The NIEP: Some Results on Johnson's Conjecture

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Overview

Some Terminology

1. A matrix $A = [a_{ij}]_{1 \leq i,j \leq n} \in M_n(\mathbb{R})$ is nonnegative if $a_{ij} \geq 0$ for each $i, j \in \{1, ..., n\}$.
2. The spectrum of A, $\sigma(A)$, is the list of roots (including repetitions) of the polynomial $p(x) = det(xI - A)$ (characteristic polynomial), also known as the list of eigenvalues of A.
3. A list $\sigma = (\lambda_1, ..., \lambda_n)$ of complex numbers is realisable if there exists a nonnegative matrix of which it is the spectrum.
4. For $\sigma = (\lambda_1, ..., \lambda_n)$, $s_k = \lambda_1^k + ... + \lambda_n^k$ is the k^{th} moment of σ.
5. When we say "moment" or "realisability" of a polynomial, we are using these terms as above, but using the roots of the polynomial as the list σ.

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Theorem 1 (Newton's Identities)

Let $\sigma = (x_1, x_2, \ldots, x_n)$ be a list of variables, and let e_k be its k^{th} symmetric polynomial. Then the following holds:

$$ke_k = \sum_{i=1}^{k} (-1)^{i-1} e_{k-i} s_i$$

where s_k denotes the k^{th} moment of σ.

Example

- $e_1 = s_1$
- $2e_2 = s_1^2 - s_2$
- $3e_3 = \frac{1}{2}s_1^3 - \frac{3}{2}s_1 s_2 + s_3$

Necessary and Sufficient Conditions

The NIEP specifically asks for necessary and sufficient conditions for the realisability of a finite list of complex numbers.

General Cases

The NIEP has only been fully solved for lists of length 4 and less.

Special Cases

Cases for longer lists which have certain specifications have been solved. An example of this is a list of length 5 that sums to zero, which was solved by Laffey and Meehan in 1999.

Necessary Conditions

Let $\sigma = (\lambda_1, ..., \lambda_n)$ be a list of n complex numbers. If σ is realisable, then the following properties hold.

1. $\lambda_1 + ... + \lambda_n \geq 0$.
2. σ is closed under complex conjugation.
3. For any $k, m \in \mathbb{N}$, $n^{m-1} s_{km} \geq s_k^m$ (the JLL inequalities, 1978-1981).
4. $n^2 s_3 + 2 s_1^3 - 3 n s_1 s_2 + \frac{n-2}{\sqrt{n-1}}(n s_2 - s_1^2)^{3/2} \geq 0$ (Cronin, 2012).

There are possibly many more necessary conditions for this which haven't been discovered yet.

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Sufficient Conditions

Sufficient conditions are treated on a case by case basis, as they vary depending on the length of the list in question (for example, realisable lists of length 3 can contain non-real entries, while realisable lists of length 2 cannot).

Theorem 2 (Loewy and London, 1978)

For a list of complex numbers $\sigma = (\lambda_1, \lambda_2, \lambda_3)$, σ is realisable if and only if:

- $max\{|\lambda_k| : \lambda_k \in \sigma\} \in \sigma$.
- $\sigma = \bar{\sigma}$.
- $s_1 \geq 0$.
- $3s_2 \geq s_1^2$.

Johnson's Conjecture

Given that a list $(\lambda_1, \ldots, \lambda_n)$ of complex numbers is realisable, with characteristic polynomial $p(x) = \prod_{i=1}^{n}(x - \lambda_i)$, the polynomial $q(x) = \frac{p'(x)}{n}$ is the characteristic polynomial of an $(n-1) \times (n-1)$ nonnegative matrix.

- For $n \leq 4$, Johnson's Conjecture has been proven in the affirmative by Cronin.
- It has also been proven in the affirmative by Cronin for the $n = 5, 6$ with zero sum case.
- It is not unanimously believed that Johnson's Conjecture holds for all n.

I will here present some work towards a proof of Johnson's Conjecture for $n = 5$, for which some results by Torre-Mayo et al are needed.

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Theorem 3 (Torre-Mayo et al., 2007)

Let $p(x) = x^n + k_1 x^{n-1} + \ldots + k_n$ be the characteristic polynomial of a nonnegative matrix with $deg(p(x)) \geq 3$. The following properties then hold:

- $k_1 \leq 0$
- $k_2 \leq \frac{n-1}{2n} k_1^2$
- $k_3 \leq \begin{cases} \frac{n-2}{n}(k_1 k_2 + \frac{n-1}{3n}((k_1^2 - \frac{2nk_2}{n-1})^{3/2} - k_1^3)), & \text{if } \frac{(n-4)(n-1)}{2(n-2)^2} k_1^2 < k_2 \\ k_1 k_2 - \frac{(n-3)(n-1)}{3(n-2)^2} k_1^3, & \text{otherwise.} \end{cases}$

Corollary 1 (Fulcher, 2021)

If $p(x)$ is a realisable polynomial of degree $n \geq 4$, then $q(x) = \frac{p'(x)}{n}$ satisfies the conditions of Theorem (3).

EBL Matrices

An EBL matrix is a lower Hessenberg matrix with 1 for each of its superdiagonal positions, i.e. of the form

$$\begin{pmatrix} a_{11} & 1 & 0 & \dots & 0 \\ a_{21} & a_{22} & 1 & \ddots & 0 \\ \vdots & \vdots & \ddots & \ddots & 0 \\ \vdots & \vdots & & \ddots & 1 \\ a_{n1} & a_{n2} & \dots & \dots & a_{nn} \end{pmatrix}.$$

Theorem 4 (Torre-Mayo et al., 2007)

Every realisable polynomial of degree 4 has an EBL realisation.

A Property of EBL Matrices

Let $A = \begin{pmatrix} a_{11} & 1 & 0 & 0 \\ a_{21} & a_{22} & 1 & 0 \\ a_{31} & a_{32} & a_{33} & 1 \\ a_{41} & a_{42} & a_{43} & a_{44} \end{pmatrix}$, and let $p(x) = |xI - A|$, and let

$B = \begin{pmatrix} a_{11} & 1 & 0 & 0 \\ a_{21} & a_{22} & 1 & 0 \\ a_{31} & a_{32} & a_{33} & 1 \\ a_{41} + b & a_{42} & a_{43} & a_{44} \end{pmatrix}$. Then $|xI - B| = p(x) - b$.

Let

$$f(x) := x^5 + p_1x^4 + p_2x^3 + p_3x^2 + p_4x + p_5$$

be a realisable polynomial, and

$$g(x) := \frac{f'(x)}{5} = x^4 + k_1x^3 + k_2x^2 + k_3x + k_4$$

.

Letting s_k denote the k^{th} moment of $f(x)$, using Newton's Identities we have that

$$\begin{aligned}
k_1 &= \frac{4p_1}{5} = -\frac{4s_1}{5} \\
k_2 &= \frac{3p_2}{5} = \frac{3(s_1^2 - s_2)}{10} \\
k_3 &= \frac{2p_3}{5} = \frac{3s_1s_2 - s_1^3 - 2s_3}{15} \\
k_4 &= \frac{p_4}{5} = \frac{s_1^4 - 6s_1^2s_2 + 8s_1s_3 + 3s_2^2 - 6s_4}{120}
\end{aligned}$$

.

Theorem 5 (Torre-Mayo et al., 2007)

Let $g(x) = x^4 + k_1x^3 + k_2x^2 + k_3x + k_4$ be a polynomial which satisfies the necessary conditions of Theorem (3). Then there is an EBL realisation of

$$x^4 + k_1x^3 + k_2x^2 + k_3x + k_4^{max} \tag{1}$$

where k_4^{max} is a function of k_1, k_2, k_3.

An EBL matricial realisation of (1) will, in general terms, be of the form

$$\begin{pmatrix} a_{11} & 1 & 0 & 0 \\ a_{21} & a_{22} & 1 & 0 \\ a_{31} & a_{32} & a_{33} & 1 \\ 0 & a_{42} & a_{43} & a_{44} \end{pmatrix},$$

and thus, if $k_4 = \frac{p_4}{5} = \frac{s_1^4 - 6s_1^2 s_2 + 8s_1 s_3 + 3s_2^2 - 6s_4}{120} \leq k_4^{max} = m(k_1, k_2, k_3)$ for some function m,

then $\begin{pmatrix} a_{11} & 1 & 0 & 0 \\ a_{21} & a_{22} & 1 & 0 \\ a_{31} & a_{32} & a_{33} & 1 \\ k_4^{max} - k_4 & a_{42} & a_{43} & a_{44} \end{pmatrix}$ is a nonnegative matrix that realises $g(x) = \frac{f'(x)}{5}$.

A Simplified Case (Fulcher, 2021)

Let $f(x) = x^5 + p_1x^4 + p_2x^3 + p_3x^2 + p_4x + p_5$ be a realisable polynomial. With the restriction $5s_2 = s_1^2$, we get a matricial realisation of $f(x)$ with the form

$$A = \begin{pmatrix} h & m_{12} & m_{13} & m_{14} & m_{15} \\ m_{21} & h & m_{23} & m_{24} & m_{25} \\ m_{31} & m_{32} & h & m_{34} & m_{35} \\ m_{41} & m_{42} & m_{43} & h & m_{45} \\ m_{51} & m_{52} & m_{53} & m_{54} & h \end{pmatrix}.$$

Note that if $A = D + C$ where $D, C \geq 0$, D is a diagonal matrix, and $tr(C) = 0$, it can be shown that $tr(C^2) = 0$, meaning that $m_{ij}m_{ji} = 0$ for $1 \leq i < j \leq 5$.

Simplified Case ctd.

Letting $g(x) = \frac{f'(x)}{5} = x^4 + k_1x^3 + k_2x^2 + k_3x + k_4$, we get the equalities:

$$\begin{aligned} k_1 &= -4h \\ k_2 &= 6h^2 \\ k_3 &= -4h^3 - \frac{2}{5}\tau \\ k_4 &= h^4 + \frac{2}{5}h\tau - \frac{1}{5}\rho. \end{aligned}$$

Since $g(x)$ satisfies the Torre-Mayo inequalities, we have that the polynomial $h(x) = x^4 + k_1x^3 + k_2x^2 + k_3x + k_4^{max}$ has an EBL realisation where

$$k_4^{max} = \frac{k_1k_3}{4} - 3\left(\frac{k_1}{4}\right)^4 = h^4 + \frac{2}{5}h\tau \geq k_4.$$

The Appeal of EBL

$$E = \begin{pmatrix} c_1 & 1 & 0 & 0 \\ c_{12} & c_2 & 1 & 0 \\ c_{123} & c_{23} & c_3 & 1 \\ c_{1234} & c_{234} & c_{34} & c_4 \end{pmatrix}$$

with

$$\begin{aligned} |xI - E| = x^4 &+ (-c_1 - c_2 - c_3 - c_4)x^3 + (c_1 c_2 + c_1 c_3 + c_1 c_4 + c_2 c_3 + c_2 c_4 + c_3 c_4 - c_{12} \\ &- c_{23} - c_{13})x^2 + (-c_1 c_2 c_3 - c_1 c_2 c_4 - c_1 c_3 c_4 - c_2 c_3 c_4 + c_1 c_{23} + c_1 c_{34} + c_2 c_{34} \\ &+ c_3 c_{12} - c_{123} - c_{234})x + c_1 c_2 c_3 c_4 - c_1 c_2 c_{34} - c_1 c_{23} c_4 - c_{12} c_3 c_4 + c_1 c_{234} \\ &+ c_{12} c_{34} + c_{123} c_4 - c_{1234}. \end{aligned} \tag{2}$$

Meehan's n=4 NIEP Solution

Both Meehan and Torre-Mayo et al. (independently of each other) solved the n=4 NIEP using EBL matrices. Corollary 1 can be leveraged to prove some subcases of Johnson's Conjecture for n=5 using Meehan's solution to the NIEP for n=4.

Example

If σ is a realisable 5-list of complex numbers such that $\sigma' = (r, -\lambda, -a+ib, -a-ib)$ is its critical points, where $\lambda, a \geq 0$, and $b > 0$, then σ' is realised by the nonnegative matrix

$$\frac{S_1}{4}I + \begin{pmatrix} 0 & 1 & 0 & 0 \\ \frac{4S_2 - S_1^2}{16} & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \frac{16S_4 - 4S_2^2 - S_1^4 - 16S_1S_3 + 8S_1^2S_2}{64} & \frac{8S_3 - 6S_1S_2 + S_1^3}{24} & \frac{4S_2 - S_1^2}{16} & 0 \end{pmatrix}.$$

References

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The nonnegative inverse eigenvalue problem from the coefficients of the characteristic polynomial. EBL digraphs

T. Laffey, E. Meehan (1999)
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