorem 4.*

*Proof.* Let $g(x) = x^{n-1} + k_1x^{n-2} + ... + k_{n-1}$ where $k_j = \frac{(n-j)p_j}{n}$. Letting $s_j$ refer to the $j^{th}$ moment of the roots of $f(x)$, and using the Newton identities

$$k_1 = \frac{(n-1)p_1}{n} = -\frac{(n-1)s_1}{n}$$

$$k_2 = \frac{(n-2)p_2}{n} = \frac{(n-2)(s_1^2 - s_2)}{2n}$$
$$k_3 = \frac{(n-3)p_3}{n} = -\frac{(n-3)(s_1^3 - 3s_1s_2 + 2s_3)}{6n},$$
we will see that the three criteria hold. Clearly, as $s_1 \geq 0$, $k_1 \leq 0$. For the second criterion,
$$\frac{n-2}{2(n-1)}k_1^2 = \frac{(n-1)(n-2)s_1^2}{2n^2},$$
then subtracting $k_2$ gives
$$\begin{aligned}\frac{n-2}{2(n-1)}k_1^2 - k_2 &= \frac{(n-1)(n-2)s_1^2}{2n^2} - \frac{(n-2)(s_1^2 - s_2)}{2n} \\ &= \frac{((n-1)(n-2) - n(n-2))s_1^2 + n(n-2)s_2}{2n^2} \\ &= \frac{n-2}{2n^2}(ns_2 - s_1^2) \geq 0\end{aligned}$$
by the JLL inequalities (3). Thus the second criterion holds. For the third criterion, we will first look at the case for $\frac{(n-5)(n-2)}{2(n-3)^2}k_1^2 < k_2$. Considering $n$ as the degree of $f(x)$, and $n-1$ as the degree of $g(x)$, we here convert the supposed upper bound of $k_3$ into terms of the moments of $f(x)$:
$$\begin{aligned}&\frac{n-3}{n-1}\left(k_1k_2 + \frac{n-2}{3(n-1)}\left(\left(k_1^2 - \frac{2(n-1)k_2}{n-2}\right)^{3/2} - k_1^3\right)\right) \\ &= \frac{n-3}{n-1}\left(\frac{(n-1)(n-2)(s_1s_2 - s_1^3)}{2n^2} + \frac{n-2}{3(n-1)}\left(\left(\frac{(n-1)^2s_1^2}{n^2} - \frac{(n-1)(s_1^2 - s_2)}{n}\right)^{3/2} + \frac{(n-1)^3s_1^3}{n^3}\right)\right) \\ &= \frac{n-3}{n-1}\left(\left(\frac{(n-1)^2(n-2)}{3n^3} - \frac{(n-1)(n-2)}{2n^2}\right)s_1^3 + \frac{(n-1)(n-2)s_1s_2}{2n^2} + \frac{n-2}{3(n-1)}\left(\frac{n-1}{n^2}\left(ns_2 - s_1^2\right)\right)^{3/2}\right) \\ &= (n-3)\left(\frac{(-n-2)(n-2)s_1^3}{6n^3} + \frac{(n-2)s_1s_2}{2n^2} + \frac{n-2}{3n^3\sqrt{n-1}}(ns_2 - s_1^2)^{3/2}\right).\end{aligned}$$
Then subtracting $k_3$, we get
$$\begin{aligned}&(n-3)\left(\frac{(-n-2)(n-2)s_1^3}{6n^3} + \frac{(n-2)s_1s_2}{2n^2} + \frac{n-2}{3n^3\sqrt{n-1}}(ns_2 - s_1^2)^{3/2}\right) + \frac{(n-3)(s_1^3 - 3s_1s_2 + 2s_3)}{6n} \\ &= (n-3)\left(\frac{(n^2 - (n+2)(n-2))s_1^3}{6n^3} + \frac{(3(n-2) - 3n)s_1s_2}{6n^2} + \frac{s_3}{3n} + \frac{n-2}{3n^3\sqrt{n-1}}(ns_2 - s_1^2)^{3/2}\right) \\ &= (n-3)\left(\frac{2s_1^3}{3n^3} - \frac{s_1s_2}{n^2} + \frac{s_3}{3n} + \frac{n-2}{3n^3\sqrt{n-1}}(ns_2 - s_1^2)^{3/2}\right) \\ &= \frac{n-3}{3n^3}\left(2s_1^3 - 3ns_1s_2 + n^2s_3 + \frac{n-2}{\sqrt{n-1}}(ns_2 - s_1^2)^{3/2}\right) \geq 0\end{aligned}$$
by theorem 2. Now we look at the case for $k_2 \leq \frac{(n-5)(n-2)}{2(n-3)^2}k_1^2$. First note that this inequality gives
$$\begin{aligned}&\frac{(n-2)(s_1^2 - s_2)}{2n} \leq \frac{(n-5)(n-2)(n-1)^2s_1^2}{2n^2(n-3)^2} \\ \implies &\frac{(n-2)s_2}{2n} \geq -\frac{(n-5)(n-2)(n-1)^2s_1^2}{2n^2(n-3)^2} + \frac{(n-2)s_1^2}{2n} \\ &= \frac{(n(n-3)^2(n-2) - (n-5)(n-2)(n-1)^2)s_1^2}{2n^2(n-3)^2} \\ \implies &s_2 \geq \frac{(n(n-3)^2 - (n-5)(n-1)^2)s_1^2}{n(n-3)^2} = \frac{n^2 - 2n + 5}{n(n-3)^2}s_1^2.\end{aligned}$$

Here the supposed upper bound of $k_3$ is

$$\begin{aligned}
k_1k_2 - \frac{(n-4)(n-2)}{3(n-3)^2}k_1^3 &= \frac{(n-1)(n-2)(s_1s_2 - s_1^3)}{2n^2} + \frac{(n-4)(n-2)(n-1)^3 s_1^3}{3(n-3)^2 n^3} \\
&= \frac{(n-1)(n-2)s_1s_2}{2n^2} + \frac{(-3n(n-1)(n-2)(n-3)^2 + 2(n-1)^3(n-2)(n-4))s_1^3}{6n^3(n-3)^2} \\
&= \frac{(n-1)(n-2)s_1s_2}{2n^2} - \frac{(n-1)(n-2)(n^3-6n^2+9n+8)s_1^3}{6(n-3)^2n^3},
\end{aligned}$$

then subtracting $k_3$ gives

$$\begin{aligned}
&\left(\frac{n-1}{6n} - \frac{(n-1)(n-2)(n^3-6n^2+9n+8)}{6(n-3)^2n^3}\right)s_1^3 + \left(\frac{(n-1)(n-2)}{2n^2} - \frac{3(n-3)}{6n}\right)s_1s_2 + \frac{(n-3)s_3}{3} \\
&= \frac{(n-1)(n^3-6n^2+5n+8)s_1^3}{6(n-3)^2n^3} + \frac{3s_1s_2}{2} + \frac{(n-3)s_3}{3} \\
&\geq \left(\frac{(n-1)(n^3-6n^2+5n+8)}{6(n-3)^2n^3} + \frac{3n^2-6n+15}{n(n-3)^2}\right)s_1^3 \\
&= \frac{(19n^4 - 43n^3 + 11n^2 + 93n - 8)s_1^3}{6(n-3)^2n^3} \geq 0 \qquad \forall n \geq 4.
\end{aligned}$$

□

**Corollary 2.** *(Torre-Mayo et al.) Let $\sigma = (\lambda_1, \lambda_2, \lambda_3)$ be a list of complex numbers such that $\sigma = \bar{\sigma}$ and let $p(x) = (x - _1)(x - \lambda_2)(x - \lambda_3) = x^3 + k_1x^2 + k_3x + k_4$. Then the properties listed in Theorem 3 are equivalent to those listed in Theorem 4 for $\sigma$ and $p(x)$ respectively.*

Considering the above results, the $n = 4$ case of Johnson's Conjecture follows easily:

Let $f(x)$ be the characteristic polynomial of a nonnegative $4 \times 4$ matrix, and $g(x) = \frac{f'(x)}{4}$. Since the coefficients of $f(x)$ are real, the coefficients of $g(x)$ are real, so its list of roots is closed under complex conjugation. Applying Corollary 1 and Corollary 2, the result that $g(x)$ is realisable follows immediately.

It is clear that this current strategy for proving Johnson's Conjecture goes as follows:

Let $f(x) = x^n + p_1x^{n-1} + ... + p_n$ be a realisable polynomial, and $g(x) = \frac{f'(x)}{n} = x^{n-1} + q_1x^{n-2} + ... + q_{n-1}$. Using the information inherited from $f(x)$, check if $g(x)$ satisfies the necessary and sufficient conditions for realisability.

Thus, using this strategy requires the NIEP to be solved for the $n - 1$ case of the above statement. Complete solutions to the NIEP only exist up to the $n = 4$ case, so currently this strategy is only possible up to the $n = 5$ case of Johnson's Conjecture. Using this strategy, I will attempt to make use of the complete solution of the $n = 4$ case of the NIEP provided by Torre-Mayo et al in 2007. This solution is given in terms of various different cases determined by the ranges of values that the coefficients of the polynomial in question fall in. In order to proceed further, the following definition is needed.

**Definition 4.** *An EBL matrix is a nonnegative matrix of the form*

$$\begin{pmatrix} a_{11} & 1 & 0 & \dots & 0 \\ a_{21} & a_{22} & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ \vdots & \vdots & & \ddots & 1 \\ a_{n1} & a_{n2} & \dots & \dots & a_{nn} \end{pmatrix}.$$

**Theorem 6.** *(Torre-Mayo et al.) Every realisable polynomial of degree 4 has an EBL realisation.*

EBL is an abreviation of a Spanish phrase that translates to Lineal Basic Structure. The inspiration for this name comes from the type of weighted digraph of which it is the adjacency matrix, and it is called an EBL graph.

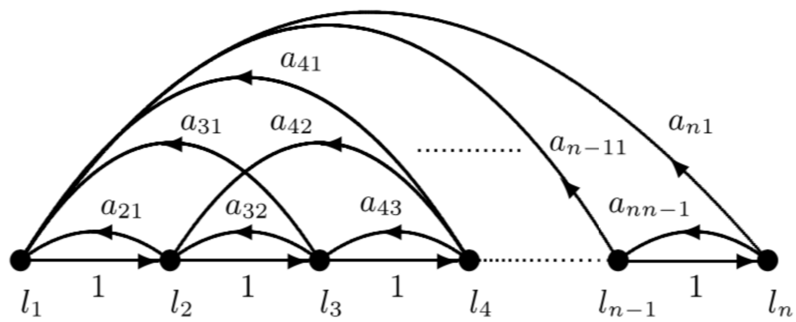


Figure 1 - The weighted EBL digraph

The above weighted digraph is in the form of an EBL graph with $n$ vertices. Note that the loops have not been drawn in, but rather have been represented by $l_i$ for the loop at the $i^{th}$ vertex. The structure of the EBL graph is such that each cycle from vertex $v_i$ consists entirely of edges with weight one, as well as one edge of weight $a_{ji}$ for some $j \in \{1, ..., n\}$. An upshot of this is manifest in the Coefficient Theorem for weighted digraphs given below.

**Theorem 7.** *Let $G$ be a weighted digraph, $A$ its adjacency matrix, and let*

$$|xI - A| = x^n + k_1 x^{n-1} + ... + k_n.$$

*Then, for each $i = 1, ...n$,*

$$k_i = \sum_{L \in \mathscr{L}_i} (-1)^{p(L)} \Pi(L)$$

*where $\mathscr{L}_i$ is the set subgraphs of $G$ with $i$ vertices, comprised of disjoint cycles; $p(L)$ denotes the number of cycles in $L$; $\Pi(L)$ denotes the product of the weights of all arcs belonging to $L$.*

Let $r \geq 1$, and $c_{i_1 i_2 ... i_r}$ denote the product of the weights of the cycle $v_{i_1} v_{i_r} ... v_{i_r} v_{i_1}$ in a graph $G$. Considering the structure of the EBL graph, its adjacency matrix

$$\begin{pmatrix} a_{11} & 1 & 0 & \dots & 0 \\ a_{21} & a_{22} & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ \vdots & \vdots & & \ddots & 1 \\ a_{n1} & a_{n2} & \dots & \dots & a_{nn} \end{pmatrix}$$

is the same as

$$\begin{pmatrix} l_1 & 1 & 0 & \dots & \dots & 0 \\ c_{12} & l_2 & \ddots & \ddots & & \vdots \\ c_{123} & c_{23} & \ddots & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \ddots & 0 \\ \vdots & \vdots & & \ddots & \ddots & 1 \\ c_{1...n} & c_{2...n} & \dots & \dots & c_{n-1n} & l_n \end{pmatrix}.$$

Torre-Mayo et al. use the Coefficient Theorem extensively for analysing and adjusting values in the weighted digraph to identify a variety of useful properties of the coefficients of characteristic polynomials. We will not be looking explicitly at those uses in this report, but I am hoping to use this kind of technique to prove an inequality that features later on in this report.
One of the useful properties of EBL matrices that I'm leveraging for my strategy for proving Johnson's Conjecture for $n = 5$ is the following.

Let $A = \begin{pmatrix} a_{11} & 1 & 0 & 0 \\ a_{21} & a_{22} & 1 & 0 \\ a_{31} & a_{32} & a_{33} & 1 \\ a_{41} & a_{42} & a_{43} & a_{44} \end{pmatrix}$, then let $p(x) = |xI - A|$. If we then let $B = \begin{pmatrix} a_{11} & 1 & 0 & 0 \\ a_{21} & a_{22} & 1 & 0 \\ a_{31} & a_{32} & a_{33} & 1 \\ a_{41}+b & a_{42} & a_{43} & a_{44} \end{pmatrix}$, then $|xI - B| = p(x) - b$.

The following theorem provided by Torre-Mayo et al. is broken into various cases in their 2007 paper. It provides a complete solution to the NIEP for the $n = 4$ case, and I am hoping to make use of it for proving Johnson's Conjecture for the $n = 5$ case.

**Theorem 8.** *(Torre-Mayo et al.) Let $k_1, k_2, k_3$ satisfy the conditions as stated in Theorem 4. There is then a matricial realisation of the polynomial $x^4 + k_1x^3 + k_2x^2 + k_3x + k_4^{max}$.*

The value $k_4^{max}$ refers to the maximum value that the constant coefficient can take, given the other three coefficients, in order for the polynomial to be realisable. Considering the property of EBL matrices given prior to Theorem 6, this theorem can be restated as follows:

Let $k_1, k_2, k_3$ satisfy the conditions as stated in Theorem 4. There then exists a $k_4^{max}$ such that $x^4 + k_1x^3 + k_2x^2 + k_3x + k_4$ is realisable for all $k_4 \leq k_4^{max}$.

The value $k_4^{max}$ is computed as a function of $k_1, k_2, k_3$, and the particular function that defines it depends on the range of values that $k_1, k_2, k_3$ occur in, so we will denote it as $k_4^{max}(k_1, k_2, k_3)$. Considering Corollary 1, proving Johnson's conjecture for $n = 5$ is then equivalent to proving, for realisable $f(x) = x^5 + p_1x^4 + p_2x^3 + p_3x^2 + p_4x + p_5$ with $g(x) = \frac{f'(x)}{5} = x^4 + k_1x^3 + k_2x^3 + k_3x + k_4$, that $k_4 \leq k_4^{max}(k_1, k_2, k_3)$. In the case that this last inequality holds, $g(x)$ has an EBL realisation with $k_4^{max} - k_4$ as its $(4, 1)$ entry.

Now we will look at a simplified situation.

Let $f(x) = x^5 + p_1x^4 + p_2x^3 + p_3x^2 + p_4x + p_5$ be a realisable polynomial and $s_k$ refer to the $k^{th}$ moment of $f(x)$. If $5s_2 = s_1^2$, then the polynomial

$$x^4 + \frac{4p_1}{5}x^3 + \frac{3p_2}{5}x^2 + \frac{2p_3}{5}x + k_4^{max}$$

is a realisable polynomial, where $k_4^{max} = \frac{k_1k_3}{4} - 3\left(\frac{k_1}{4}\right)^4$. Letting $k_j = \frac{(5-j)p_j}{5}$, an EBL realisation of this polynomial is

$$\begin{pmatrix} l & 1 & 0 & 0 \\ d & l & 1 & 0 \\ t & 0 & l & 1 \\ 0 & 0 & d & l \end{pmatrix} \quad \text{where} \quad \begin{cases} l = -\frac{k_1}{4} \\ d = \frac{6l^2 - k_2}{2} \\ t = -4l^3 + 4dl - k_3. \end{cases}$$

Thus it here suffices to show the nonnegativity of

$$\begin{aligned}
k_4^{max} - k_4 =& \frac{k_1 k_3}{4} - 3\left(\frac{k_1}{4}\right)^4 - k_4 \\
=& \frac{2p_1p_3}{25} - 3\left(\frac{p_1}{5}\right)^4 - \frac{p_4}{5} \\
=& \frac{s_1^4}{5000} + \frac{s_1^2 s_2}{100} - \frac{s_1 s_3}{25} - \frac{s_2^2}{40} + \frac{s_4}{20} \\
=& \frac{3s_1^4}{2500} - \frac{s_1 s_3}{25} + \frac{s_4}{20},
\end{aligned} \tag{10}$$

which would then give

$$\begin{pmatrix} l & 1 & 0 & 0 \\ d & l & 1 & 0 \\ t & 0 & l & 1 \\ k_4^{max} - k_4 & 0 & d & l \end{pmatrix}$$

as a nonnegative realisation of $\frac{f'(x)}{5}$.

I have not managed to show the nonnegativity of (10), but have found a way to solve this case by analysing the $5 \times 5$ nonnegative matrix that realises $f$, which is given below.

Let $f(x) = x^5 + p_1x^4 + p_2x^3 + p_3x^2 + p_4x + p_5$ be a realisable polynomial, with $s_k$ referring to its $k^{th}$ moment. With the restriction $5s_2 = s_1^2$, we get a matricial realisation of $f(x)$ with the form

$$A = \begin{pmatrix} h & m_{12} & m_{13} & m_{14} & m_{15} \\ m_{21} & h & m_{23} & m_{24} & m_{25} \\ m_{31} & m_{32} & h & m_{34} & m_{35} \\ m_{41} & m_{42} & m_{43} & h & m_{45} \\ m_{51} & m_{52} & m_{53} & m_{54} & h \end{pmatrix}.$$

First note that if $A = D + C$ where $D, C \geq 0$, $D$ is a diagonal matrix, and $tr(C) = 0$, it can be shown that $tr(C^2) = 0$, meaning that $m_{ij}m_{ji} = 0$ for $1 \leq i < j \leq 5$. Using Newton's identities, we then get the following equalities:

$$\begin{aligned}
p_1 &= -5h \\
p_2 &= 10h^2 \\
p_3 &= -10h^3 - \tau \\
p_4 &= 5h^4 + 2h\tau - \rho,
\end{aligned}$$

where $\tau$ is the sum of the 3-cycles (terms of the general form $m_{jk}m_{kl}m_{lj}$), and $\rho$ is the sum of the 4-cycles. Letting $g(x) = \frac{f'(x)}{5} = x^4 + k_1x^3 + k_2x^2 + k_3x + k_4$, we get the equalities:

$$\begin{aligned}
k_1 &= -4h \\
k_2 &= 6h^2 \\
k_3 &= -4h^3 - \frac{2}{5}\tau \\
k_4 &= h^4 + \frac{2}{5}h\tau - \frac{1}{5}\rho.
\end{aligned}$$

Since $g(x)$ satisfies the Torre-Mayo inequalities, we have that the polynomial $h(x) = x^4 + k_1x^3 + k_2x^2 + k_3x + k_4^{max}$ has an EBL realisation where

$$k_4^{max} = \frac{k_1k_3}{4} - 3\left(\frac{k_1}{4}\right)^4 = h^4 + \frac{2}{5}h\tau,$$

so consider now

$$k_4^{max} - k_4 = \frac{1}{5}\rho \geq 0,$$

which means that $g(x)$ has a matricial realisation the same as the EBL realisation for $h(x)$, but with $\frac{1}{5}\rho$ added to its $(4,1)$ entry.

The following result has already been discovered, but I give an original proof of it here.

**Remark 1.** *Theorem 5 implies Theorem 2 for $n \geq 3$.*

*Proof.* Let $p(x) = x^n + k_1x^{n-1} + k_2x^{n-2} + ... + k_n$ be the characteristic polynomial of an $n \times n$ nonnegative matrix, so the coefficients satisfy the criteria of theorem 4. Without assuming theorem 2, we will consider the 2 cases given in the criteria of theorem 4.
By Newton's inequalities, we here have

$$\begin{aligned} k_1 &= -s_1 \\ k_2 &= \frac{s_1^2 - s_2}{2} \\ k_3 &= \frac{3s_1s_2 - s_1^3 - 2s_3}{6}. \end{aligned}$$

Case 1:

$$\begin{aligned} k_2 &> \frac{(n-4)(n-1)}{2(n-2)^2}k_1^2, \qquad \text{and} \\ k_3 &\leq \frac{n-2}{n}\left(k_1k_2 + \frac{n-1}{3n}\left(\left(k_1^2 - \frac{2nk_2}{n-1}\right)^{3/2} - k_1^3\right)\right). \end{aligned}$$

$$\begin{aligned} 0 &\leq \frac{n-2}{n}\left(k_1k_2 + \frac{n-1}{3n}\left(\left(k_1^2 - \frac{2nk_2}{n-1}\right)^{3/2} - k_1^3\right)\right) - k_3 \\ &= \frac{n-2}{n}\left(\frac{s_1s_2 - s_1^3}{2} + \frac{n-1}{3n}\left(\left(s_1^2 - \frac{n(s_1^2 - s_2)}{n-1}\right)^{3/2} + s_1^3\right)\right) + \frac{s_1^3 - 3s_1s_2 + 2s_3}{6} \\ &= \frac{n-2}{n}\left(\frac{ns_3}{3(n-2)} + \left(\frac{1}{2} - \frac{n}{2(n-2)}\right)s_1s_2 + \left(\frac{n-1}{3n} + \frac{n}{6(n-2)} - \frac{1}{2}\right)s_1^3 + \frac{n-1}{3n}\left(\left(1 - \frac{n}{n-1}\right)s_1^2 + \frac{n}{n-1}s_2\right)^{3/2}\right) \\ &= \frac{n-2}{n}\left(\frac{ns_3}{3(n-2)} - \frac{s_1s_2}{n-2} + \frac{2s_1^3}{3n(n-2)} + \frac{n-1}{3n}\left(\frac{ns_2}{n-1} - \frac{s_1^2}{n-1}\right)^{3/2}\right) \\ &= \frac{1}{3n^2}\left(n^2s_3 + 2s_1^3 - 3ns_1s_2 + \frac{n-2}{\sqrt{n-1}}(ns_2 - s_1^2)^{3/2}\right). \end{aligned}$$

Case 2:

$$\begin{aligned} k_2 &\leq \frac{(n-4)(n-1)}{2(n-2)^2}k_1^2, \qquad \text{and} \\ k_3 &\leq k_1k_2 - \frac{(n-1)(n-3)}{3(n-2)^2}k_1^3. \end{aligned}$$

We will proceed by subtracting the maximum for $k_3$ in Case 2 from the maximum for $k_3$ in Case 1.

$$\begin{aligned} &\frac{n-2}{n}\left(k_1k_2 + \frac{n-1}{3n}\left(\left(k_1^2 - \frac{2nk_2}{n-1}\right)^{3/2} - k_1^3\right)\right) - k_1k_2 + \frac{(n-1)(n-3)}{3(n-2)^2}k_1^3 \\ &= \left(\frac{n-2}{n} - 1\right)k_1k_2 + \left(\frac{(n-1)(n-3)}{3(n-2)^2} - \frac{(n-1)(n-2)}{3n^2}\right)k_1^3 + \frac{(n-1)(n-2)}{3n^2}\left(k_1^2 - \frac{2nk_2}{n-1}\right)^{3/2} \qquad (11) \\ &= -\frac{2}{n}k_1k_2 + \left(\frac{3n^3 - 15n^2 + 20n - 8}{3n^2(n-2)^2}\right)k_1^3 + \frac{(n-1)(n-2)}{3n^2}\left(k_1^2 - \frac{2nk_2}{n-1}\right)^{3/2}. \end{aligned}$$

Considering the above as a function of $k_1, k_2$, and $n$,

$$\Omega(k_1, k_2, n) := -\frac{2}{n}k_1k_2 + \left(\frac{3n^3 - 15n^2 + 20n - 8}{3n^2(n-2)^2}\right)k_1^3 + \frac{(n-1)(n-2)}{3n^2}\left(k_1^2 - \frac{2nk_2}{n-1}\right)^{3/2}. \quad (12)$$

Note that $k_1^2 \geq \frac{2nk_2}{n-1}$, so $\Omega(k_1, k_2, n)$ is always real. Setting $\Omega(k_1, k_2, n)$ equal to zero and solving for $k_2$ in terms of $n$ and $k_1$ gives the roots

$$k_2^{(1)} := \frac{k_1^2(n^3 - 5n^2 + 7n - 3)}{2(n-2)^2 n}, \qquad k_2^{(2)} := \frac{k_1^2(n^2 - 5n + 4)}{2(n-2)^2} = k_2^{max}.$$

Since

$$k_2 \leq \frac{(n-4)(n-1)}{2(n-2)^2}k_1^2 = \frac{k_1^2(n^2 - 5n + 4)}{2(n-2)^2} = k_2^{(2)} < k_2^{(1)}$$

for $n \geq 3$, we know that (12) has viable roots only when $k_2^{(2)} = k_2^{max}$. Now we can just check what happens when $k_2 < k_2^{max}$, so letting $k_2 = k_2^{max} - 1$ we get

$$\Omega(k_1, k_2^{max}-1, n) = \frac{2\sqrt{2}(n-2)(2k_1^2 n + n^3 - 2k_1^2 - 4n^2 + 4n)\sqrt{\frac{2k_1^2 n + n^3 - 2k_1^2 - 4n^2 + 4n}{(n-2)^2(n-1)}} + 6\left(n^3 - 4n^2 + \left(\frac{4k_1^2}{3} + 4\right)n - \frac{4k_1^2}{3}\right)k_1}{3n^2(n-2)^2},$$

which has no real roots for $n \geq 3$, so we can just input some values from our ranges to see if it is always positive or negative, since it is continuous everywhere but the singularities. Choosing $k_1 = -1$ and $n = 3$, we get $\Omega(-1, k_2^{max} - 1, 3) = 0.1126... > 0$, meaning that $\Omega(k_1, k_2, n)$, and thus (11), is always nonnegative given our ranges of values. Therefore,

$$\begin{aligned}
&\frac{1}{3n^2}\left(n^2 s_3 + 2s_1^3 - 3ns_1s_2 + \frac{n-2}{\sqrt{n-1}}(ns_2 - s_1^2)^{3/2}\right) \\
&= \frac{n-2}{n}\left(k_1k_2 + \frac{n-1}{3n}\left(\left(k_1^2 - \frac{2nk_2}{n-1}\right)^{3/2} - k_1^3\right)\right) - k_3 \\
&\geq \frac{n-2}{n}\left(k_1k_2 + \frac{n-1}{3n}\left(\left(k_1^2 - \frac{2nk_2}{n-1}\right)^{3/2} - k_1^3\right)\right) - k_1k_2 + \frac{(n-1)(n-3)}{3(n-2)^2}k_1^3 \\
&\geq 0.
\end{aligned}$$

$\square$

# References

[1] R. Loewy, D. London, A note on the Inverse Problem for non-negative Matrices, *Linear and Multilinear Algebra*, **6**, (1978), 83-90.

[2] A. Cronin, T.J. Laffey, An Inequality for the Spectra of Nonnegative Matrices, *Linear Algebra and Its Applications*, **436**, (2012), 3225-3238.

[3] Torre-Mayo et al., The nonnegative inverse eigenvalue problem from the coefficients of the characteristic polynomial. EBL digraphs, *Linear Algebra and Its Applications*, **426**, (2007), 729–773.