ROGERS-RAMANUJAN TYPE IDENTITIES FOR ALTERNATING KNOTS

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Dedicated to Wen-Ching Winnie Li on the occasion of her birthday

Abstract. We highlight the role of q-series techniques in proving identities arising from knot theory. In particular, we prove Rogers-Ramanujan type identities for alternating knots as conjectured by Garoufalidis, Lê and Zagier.

1. Introduction

Two of the most important results in the theory of q-series are the classical Rogers-Ramanujan identities which state that

$$\sum_{n\geq 0} \frac{q^{n^2+sn}}{(q)_n} = \frac{1}{(q^{1+s};q^5)_\infty (q^{4-s};q^5)_\infty} \tag{1.1}$$

where $s = 0$ or 1 and

$$(a)_n = (a;q)_n = \prod_{k=1}^{n} (1 - aq^{k-1}),$$

valid for $n \in \mathbb{N} \cup \{\infty\}$. In 1974, Andrews [1] obtained a generalization of (1.1) to odd moduli, namely for all $k \geq 2$, $1 \leq i \leq k$,

$$\sum_{n_1,n_2,\dots,n_{k-1}\geq 0} \frac{q^{N_1^2+N_2^2+\cdots+N_{k-1}^2+N_i+N_{i+1}+\cdots+N_{k-1}}}{(q)_{n_1}(q)_{n_2}\cdots(q)_{n_{k-1}}} = \frac{(q^i;q^{2k+1})_\infty (q^{2k+1-i};q^{2k+1})_\infty (q^{2k+1};q^{2k+1})_\infty}{(q)_\infty} \tag{1.2}$$

where $N_j = n_j + n_{j+1} + \cdots + n_{k-1}$. There has been recent interest in the appearance of these (and similar) identities in knot theory. For example, Hikami [14] considered (1.1) from the perspective of the colored Jones polynomial of torus knots while Armond and Dasbach [6] gave a skein-theoretic proof of (1.2). For similar identities related to false theta series, see [13] and for other connections between q-series and quantum invariants of knots, see [7]–[9], [11], [15] and [16].

In this paper, we consider recent work in [10] whereby the q-multisums $\Phi_K(q)$ and $\Phi_{-K}(q)$ were associated to a given alternating knot K and its mirror $-K$. The q-multisum $\Phi_K(q)$ occurs as the 0-limit (or "tail") of the colored Jones polynomial of K (see Theorem 1.10 in [10]). In

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Date: February 4, 2015.
2010 *Mathematics Subject Classification.* Primary: 33D15; Secondary: 05A30, 57M25.
Key words and phrases. q-series identities, q-series transformations, Bailey pairs, alternating knots.

Appendix D of [10], Garoufalidis and Lê (with Zagier) conjectured evaluations of $\Phi_K(q)$ for 21 knots and of $\Phi_{-K}(q)$ for 22 knots in terms of modular forms and false theta series and state "every such guess is a q-series identity whose proof is unknown to us". Before stating these conjectures, we recall some notation from [10]. For a positive integer b, we define

$$h_b = h_b(q) = \sum_{n \in \mathbb{Z}} \epsilon_b(n) q^{\frac{bn(n+1)}{2} - n}$$

where

$$\epsilon_b(n) = \begin{cases} (-1)^n & \text{if } b \text{ is odd,} \\ 1 & \text{if } b \text{ is even and } n \geq 0, \\ -1 & \text{if } b \text{ is even and } n < 0. \end{cases}$$

Note that $h_1(q) = 0$, $h_2(q) = 1$ and $h_3(q) = (q)_\infty$. For an integers p, a and b, let K_p denote the pth twist knot obtained by $-1/p$ surgery on the Whitehead link and $T(a, b)$ the left-handed (a, b) torus knot. The 43 conjectures from [10] are as follows:

K	$\Phi_K(q)$	$\Phi_{-K}(q)$
3_1	h_3	1
4_1	h_3	h_3
5_1	h_5	1
5_2	h_4	h_3
6_1	h_5	h_3
6_2	h_3h_4	h_3
6_3	h_3^2	h_3^2
7_1	h_7	1
7_2	h_6	h_3
7_3	h_5	h_4
7_4	h_4^2	h_3
7_5	h_3h_4	h_4
7_6	h_3h_4	h_3^2
7_7	h_3^3	h_3^2
8_1	h_7	h_3
8_2	h_3h_6	h_3
8_3	h_5	h_5
8_4	h_3	h_4h_5
8_5	?	h_3
K_p, $p > 0$	h_{2p}	h_3
K_p, $p < 0$	$h_{2\lvert p\rvert+1}$	h_3
$T(2, p)$, $p > 0$	h_{2p+1}	1

TABLE 1.

Here, we have corrected the entries for 6_1, 7_3, 8_1, 8_4, 8_5, K_p, $p < 0$ (and their mirrors) and 7_5 in Appendix D of [10]. Three of these Rogers-Ramanujan type identities, namely

$$\Phi_{3_1}(q) = h_3, \quad \Phi_{4_1}(q) = h_3 \quad \text{and} \quad \Phi_{6_3}(q) = h_3^2 \tag{1.3}$$

have been proven by Andrews [4]. Motivated by his work (and in conjunction with (1.3)), we prove the following result.

Theorem 1.1. *The identities in Table 1 are true.*

In principle, one can use either Theorem 5.1 of [6] or Theorem 4.12 of [13] to give a skein-theoretic proof of Theorem 1.1. Here, we have chosen to highlight the role of q-series techniques in proving such identities. For example, one can use the Bailey machinery to quickly prove identity (2.7) in [13]. The paper is organized as follows. In Section 2, we provide the necessary background on q-series identities and the Bailey machinery. In Section 3, we clarify the construction of the q-multisums $\Phi_K(q)$ and $\Phi_{-K}(q)$ from [10] (see also [11]). In Section 4, we prove Theorem 1.1. It is interesting to note that the proofs for 5_1 and -8_4 require (1.1) while those for 7_1 and $T(2,p)$ utilize (1.2). Although, one can simplify $\Phi_{8_5}(q)$ using the techniques in this paper, a conjectural evaluation is still currently unknown. Moreover, it is not known for a general alternating knot K if $\Phi_K(q)$ reduces as in the current pleasant situation.

2. Preliminaries

We first recall five q-series identities. The first two are due to Euler (see II.1 and II.2, page 236 in [12]), the third is the $z = 1$ case of Lemma 2 in [4], the fourth is the q-binomial theorem (see II.4, page 236 in [12]) and the fifth is the Jacobi triple product (see II.28, page 239 in [12]):

$$\sum_{n=0}^{\infty} \frac{t^n}{(q)_n} = \frac{1}{(t)_\infty}, \tag{2.1}$$

$$\sum_{n=0}^{\infty} \frac{(-1)^n t^n q^{n(n-1)/2}}{(q)_n} = (t)_\infty, \tag{2.2}$$

$$\sum_{n=0}^{\infty} \frac{q^{n^2+An}}{(q)_n (q)_{n+A}} = \frac{1}{(q)_\infty} \tag{2.3}$$

for any integer A,

$$\sum_{n=0}^{\infty} \frac{(-1)^n t^n q^{\frac{n(n-1)}{2}}}{(q)_n (q)_{K-n}} = \frac{(t)_K}{(q)_K} \tag{2.4}$$

and

$$\sum_{n\in\mathbb{Z}} z^n q^{n^2} = (-zq;q^2)_\infty (-q/z;q^2)_\infty (q^2;q^2)_\infty. \tag{2.5}$$

Here and throughout, we use the convention that

$$\frac{1}{(q)_n} = 0$$

for $n < 0$. In addition, one can easily check that for a, $b \geq 0$,

$$\frac{(q^{-a-b})_a}{(q)_a} = (-1)^a q^{-\frac{a(a+1)}{2}-ab} \frac{(q)_{a+b}}{(q)_a (q)_b}. \tag{2.6}$$

We now derive a key result which follows from a generalization of Sears' transformation (see III.15, page 242 in [12]).

Lemma 2.1. *For any $n > 2$ and integers c_k,*

$$\sum_{a\geq 0} (-1)^{na} \frac{q^{\frac{na(a+1)}{2}-a+a\sum\limits_{k=1}^{n-1} c_k}}{(q)_a \prod\limits_{k=1}^{n-1} (q)_{a+c_k}} = \frac{1}{(q)_\infty} \sum_{i_1,\ldots,i_{n-2}\geq 0} (-1)^{\sum\limits_{k=1}^{n-2}\sum\limits_{j=1}^{k} i_j} \frac{q^{\frac{1}{2}\sum\limits_{k=1}^{n-2}\left(\sum\limits_{j=1}^{k} i_j\right)\left(1+\sum\limits_{j=1}^{k} i_j\right)+\sum\limits_{k=2}^{n-1}\sum\limits_{j=1}^{k-1} c_k i_j}}{\prod\limits_{k=1}^{n-2} (q)_{i_k} \prod\limits_{k=1}^{n-2} (q)_{c_k+\sum\limits_{j=1}^{k} i_j}}.$$

Proof. We first use that

$$\lim_{t\to 0} \left(\frac{1}{t}\right)_n t^n = (-1)^n q^{\frac{n(n-1)}{2}},$$

then apply Corollary 1 in [5] and simplify to obtain

$$\sum_{a\geq 0} (-1)^{na} \frac{q^{\frac{na(a+1)}{2}-a+a\sum\limits_{k=1}^{n-1} c_k}}{(q)_a \prod\limits_{k=1}^{n-1} (q)_{a+c_k}} = \frac{1}{\prod\limits_{k=1}^{n-1} (q)_{c_k}} \lim_{t\to 0} \sum_{a\geq 0} \frac{(\frac{1}{t})_a^n\, t^{na}\, q^{a\left(n-1+\sum\limits_{k=1}^{n-1} c_k\right)}}{(q)_a \prod\limits_{k=1}^{n-1} (q^{c_k+1})_a}$$

$$= \frac{1}{\prod\limits_{k=1}^{n-1} (q)_{c_k}} \lim_{t\to 0} \frac{(tq^{c_{n-1}+1})_\infty (t^{n-1} q^{n-1+\sum\limits_{k=1}^{n-1} c_k})_\infty}{(q^{c_{n-1}+1})_\infty (t^n q^{n-1+\sum\limits_{k=1}^{n-1} c_k})_\infty}$$

$$\times \sum_{i_1,\ldots,i_{n-2}\geq 0} \frac{(tq^{c_2+1})^{i_1} (tq^{c_3+1})^{i_1+i_2} \cdots (tq^{c_{n-1}+1})^{i_1+i_2+\ldots+i_{n-2}}}{(q)_{i_1} (q)_{i_2} \cdots (q)_{i_{n-2}}}$$

$$\times \frac{(\frac{1}{t})_{i_1} (\frac{1}{t})_{i_1+i_2} \cdots (\frac{1}{t})_{i_1+i_2+\ldots+i_{n-2}}}{(q^{c_1+1})_{i_1} (q^{c_2+1})_{i_1+i_2} \cdots (q^{c_{n-2}+1})_{i_1+i_2+\ldots+i_{n-2}}}$$

$$\times \frac{(tq^{c_1+1})_{i_1} \cdots (tq^{c_{n-2}+1})_{i_{n-2}} (tq^{c_1+1})_{i_1} (t^2 q^{2+c_1+c_2+i_1})_{i_2} \cdots (t^{n-2} q^{n-2+c_1+\ldots+c_{n-2}+i_1+\ldots+i_{n-3}})_{i_{n-2}}}{(t^{n-1} q^{n-1+c_1+\ldots+c_{n-1}})_{i_1+\ldots+i_{n-2}}}$$

$$= \frac{1}{(q)_\infty} \sum_{i_1,\ldots,i_{n-2}\geq 0} (-1)^{\sum\limits_{k=1}^{n-2}\sum\limits_{j=1}^{k} i_j} \frac{q^{\frac{1}{2}\sum\limits_{k=1}^{n-2}\left(\sum\limits_{j=1}^{k} i_j\right)\left(1+\sum\limits_{j=1}^{k} i_j\right)+\sum\limits_{k=2}^{n-1}\sum\limits_{j=1}^{k-1} c_k i_j}}{\prod\limits_{k=1}^{n-2} (q)_{i_k} \prod\limits_{k=1}^{n-2} (q)_{c_k+\sum\limits_{j=1}^{k} i_j}}.$$

□

We now recall the Bailey machinery as initiated by Bailey and Slater in the 1940's and 50's and perfected by Andrews in the 1980's (for further details, see [2], [3] or [18]). A pair of sequences $(\alpha_n, \beta_n)_{n\geq 0}$ satisfying

$$\beta_n = \sum_{k=0}^{n} \frac{\alpha_k}{(q)_{n-k}(aq)_{n+k}} \tag{2.7}$$

is called a *Bailey pair relative to a*. If $(\alpha_n, \beta_n)_{n\geq 0}$ is a Bailey pair relative to a, then so is $(\alpha_n^{'}, \beta_n^{'})_{n\geq 0}$ where

$$\alpha_n^{'} = \frac{(b)_n(c)_n(aq/bc)^n}{(aq/b)_n(aq/c)_n}\alpha_n \tag{2.8}$$

and

$$\beta_n^{'} = \sum_{k=0}^{n} \frac{(b)_k(c)_k(aq/bc)_{n-k}(aq/bc)^k}{(aq/b)_n(aq/c)_n(q)_{n-k}}\beta_k. \tag{2.9}$$

Iterating (2.8) and (2.9) leads to a sequence of Bailey pairs, called the *Bailey chain*. Putting (2.8) and (2.9) into (2.7) and letting $n \to \infty$ gives

$$\sum_{n\geq 0}(b)_n(c)_n(aq/bc)^n\beta_n = \frac{(aq/b)_\infty(aq/c)_\infty}{(aq)_\infty(aq/bc)_\infty}\sum_{n\geq 0}\frac{(b)_n(c)_n(aq/bc)^n}{(aq/b)_n(aq/c)_n}\alpha_n. \tag{2.10}$$

For example, if we consider the Bailey pair relative to q (see B(3) in [17])

$$\alpha_n = \frac{(1-q^{2n+1})(-1)^n q^{\frac{3}{2}n^2+\frac{1}{2}n}}{1-q} \tag{2.11}$$

and

$$\beta_n = \frac{1}{(q)_n}, \tag{2.12}$$

then one application of (2.8) and (2.9) with b, $c \to \infty$ yields

$$\alpha_n^{'} = \frac{(1-q^{2n+1})(-1)^n q^{\frac{5}{2}n^2+\frac{3}{2}n}}{1-q} \tag{2.13}$$

and

$$\beta_n^{'} = \sum_{k=0}^{n} \frac{q^{k(k+1)}}{(q)_k(q)_{n-k}} \tag{2.14}$$

while $l-2$ applications, $l > 2$, of (2.8) and (2.9) with b, $c \to \infty$ at each step produces

$$\alpha_n^{(l-2)} = \frac{(1-q^{2n+1})(-1)^n q^{\frac{2l-1}{2}n^2+\frac{2l-3}{2}n}}{1-q} \tag{2.15}$$

and

$$\beta_n^{(l-2)} = \sum_{n=n_{l-1},n_{l-2},\dots,n_1\geq 0} \frac{q^{\sum\limits_{k=1}^{l-2} n_k(n_k+1)}}{(q)_{n_1}\prod\limits_{k=2}^{l-1}(q)_{n_k-n_{k-1}}}. \tag{2.16}$$

Inserting (2.13) and (2.14) into (2.10), then letting $b \to \infty$ and $c = q$ gives

$$\sum_{n,k\geq 0} (-1)^n \frac{q^{k(k+1)+\frac{n(n+1)}{2}}(q)_n}{(q)_k(q)_{n-k}} = \sum_{n\geq 0} q^{3n^2+2n}(1-q^{2n+1}) \tag{2.17}$$

while substituting (2.15) and (2.16) into (2.10), then letting $b \to \infty$ and $c = q$ leads to

$$\sum_{n_{l-1},n_{l-2},\dots,n_1\geq 0} (-1)^{n_{l-1}} \frac{q^{\sum\limits_{k=1}^{l-2} n_k(n_k+1)+\frac{n_{l-1}(n_{l-1}+1)}{2}}(q)_{n_{l-1}}}{(q)_{n_1}\prod\limits_{k=2}^{l-1}(q)_{n_k-n_{k-1}}} = \sum_{n\geq 0} q^{ln^2+(l-1)n}(1-q^{2n+1}). \tag{2.18}$$

3. $\Phi_K(q)$ and $\Phi_{-K}(q)$

Let K be an alternating knot with c crossings and D its associated diagram. We checkerboard D with colors A and B such that the exterior X is colored A (here, we identify D with the planar graph obtained by placing a vertex at each crossing and an edge at each arc) and let $\mathcal{T}_K$ be the Tait graph of K (or, equivalently, of D). The reduced Tait graph $\mathcal{T}_K'$ is obtained from $\mathcal{T}_K$ by replacing every set of two edges that connect the same two vertices by a single edge. Let $E(D)$ be the set of edges, R the set of faces, R_A the set of A-colored faces and R_B the set of B-colored faces in D. The idea is to assign variables to each face of D, including X. Thus, we let

$$S = \{s : R \to \mathbb{Z} : s(X) = 0\}.$$

For F, F_i and $F_j \in R$, define $e(F)$ to be the number of edges of F, $cv(F_i, F_j)$ the number of common vertices and $ce(F_i, F_j)$ the number of common edges between F_i and F_j. We now consider the functions $L : R \to \frac{1}{2}\mathbb{Z}$ and $Q : R \times R \to \mathbb{Z}$ given by

$$L(F) := \begin{cases} 1 & \text{if } F \in R_B, \\ \frac{e(F)}{2} - 1 & \text{if } F \in R_A \end{cases}$$

and

$$Q(F_i, F_j) := \begin{cases} 0 & \text{if } i = j,\ F_i \in R_B \text{ or } i \neq j,\ F_i,\ F_j \in R_A, \\ e(F_i) & \text{if } i = j,\ F_i \in R_A, \\ cv(F_i, F_j) & \text{if } i \neq j,\ F_i,\ F_j \in R_B, \\ ce(F_i, F_j) & \text{if } i \neq j,\ F_i \in R_B,\ F_j \in R_A \text{ or } F_i \in R_A,\ F_j \in R_B. \end{cases}$$

We extend $s \in S$ to $E(D)$ by defining $s(e)$ to be the sum of the variables in adjacent faces. Furthermore, suppose $F \in R_B$ shares a common edge with the maximum number of faces in R_A. If F is not unique, choose a face in R_B that shares a common edge with the maximum number of faces in $R_A \setminus \{X\}$. If this latter face is not unique, choose from any of the remaining candidates of faces and let F^* denote this choice. Finally, we let

$$\Lambda := \{s \in S : s(e) \geq 0, \forall e \in E(D) \text{ and } s(F^*) = 0\}$$

and consider the functions $L' : \Lambda \to \frac{1}{2}\mathbb{Z}^{|R|-1}$ and $Q' : \Lambda \to \frac{1}{2}\mathbb{Z}^{|R|-1}$ defined by

$$L'(s) = \sum_{i=1}^{|R|-1} L(F_i)s(F_i)$$

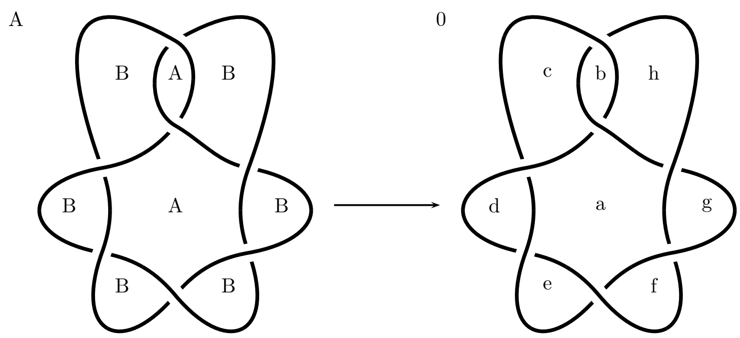
and

$$Q'(s) = \frac{1}{2} \sum_{1 \leq i,j \leq |R|-1} Q(F_i, F_j)s(F_i)s(F_j).$$

The q-multisum $\Phi_K(q)$ is now given by (see Theorem 1.10 in [10])

$$\Phi_K(q) = (q)_\infty^c S_K := (q)_\infty^c \sum_{s \in \Lambda} (-1)^{2L'(s)} \frac{q^{Q'(s)+L'(s)}}{\prod_{e \in E(D)} (q)_{s(e)}}.$$

Let us illustrate this construction for $K = 7_2$. We first consider



In matrix notation, we have

$$s = [c, d, e, f, g, h, a, b]^T, \quad L' = [1, 1, 1, 1, 1, 1, 2, 0],$$

$$Q' = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 2 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 \\ 2 & 0 & 0 & 0 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 6 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & 0 & 2 \end{pmatrix} \tag{3.1}$$

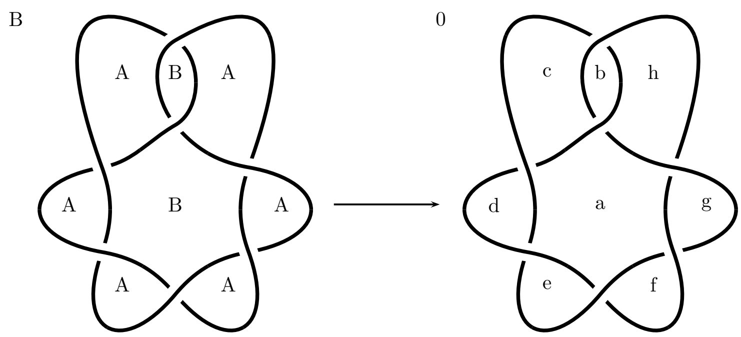
and

$$\Lambda = \{[c,d,e,f,g,h,a,b] \in \mathbb{Z}^8 : a,b,c,d,e,f,g \geq 0, h = 0\}.$$

Thus, in matrix notation,

$$\begin{aligned}\Phi_{7_2}(q) = (q)_\infty^7 S_{7_2} &= (q)_\infty^7 \sum_{s\in\Lambda} (-1)^{2L's} \frac{q^{s^T Q' s + L's}}{\prod\limits_{e\in E(D)} (q)_{s(e)}} \\ &= (q)_\infty^7 \sum_{a,b,c,d,e,f,g\geq 0} \frac{q^{3a^2+2a+b^2+bc+ac+ad+ae+af+ag+cd+de+ef+fg+c+d+e+f+g}}{(q)_a(q)_b(q)_c(q)_d(q)_e(q)_f(q)_g(q)_{b+c}(q)_{a+c}(q)_{a+d}(q)_{a+e}(q)_{a+f}(q)_{a+g}}.\end{aligned}$$

To compute $\Phi_{-K}(q)$, we repeat the above process but swap A and B faces while still imposing the condition that $s(X) = 0$ and choosing $F^* \in R_A$. So, for $-K = -7_2$,



Here,

$$s = [c,d,e,f,g,h,a,b]^T, \quad L' = \left[\frac{1}{2}, 0, 0, 0, 0, \frac{1}{2}, 1, 1\right],$$

$$Q' = \begin{pmatrix} 3 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 2 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 2 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \end{pmatrix}$$

and

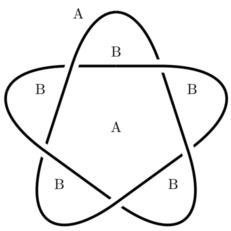
$$\Lambda = \{[a,b,c,d,e,f,g,h] \in \mathbb{Z}^8 : a,b,c,d,e,f,g \geq 0, h = 0\}.$$

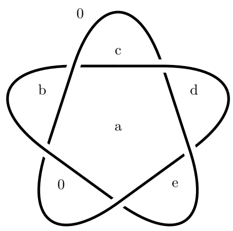
This gives us

$$\Phi_{-7_2}(q) = (q)_\infty^7 S_{-7_2} = (q)_\infty^7 \sum_{s\in\Lambda} (-1)^{2L's} \frac{q^{s^T Q' s + L's}}{\prod\limits_{e\in\epsilon} (q)_{s(e)}}$$
$$= (q)_\infty^7 \sum_{a,b,c,d,e,f,g\geq 0} \frac{q^{a+b+ab+ac+ad+ae+af+ag+bc+\frac{c(3c+1)}{2}+d^2+e^2+f^2+g^2}}{(q)_a(q)_b(q)_c(q)_d(q)_e(q)_f(q)_g(q)_{a+c}(q)_{a+d}(q)_{a+e}(q)_{a+f}(q)_{a+g}(q)_{b+c}}.$$

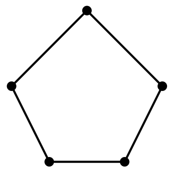
Finally, by Theorem 2 in [6] or Corollary 1.12 in [10], if the reduced Tait graphs of two alternating knots K and K' are isomorphic, then $\Phi_K(q) = \Phi_{K'}(q)$. Thus, in order to deduce Theorem 1.1, it suffices to verify the conjectural identities in the following cases: 5_1, 5_2, 6_2, 7_1, 7_2, 7_4, 7_7, 8_2, 8_4, K_p, $p > 0$, $T(2,p)$, -3_1, -7_7 and -8_4. For each of these 14 knots, we provide the checkerboard coloring, assignment of variables and (reduced) Tait graph.

5_1





$\mathcal{T}_{5_1}$



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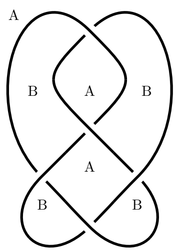
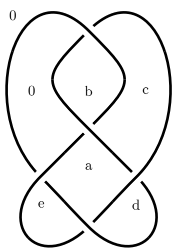
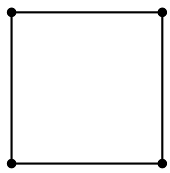
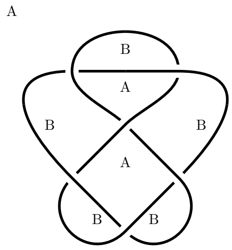


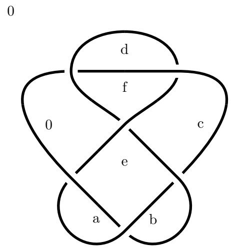


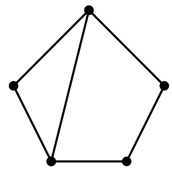
$\mathcal{T}_{5_2}$



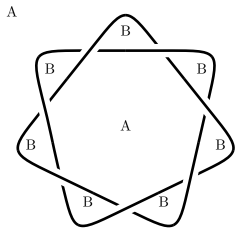
6_2 $\mathcal{T}_{6_2}$

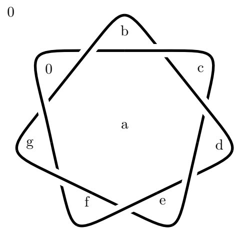


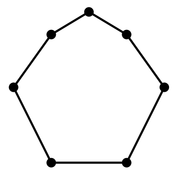




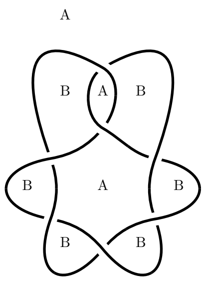
7_1 $\mathcal{T}_{7_1}$

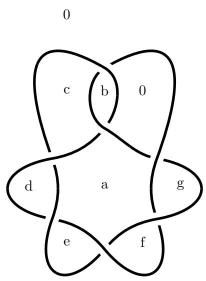


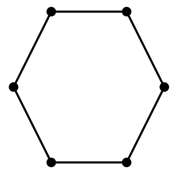




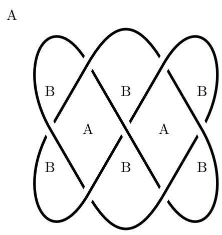
7_2 $\mathcal{T}_{7_2}$

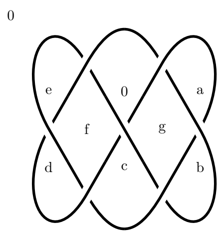


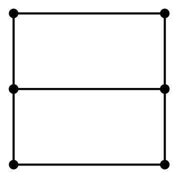




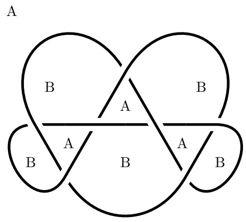
7_4 $\mathcal{T}_{7_4}$

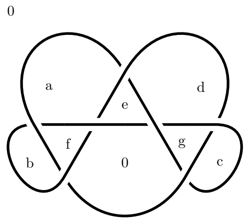


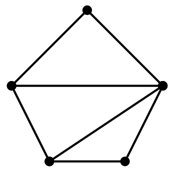




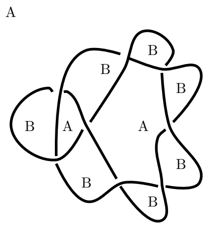
7_7 $\mathcal{T}_{7_7}$

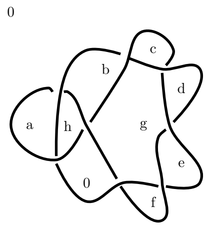


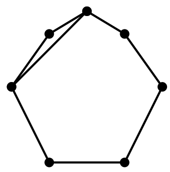




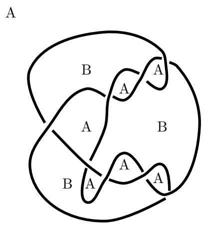
8_2 $\mathcal{T}_{8_2}$

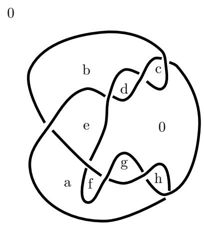


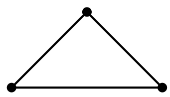




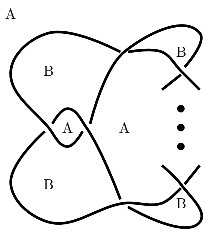
8_4 $\mathcal{T}_{8_4}$

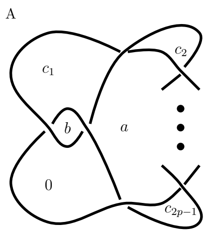


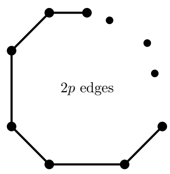




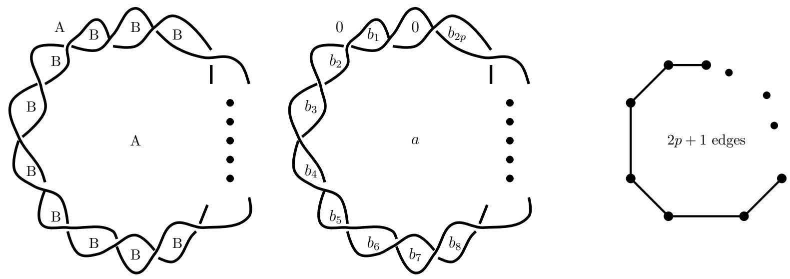
$K_p, p > 0$ $\mathcal{T}_{K_p},\ p > 0$



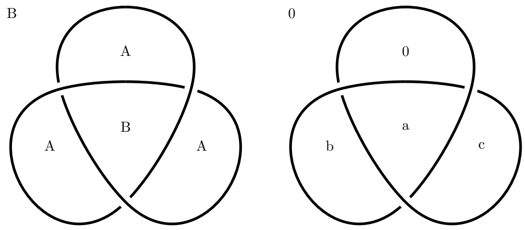




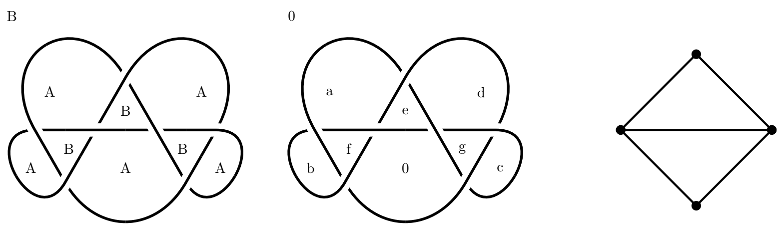
$T(2,p), p>0$ $\qquad$ $\mathcal{T}_{T(2,p)},\ p>0$



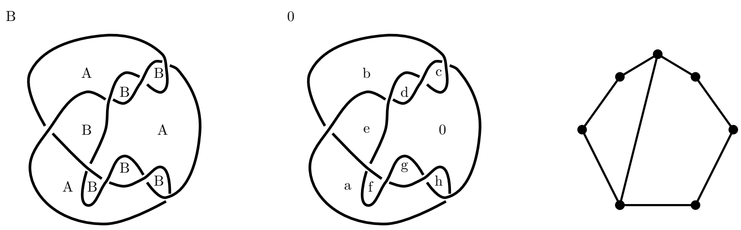
-3_1 $\qquad$ $\mathcal{T}_{-3_1}$



-7_7 $\qquad$ $\mathcal{T}_{-7_7}$



-8_4 $\qquad$ $\mathcal{T}_{-8_4}$



4. Proof of Theorem 1.1

We can now prove Theorem 1.1.

Proof of Theorem 1.1. For $\Phi_{5_1}(q)$, it suffices to prove

$$S_{5_1} := \sum_{a,b,c,d,e\geq 0} (-1)^a \frac{q^{\frac{a(5a+3)}{2}+ab+ac+ad+ae+bc+cd+de+b+c+d+e}}{(q)_a(q)_b(q)_c(q)_d(q)_e(q)_{a+b}(q)_{a+c}(q)_{a+d}(q)_{a+e}} = \frac{1}{(q)_\infty^5} h_5. \tag{4.1}$$

We now have

$$\begin{aligned}
S_{5_1} &= \frac{1}{(q)_\infty} \sum_{i,j,k,b,c,d,e\geq 0} (-1)^{i+k} \frac{q^{\frac{3i(i+1)}{2}+j^2+j+\frac{k(k+1)}{2}+2ij+jk+ki+b+bc+c+ci+cd+d+di+dj+de+e+ei+ej+ek}}{(q)_i(q)_j(q)_k(q)_b(q)_c(q)_d(q)_e(q)_{i+b}(q)_{i+j+c}(q)_{i+j+k+d}} \\
&\quad \text{(apply Lemma 2.1 to the a-sum with $n=5$)} \\
&= \frac{1}{(q)_\infty^2} \sum_{i,j,k,b,c,d\geq 0} (-1)^{i+k} \frac{q^{\frac{3i(i+1)}{2}+j^2+j+\frac{k(k+1)}{2}+2ij+jk+ki+b+bc+c+ci+cd+d+di+dj}}{(q)_i(q)_j(q)_k(q)_b(q)_c(q)_d(q)_{i+b}(q)_{i+j+c}} \\
&\quad \text{(evaluate the e-sum with (2.1))} \\
&= \frac{1}{(q)_\infty^5} \sum_{i,j,k\geq 0} (-1)^{i+k} \frac{q^{\frac{3i(i+1)}{2}+j^2+j+\frac{k(k+1)}{2}+2ij+jk+ki}}{(q)_i(q)_j(q)_k} \\
&\quad \text{(evaluate the d-sum, c-sum and b-sum with (2.1))} \\
&= \frac{1}{(q)_\infty^5} \sum_{i,j,k\geq 0} (-1)^{i+k} \frac{q^{\frac{i(i+1)}{2}+j^2+j+\frac{k(k+1)}{2}+jk}}{(q)_i(q)_{j-i}(q)_k} \quad \text{(shift $j \to j-i$)} \\
&= \frac{1}{(q)_\infty^5} \sum_{j,k\geq 0} (-1)^{k} \frac{q^{j^2+j+\frac{k(k+1)}{2}+jk}}{(q)_k} \quad \text{(apply (2.4) to the i-sum)} \\
&= \frac{1}{(q)_\infty^4} \sum_{j\geq 0} \frac{q^{j^2+j}}{(q)_j} \quad \text{(apply (2.2) to the k-sum)} \\
&= \frac{(q;q^5)_\infty(q^4;q^5)_\infty(q^5;q^5)_\infty}{(q)_\infty^5} \quad \text{(by (1.1))} \\
&= \frac{1}{(q)_\infty^5} h_5 \quad \text{(apply (2.5) with $q \to q^{5/2}$, $z=-q^{3/2}$).}
\end{aligned}$$

For $\Phi_{5_2}(q)$, it suffices to prove

$$S_{5_2} := \sum_{a,b,c,d,e\geq 0} \frac{q^{2a^2+b^2+ac+ad+ae+bc+cd+de+a+c+d+e}}{(q)_a(q)_b(q)_c(q)_d(q)_e(q)_{b+c}(q)_{a+c}(q)_{a+d}(q)_{a+e}} = \frac{1}{(q)_\infty^5} h_4. \tag{4.2}$$

Thus,

$$
\begin{aligned}
S_{5_2} &= \frac{1}{(q)_\infty} \sum_{a,c,d,e\geq 0} \frac{q^{2a^2+ac+ad+ae+cd+de+a+c+d+e}}{(q)_a(q)_c(q)_d(q)_e(q)_{a+c}(q)_{a+d}(q)_{a+e}} \qquad (4.3)\\
&\quad \text{(evaluate the b-sum with (2.3))}\\
&= \frac{1}{(q)_\infty^2} \sum_{i,j,c,d,e\geq 0} (-1)^j \frac{q^{i^2+i+\frac{j^2+j}{2}+ij+di+e(i+j)+cd+de+c+d+e}}{(q)_i(q)_j(q)_c(q)_d(q)_e(q)_{i+c}(q)_{i+j+d}}\\
&\quad \text{(apply Lemma 2.1 to the a-sum with $n=4$)}\\
&= \frac{1}{(q)_\infty^3} \sum_{i,j,c,d\geq 0} (-1)^j \frac{q^{i^2+i+\frac{j^2+j}{2}+ij+di+cd+c+d}}{(q)_i(q)_j(q)_c(q)_d(q)_{i+c}} \quad \text{(evaluate the e-sum with (2.1))}\\
&= \frac{1}{(q)_\infty^5} \sum_{i,j\geq 0} (-1)^j \frac{q^{i^2+i+\frac{j^2+j}{2}+ij}}{(q)_i(q)_j} \quad \text{(evaluate the d-sum and c-sum with (2.1))}\\
&= \frac{1}{(q)_\infty^5} \sum_{i,j\geq 0} (-1)^j \frac{q^{i^2+i+\frac{j^2-j}{2}-ij}}{(q)_{i-j}(q)_j} \quad \text{(shift $i \to i-j$)}\\
&= \frac{1}{(q)_\infty^5} \sum_{i\geq 0} (-1)^i q^{\frac{i^2+i}{2}} \quad \text{(apply (2.4) to the j-sum, then use (2.6))}\\
&= \frac{1}{(q)_\infty^5} h_4\\
&\quad \text{(consider $i=2n$, $i=2n+1$, then let $n\to -n-1$ in the second resulting sum).}
\end{aligned}
$$

For $\Phi_{6_2}(q)$, it suffices to prove

$$
S_{6_2} := \sum_{a,b,c,d,e,f\geq 0} (-1)^e \frac{q^{2f^2+f+\frac{e(3e+1)}{2}+ab+af+bc+bf+cd+ce+cf+de+a+b+c+d}}{(q)_a(q)_b(q)_c(q)_d(q)_e(q)_f(q)_{a+f}(q)_{b+f}(q)_{c+e}(q)_{c+f}(q)_{d+e}} = \frac{1}{(q)_\infty^5} h_4.
$$

Thus,

$$
\begin{aligned}
S_{6_2} &= \frac{1}{(q)_\infty} \sum_{a,b,c,d,e,f\geq 0} (-1)^e \frac{q^{2f^2+f+\frac{e(e+1)}{2}+ab+af+bc+bf+cd+cf+de+a+b+c+d}}{(q)_a(q)_b(q)_c(q)_d(q)_e(q)_f(q)_{a+f}(q)_{b+f}(q)_{c+e}(q)_{c+f}}\\
&\quad \text{(apply Lemma 2.1 to the e-sum with $n=3$)}\\
&= \frac{1}{(q)_\infty^2} \sum_{a,b,c,e,f\geq 0} (-1)^e \frac{q^{2f^2+f+\frac{e(e+1)}{2}+ab+af+bc+bf+cf+a+b+c}}{(q)_a(q)_b(q)_c(q)_e(q)_f(q)_{a+f}(q)_{b+f}(q)_{c+f}}\\
&\quad \text{(evaluate the d-sum with (2.1))}
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{1}{(q)_\infty} \sum_{a,b,c,f\geq 0} \frac{q^{2f^2+f+ab+af+bc+bf+cf+a+b+c}}{(q)_a(q)_b(q)_c(q)_f(q)_{a+f}(q)_{b+f}(q)_{c+f}} \quad \text{(evaluate the e-sum with (2.2))}\\
&= \frac{1}{(q)_\infty^5} h_4 \quad \text{(let $(a,b,c,f)\to(c,d,e,a)$, then proceed with (4.3)).}
\end{aligned}
$$

For $\Phi_{7_1}(q)$, it suffices to prove

$$
\begin{aligned}
S_{7_1} &:= \sum_{a,b,c,d,e,f,g\geq 0} (-1)^a \frac{q^{\frac{a(7a+5)}{2}+ab+ac+ad+ae+af+ag+bc+cd+de+ef+fg+b+c+d+e+f+g}}{(q)_a(q)_b(q)_c(q)_d(q)_e(q)_f(q)_g(q)_{a+b}(q)_{a+c}(q)_{a+d}(q)_{a+e}(q)_{a+f}(q)_{a+g}}\\
&= \frac{1}{(q)_\infty^7} h_7.
\end{aligned}
$$

Thus,

$$
\begin{aligned}
S_{7_1} &= \frac{1}{(q)_\infty} \sum_{i,j,k,l,m,b,c,d\geq 0} (-1)^{i+k+m} \frac{q^{\frac{5i(i+1)}{2}+2j(j+1)+\frac{3k(k+1)}{2}+l(l+1)+\frac{m(m+1)}{2}}}{(q)_i(q)_j(q)_k(q)_l(q)_m}\\
&\times \frac{q^{bc+cd+de+ef+fg+b+c+d+e+f+g}}{(q)_b(q)_c(q)_d(q)_e(q)_f(q)_g}\\
&\times \frac{q^{4ij+3ik+2il+im+3jk+2jl+jm+2kl+km+lm+ci+d(i+j)+e(i+j+k)+f(i+j+k+l)+g(i+j+k+l+m)}}{(q)_{b+i}(q)_{c+i+j}(q)_{d+i+j+k}(q)_{e+i+j+k+l}(q)_{f+i+j+k+l+m}}\\
&\text{(apply Lemma 2.1 to the a-sum with $n=7$)}\\
&= \frac{1}{(q)_\infty^7} \sum_{i,j,k,l,m\geq 0} (-1)^{i+k+m} \frac{q^{\frac{5i(i+1)}{2}+2j(j+1)+\frac{3k(k+1)}{2}+l(l+1)+\frac{m(m+1)}{2}}}{(q)_i(q)_j(q)_k}\\
&\times \frac{q^{4ij+3ik+2il+im+3jk+2jl+jm+2kl+km+lm}}{(q)_l(q)_m}\\
&\text{(evaluate the g-sum, f-sum, e-sum, d-sum, c-sum and b-sum with (2.1))}\\
&= \frac{1}{(q)_\infty^7} \sum_{i,j,k,l,m\geq 0} (-1)^{i+k+m} \frac{q^{\frac{i(i+1)}{2}+2j(j+1)+\frac{3k(k+1)}{2}+l(l+1)+\frac{m(m+1)}{2}+3jk+2jl+jm+2kl+km+lm}}{(q)_i(q)_{j-i}(q)_k(q)_l(q)_m}\\
&\text{(shift $j\to j-i$)}\\
&= \frac{1}{(q)_\infty^7} \sum_{j,k,l,m\geq 0} (-1)^{k+m} \frac{q^{2j(j+1)+\frac{3k(k+1)}{2}+l(l+1)+\frac{m(m+1)}{2}+3jk+2jl+jm+2kl+km+lm}}{(q)_k(q)_l(q)_m}\\
&\text{(evaluate the i-sum with (2.4))}
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{1}{(q)_\infty^7} \sum_{j,k,l,m\geq 0} (-1)^{k+m} \frac{q^{2j(j+1)+\frac{k(k+1)}{2}+l(l+1)+\frac{m(m+1)}{2}+jk+2jl+jm+lm}}{(q)_k(q)_{l-k}(q)_m} \quad \text{(shift } l \to l-k\text{)} \\
&= \frac{1}{(q)_\infty^7} \sum_{j,l,m\geq 0} (-1)^{m} \frac{q^{2j(j+1)+l(l+1)+\frac{m(m+1)}{2}+2jl+jm+lm}(q^{1+j})_l}{(q)_l(q)_m} \\
&\text{(evaluate the } k\text{-sum with (2.4))} \\
&= \frac{1}{(q)_\infty^6} \sum_{j,l\geq 0} \frac{q^{2j(j+1)+l(l+1)+2jl}}{(q)_j(q)_l} \quad \text{(evaluate the } m\text{-sum with (2.2) and simplfy)} \\
&= \frac{(q;q^7)_\infty(q^6;q^7)_\infty(q^7;q^7)_\infty}{(q)_\infty^7} \quad \text{(by (1.2) with } k=3,\ n_1=l,\ n_2=j\text{)} \\
&= \frac{1}{(q)_\infty^7} h_7 \quad \text{(by (2.5) with } q \to q^{7/2},\ z=-q^{5/2}\text{).}
\end{aligned}
$$

For $\Phi_{7_2}(q)$, it suffices to prove

$$
\begin{aligned}
S_{7_2} &:= \sum_{a,b,c,d,e,f,g\geq 0} \frac{q^{3a^2+2a+b^2+bc+ac+ad+ae+af+ag+cd+de+ef+fg+c+d+e+f+g}}{(q)_a(q)_b(q)_c(q)_d(q)_e(q)_f(q)_g(q)_{b+c}(q)_{a+c}(q)_{a+d}(q)_{a+e}(q)_{a+f}(q)_{a+g}} \\
&= \frac{1}{(q)_\infty^7} h_6.
\end{aligned}
\tag{4.4}
$$

Thus,

$$
\begin{aligned}
S_{7_2} &= \frac{1}{(q)_\infty^2} \sum_{i,j,k,l,c,d,e,f,g\geq 0} (-1)^{j+l} \frac{q^{2i(i+1)+\frac{3j(j+1)}{2}+k(k+1)+\frac{l(l+1)}{2}}}{(q)_i(q)_j(q)_k(q)_l} \\
&\quad \times \frac{q^{3ij+2ik+il+2jk+jl+kl+di+e(i+j)+f(i+j+k)+g(i+j+k+l)+cd+de+ef+fg+c+d+e+f+g}}{(q)_{c+i}(q)_{d+i+j}(q)_{e+i+j+k}(q)_{f+i+j+k+l}(q)_c(q)_d(q)_e(q)_f(q)_g} \\
&\quad \text{(evaluate the b-sum with (2.3) and apply Lemma 2.1 to the } a\text{-sum with } n=6\text{)} \\
&= \frac{1}{(q)_\infty^7} \sum_{i,j,k,l\geq 0} (-1)^{j+l} \frac{q^{2i(i+1)+\frac{3j(j+1)}{2}+k(k+1)+\frac{l(l+1)}{2}+3ij+2ik+il+2jk+jl+kl}}{(q)_i(q)_j(q)_k(q)_l} \\
&\quad \text{(evaluate the } g\text{-sum, } f\text{-sum, } e\text{-sum, } d\text{-sum and } c\text{-sum with (2.1))}
\end{aligned}
$$

$$= \frac{1}{(q)_\infty^7} \sum_{i,j,k,l\geq 0} (-1)^{j+l} \frac{q^{2i(i+1)+\frac{j(j-1)}{2}+k(k+1)+\frac{l(l+1)}{2}-ij+2ik+il+kl}}{(q)_{i-j}(q)_j(q)_k(q)_l} \quad \text{(shift } i \to i-j\text{)}$$

$$= \frac{1}{(q)_\infty^7} \sum_{i,k,l\geq 0} (-1)^{i+l} \frac{q^{\frac{3i(i+1)}{2}+k(k+1)+\frac{l(l+1)}{2}+2ik+il+kl}}{(q)_k(q)_l}$$

(evaluate the j-sum with (2.4), then use (2.6))

$$= \frac{1}{(q)_\infty^7} \sum_{i,k,l\geq 0} (-1)^{i+l} \frac{q^{\frac{3i(i+1)}{2}+k(k+1)+\frac{l(l-1)}{2}+2ik-il-kl}}{(q)_{k-l}(q)_l} \quad \text{(shift } k \to k-l\text{)}$$

$$= \frac{1}{(q)_\infty^7} \sum_{i,k\geq 0} (-1)^{i+k} \frac{q^{\frac{3i(i+1)}{2}+\frac{k(k+1)}{2}+ik}(q)_{k+i}}{(q)_i(q)_k}$$

(evaluate the l-sum with (2.4), then use (2.6) and simplify)

$$= \frac{1}{(q)_\infty^7} \sum_{i,k\geq 0} (-1)^{k} \frac{q^{i(i+1)+\frac{k(k+1)}{2}}(q)_k}{(q)_i(q)_{k-i}} \quad \text{(shift } k \to k-i\text{)}$$

$$= \frac{1}{(q)_\infty^7} \sum_{n\geq 0} q^{3n^2+2n}(1-q^{2n+1}) \quad \text{(apply (2.17))}$$

$$= \frac{1}{(q)_\infty^7} h_6 \quad \text{(let } n \to -n-1 \text{ in the second sum).}$$

For $\Phi_{7_4}(q)$, it suffices to prove

$$S_{7_4} := \sum_{a,b,c,d,e,f,g\geq 0} \frac{q^{2f^2+f+2g^2+g+ab+ag+bc+bg+cd+cf+cg+de+df+ef+a+b+c+d+e}}{(q)_a(q)_b(q)_c(q)_d(q)_e(q)_f(q)_g(q)_{a+g}(q)_{b+g}(q)_{c+f}(q)_{c+g}(q)_{d+f}(q)_{e+f}}$$
$$= \frac{1}{(q)_\infty^7} h_4^2.$$

Thus,

$$S_{7_4} = \frac{1}{(q)_\infty^2} \sum_{a,b,c,d,e,i,j,k,l\geq 0} (-1)^{j+l} \frac{q^{i^2+i+\frac{j(j+1)}{2}+k^2+k+\frac{l(l+1)}{2}+ij+kl+di+e(i+j)+bk+c(k+l)+ab+bc+cd+de+a+b+c+d+e}}{(q)_a(q)_b(q)_c(q)_d(q)_e(q)_i(q)_j(q)_k(q)_l(q)_{a+k}(q)_{b+k+l}(q)_{c+i}(q)_{d+i+j}}$$

(apply Lemma 2.1 to the f-sum and g-sum with $n=4$)

$$= \frac{1}{(q)_\infty^7} \sum_{i,j,k,l\geq 0} (-1)^{j+l} \frac{q^{i^2+i+\frac{j(j+1)}{2}+k^2+k+\frac{l(l+1)}{2}+ij+kl}}{(q)_i(q)_j(q)_k(q)_l}$$

(evaluate the e-sum, d-sum, c-sum, b-sum and a-sum with (2.1))

$$= \frac{1}{(q)_\infty^7} \sum_{i,j,k,l\geq 0} (-1)^{j+l} \frac{q^{i^2+i+\frac{j(j-1)}{2}+k^2+k+\frac{l(l-1)}{2}-ij-kl}}{(q)_{i-j}(q)_j(q)_{k-l}(q)_l} \quad \text{(shift } i \to i-j \text{ and } k \to k-l\text{)}$$
$$= \frac{1}{(q)_\infty^7} \sum_{i,k\geq 0} (-1)^{i+k} q^{\frac{i(i+1)}{2}+\frac{k(k+1)}{2}} \quad \text{(evaluate the } j\text{-sum and } l\text{-sum with (2.4), then use (2.6))}$$
$$= \frac{1}{(q)_\infty^7} h_4^2 \quad \text{(as in the proof of (4.2)).}$$

For $\Phi_{7_7}(q)$, it suffices to prove

$$S_{7_7} := \sum_{a,b,c,d,e,f,g\geq 0} (-1)^{e+f+g} \frac{q^{\frac{3e^2}{2}+\frac{e}{2}+\frac{3f^2}{2}+\frac{f}{2}+\frac{3g^2}{2}+\frac{g}{2}+ab+ad+ae+af+bf+cd+cg+de+dg+a+b+c+d}}{(q)_a(q)_b(q)_c(q)_d(q)_e(q)_f(q)_g(q)_{a+e}(q)_{d+e}(q)_{a+f}(q)_{b+f}(q)_{c+g}(q)_{d+g}}$$
$$= \frac{1}{(q)_\infty^4}.$$

Thus,

$$S_{7_7} = \frac{1}{(q)_\infty^3} \sum_{a,b,c,d,e,f,g\geq 0} (-1)^{e+f+g} \frac{q^{\frac{e^2}{2}+\frac{e}{2}+\frac{f^2}{2}+\frac{f}{2}+\frac{g^2}{2}+\frac{g}{2}+ab+ad+ae+bf+cd+cg+a+b+c+d}}{(q)_a(q)_b(q)_c(q)_d(q)_e(q)_f(q)_g(q)_{d+e}(q)_{a+f}(q)_{d+g}}$$
(apply Lemma 2.1 to e-sum, f-sum and g-sum with $n = 3$)
$$= \frac{1}{(q)_\infty^7} \sum_{e,f,g\geq 0} (-1)^{e+f+g} \frac{q^{\frac{e(e+1)}{2}+\frac{f(f+1)}{2}+\frac{g(g+1)}{2}}}{(q)_e(q)_f(q)_g}$$
(evaluate the c-sum, b-sum, a-sum and d-sum using (2.1))
$$= \frac{1}{(q)_\infty^4} \quad \text{(evaluate the } e\text{-sum, } f\text{-sum and } g\text{-sum using (2.2)).}$$

For $\Phi_{8_2}(q)$, it suffices to prove

$$S_{8_2} := \sum_{a,b,c,d,e,f,g,h\geq 0} (-1)^{b} \frac{q^{3a^2+2a+\frac{b(3b+1)}{2}+ad+ae+af+ag+ah+bc+bd+cd+de+ef+fg+gh+c+d+e+f+g+h}}{(q)_a(q)_b(q)_c(q)_d(q)_e(q)_f(q)_g(q)_h(q)_{b+c}(q)_{b+d}(q)_{a+d}(q)_{a+e}(q)_{a+f}(q)_{a+g}(q)_{a+h}}$$
$$= \frac{1}{(q)_\infty^7} h_6.$$

Thus,

$$S_{8_2} = \frac{1}{(q)_\infty} \sum_{a,b,c,d,e,f,g,h\geq 0} (-1)^b \frac{q^{3a^2+2a+\frac{b(b+1)}{2}+ad+ae+af+ag+ah+bc+cd+de+ef+fg+gh+c+d+e+f+g+h}}{(q)_a(q)_b(q)_c(q)_d(q)_e(q)_f(q)_g(q)_h(q)_{b+d}(q)_{a+d}(q)_{a+e}(q)_{a+f}(q)_{a+g}(q)_{a+h}}$$

(apply Lemma 2.5 to the b-sum with $n=3$)

$$= \frac{1}{(q)_\infty^2} \sum_{a,b,d,e,f,g,h\geq 0} (-1)^b \frac{q^{3a^2+2a+\frac{b(b+1)}{2}+ad+ae+af+ag+ah+de+ef+fg+gh+d+e+f+g+h}}{(q)_a(q)_b(q)_d(q)_e(q)_f(q)_g(q)_h(q)_{a+d}(q)_{a+e}(q)_{a+f}(q)_{a+g}(q)_{a+h}}$$

(evaluate the c-sum with (2.1))

$$= \frac{1}{(q)_\infty} \sum_{a,d,e,f,g,h\geq 0} \frac{q^{3a^2+2a+ad+ae+af+ag+ah+de+ef+fg+gh+d+e+f+g+h}}{(q)_a(q)_d(q)_e(q)_f(q)_g(q)_h(q)_{a+d}(q)_{a+e}(q)_{a+f}(q)_{a+g}(q)_{a+h}}$$

(evaluate the b-sum with (2.2))

$$= \frac{1}{(q)_\infty^7} h_6 \quad \text{(let } (a,d,e,f,g,h) \to (a,c,d,e,f,g)\text{, then follow the proof of (4.4)).}$$

For $\Phi_{8_4}(q)$, it suffices to prove

$$S_{8_4} := \sum_{a,b,c,d,e,f,g,h\geq 0} (-1)^e \frac{q^{\frac{e(3e+1)}{2}+ae+be+ab+a+b+c^2+bc+d^2+bd+f^2+af+g^2+ag+h^2+ah}}{(q)_a(q)_b(q)_c(q)_d(q)_e(q)_f(q)_g(q)_h(q)_{a+e}(q)_{a+f}(q)_{a+g}(q)_{a+h}(q)_{b+c}(q)_{b+d}(q)_{b+e}}$$
$$= \frac{1}{(q)_\infty^7}.$$

Thus,

$$S_{8_4} = \frac{1}{(q)_\infty^5} \sum_{a,b,e\geq 0} (-1)^e \frac{q^{\frac{e(3e+1)}{2}+ae+be+ab+a+b}}{(q)_a(q)_b(q)_e(q)_{a+e}(q)_{b+e}}$$

(evaluate the c-sum, d-sum, f-sum, g-sum and h-sum with (2.3))

$$= \frac{1}{(q)_\infty^6} \sum_{a,b,e\geq 0} (-1)^e \frac{q^{\frac{e(e+1)}{2}+be+ab+a+b}}{(q)_a(q)_b(q)_e(q)_{a+e}} \quad \text{(apply Lemma 2.1 to the } e\text{-sum with } n=3)$$

$$= \frac{1}{(q)_\infty^8} \sum_{e\geq 0} (-1)^e \frac{q^{\frac{e(e+1)}{2}}}{(q)_e} \quad \text{(evaluate the } b\text{-sum and } a\text{-sum with (2.1))}$$

$$= \frac{1}{(q)_\infty^7} \quad \text{(evaluate the } e\text{-sum with (2.2)).}$$

For $\Phi_{T(2,p)}(q)$ with $p > 0$, it suffices to prove

$$S_{T(2,p)} := \sum_{a,b_1,\ldots,b_{2p}\geq 0} (-1)^a \frac{q^{\frac{a((2p+1)a+(2p-1))}{2}+a\sum\limits_{n=1}^{2p} b_n+\sum\limits_{n=1}^{2p-1} b_nb_{n+1}+\sum\limits_{n=1}^{2p} b_n}}{(q)_a \prod\limits_{n=1}^{2p} (q)_{b_n}(q)_{a+b_n}}$$
$$= \frac{1}{(q)_\infty^{2p+1}} h_{2p+1}.$$

Thus,

$$S_{T(2,p)} = \frac{1}{(q)_\infty} \sum_{i_1,\ldots,i_{2p-1},b_1,\ldots,b_{2p}\geq 0} (-1)^{\sum\limits_{k=1}^{2p-1}\sum\limits_{j=1}^{k} i_j} \frac{q^{\frac{1}{2}\sum\limits_{k=1}^{2p-1}\big(\sum\limits_{j=1}^{k} i_j\big)\big(1+\sum\limits_{j=1}^{k} i_j\big)+\sum\limits_{k=2}^{2p}\sum\limits_{j=1}^{k-1} b_k i_j+\sum\limits_{k=1}^{2p} b_k+\sum\limits_{k=1}^{2p-1} b_kb_{k+1}}}{\prod\limits_{k=1}^{2p-1} (q)_{i_k} \prod\limits_{k=1}^{2p-1} (q)_{b_k+\sum\limits_{j=1}^{k} i_j} \prod\limits_{k=1}^{2p} (q)_{b_k}}$$

(apply Lemma 2.1 to the a-sum with $n = 2p+1$)

$$= \frac{1}{(q)_\infty^{2p+1}} \sum_{i_1,\ldots,i_{2p-1}\geq 0} (-1)^{\sum\limits_{k=1}^{2p-1}\sum\limits_{j=1}^{k} i_j} \frac{q^{\frac{1}{2}\sum\limits_{k=1}^{2p-1}\big(\sum\limits_{j=1}^{k} i_j\big)\big(1+\sum\limits_{j=1}^{k} i_j\big)}}{\prod\limits_{k=1}^{2p-1} (q)_{i_k}}$$

(evaluate the b_{2p}-sum, b_{2p-1}-sum, … and b_1-sum with (2.1))

$$= \frac{1}{(q)_\infty^{2p+1}} \sum_{i_1,\ldots,i_{2p-1}\geq 0} (-1)^{\sum\limits_{k=1}^{p} i_{2k-1}} \frac{q^{\frac{1}{2}\sum\limits_{k=1}^{p} i_{2k-1}(i_{2k-1}+1)+\sum\limits_{k=1}^{p} i_{2k-1}\sum\limits_{j=1}^{k-1} i_{2j}+\sum\limits_{k=1}^{p-1}\big(\sum\limits_{j=1}^{k} i_{2j}\big)\big(\sum\limits_{j=1}^{k} i_{2j}+1\big)}}{\prod\limits_{k=1}^{p} (q)_{i_{2k-1}} \prod\limits_{k=1}^{p-1} (q)_{i_{2k}-i_{2k-1}}}$$

(shift $i_{2k} \to i_{2k} - i_{2k-1}$ for $k = 1, 2, \ldots, p-1$)

$$= \frac{1}{(q)_\infty^{2p+1}} \sum_{i_2,i_4,\ldots,i_{2p-2},i_{2p-1}\geq 0} (-1)^{i_{2p-1}} \frac{q^{\frac{i_{2p-1}(i_{2p-1}+1)}{2}+i_{2p-1}\sum\limits_{j=1}^{p-1} i_{2j}+\sum\limits_{k=1}^{p-1}\big(\sum\limits_{j=1}^{k} i_{2j}\big)\big(\sum\limits_{j=1}^{k} i_{2j}+1\big)}}{(q)_{i_{2p-1}}}$$
$$\times \prod_{k=1}^{p-1} \frac{(q)_{\sum\limits_{j=1}^{k} i_{2j}}}{(q)_{\sum\limits_{j=1}^{k-1} i_{2j}} (q)_{i_{2k}}}$$

(evaluate the i_1-sum, i_3-sum, … and i_{2p-3}-sum with (2.4), then simplify)

$$
\begin{aligned}
&= \frac{1}{(q)_\infty^{2p+1}} \sum_{i_2,i_4,\dots,i_{2p-2},i_{2p-1}\geq 0} (-1)^{i_{2p-1}} q^{\frac{i_{2p-1}(i_{2p-1}+1)}{2}+i_{2p-1}\sum\limits_{j=1}^{p-1} i_{2j}+\sum\limits_{k=1}^{p-1}\big(\sum\limits_{j=1}^{k} i_{2j}\big)\big(\sum\limits_{j=1}^{k} i_{2j}+1\big)} \\
&\times \frac{(q)_{\sum_{k=1}^{p-1} i_{2k}}}{(q)_{i_{2p-1}} \prod\limits_{k=1}^{p-1} (q)_{i_{2k}}} \quad \text{(simplify the product)} \\
&= \frac{1}{(q)_\infty^{2p}} \sum_{i_2,i_4,\dots,i_{2p-2}\geq 0} \frac{q^{\sum_{k=1}^{p-1}(\sum_{j=1}^{k} i_{2j})(\sum_{j=1}^{k} i_{2j}+1)}}{\prod\limits_{k=1}^{p-1}(q)_{i_{2k}}} \quad \text{(evaluate the } i_{2p-1}\text{-sum with (2.2))} \\
&= \frac{1}{(q)_\infty^{2p+1}} h_{2p+1}
\end{aligned}
$$

(let $n_j = i_{2j}$ and $k = p$ in (1.2) and $q \to q^{\frac{2p+1}{2}}$, $z = q^{\frac{2p-1}{2}}$ in (2.5)).

Before turning to the $\Phi_{K_p}(q)$, $p > 0$ case, we note that for any given set of indices $\{i_1, i_2, \dots, i_n\}$, if we let $i_2 \to i_2 - i_1$, $i_3 \to i_3 - i_2$, $\dots$, $i_n \to i_n - i_{n-1}$, then

$$
\sum_{k=1}^{n}\Bigg(\sum_{j=1}^{k} i_j\Bigg)\Bigg(1+\sum_{j=1}^{k} i_j\Bigg) - \frac{1}{2}\sum_{k=1}^{n} i_k(i_k+1) - \sum_{k=1}^{n} i_k \sum_{j=1}^{k-1} i_j = \sum_{k=1}^{n-1} i_k(i_k+1) + \frac{1}{2} i_n(i_n+1). \quad (4.5)
$$

For $\Phi_{K_p}(q)$ with $p > 0$, it suffices to prove

$$
\begin{aligned}
S^+_{K_p} &:= \sum_{a,b,c_1,\dots,c_{2p-1}\geq 0} \frac{q^{pa^2+(p-1)a+a\sum_{n=1}^{2p-1} c_n+b^2+bc_1+\sum_{n=1}^{2p-2} c_n c_{n+1}+\sum_{n=1}^{2p-1} c_n}}{(q)_a (q)_b (q)_{b+c_1} \prod\limits_{n=1}^{2p-1} (q)_{c_n} (q)_{a+c_n}} \\
&= \frac{1}{(q)_\infty^{2p+1}} h_{2p}.
\end{aligned}
$$

Thus,

$$S^+_{K_p} = \frac{1}{(q)_\infty} \sum_{a,c_1,\dots,c_{2p-1}\geq 0} \frac{q^{pa^2+(p-1)a+a\sum\limits_{k=1}^{2p-1} c_k + \sum\limits_{k=1}^{2p-2} c_k c_{k+1} + \sum\limits_{k=1}^{2p-1} c_k}}{(q)_a \prod\limits_{k=1}^{2p-1} (q)_{c_k} (q)_{a+c_k}}$$

(evaluate the b-sum with (2.3))

$$= \frac{1}{(q)^2_\infty} \sum_{i_1,\dots,i_{2p-2},c_1,\dots,c_{2p-1}\geq 0} (-1)^{\sum\limits_{k=1}^{2p-2}\sum\limits_{j=1}^{k} i_j} \frac{q^{\frac{1}{2}\sum\limits_{k=1}^{2p-2}\left(\sum\limits_{j=1}^{k} i_j\right)\left(1+\sum\limits_{j=1}^{k} i_j\right) + \sum\limits_{k=2}^{2p-1}\sum\limits_{j=1}^{k-1} c_k i_j + \sum\limits_{k=1}^{2p-2} c_k c_{k+1} + \sum\limits_{k=1}^{2p-1} c_k}}{\prod\limits_{k=1}^{2p-2} (q)_{i_k} \prod\limits_{k=1}^{2p-2} (q)_{c_k + \sum\limits_{j=1}^{k} i_j} \prod\limits_{k=1}^{2p-1} (q)_{c_k}}$$

(apply Lemma 2.1 to the a-sum with $n = 2p$)

$$= \frac{1}{(q)^{2p+1}_\infty} \sum_{i_1,\dots,i_{2p-2}\geq 0} (-1)^{\sum\limits_{k=1}^{2p-2}\sum\limits_{j=1}^{k} i_j} \frac{q^{\frac{1}{2}\sum\limits_{k=1}^{2p-2}\left(\sum\limits_{j=1}^{k} i_j\right)\left(1+\sum\limits_{j=1}^{k} i_j\right)}}{\prod\limits_{k=1}^{2p-2} (q)_{i_k}}$$

(evaluate the c_{2p-1}-sum, c_{2p-2}-sum, $\dots$ and c_1-sum with (2.1))

$$= \frac{1}{(q)^{2p+1}_\infty} \sum_{i_1,\dots,i_{2p-2}\geq 0} (-1)^{\sum\limits_{k=1}^{p-1} i_{2k}} \frac{q^{\sum\limits_{k=1}^{p-1}\left(\sum\limits_{j=1}^{k} i_{2j-1}\right)\left(1+\sum\limits_{j=1}^{k} i_{2j-1}\right) + \frac{1}{2}\sum\limits_{k=1}^{p-1} i_{2k}(i_{2k}-1) - \sum\limits_{k=1}^{p-1} i_{2k} \sum\limits_{j=1}^{k} i_{2j-1}}}{\prod\limits_{k=1}^{p-1} (q)_{i_{2k-1}-i_{2k}} (q)_{i_{2k}}}$$

(shift $i_{2k-1} \to i_{2k-1} - i_{2k}$ for $k = 1, 2, \dots, p-1$)

$$= \frac{1}{(q)^{2p+1}_\infty} \sum_{i_1,i_3,\dots,i_{2p-3}\geq 0} (-1)^{\sum\limits_{k=1}^{p-1} i_{2k-1}} q^{\sum\limits_{k=1}^{p-1}\left(\sum\limits_{j=1}^{k} i_{2j-1}\right)\left(1+\sum\limits_{j=1}^{k} i_{2j-1}\right) - \frac{1}{2}\sum\limits_{k=1}^{p-1} i_{2k-1}(i_{2k-1}+1) - \sum\limits_{k=1}^{p-1} i_{2k-1} \sum\limits_{j=1}^{k-1} i_{2j-1}}$$

$$\times \frac{\prod\limits_{k=1}^{p-1} (q)_{\sum\limits_{j=1}^{k} i_{2j-1}}}{\prod\limits_{k=1}^{p-1} (q)_{i_{2k-1}} (q)_{\sum\limits_{j=1}^{k-1} i_{2j-1}}}$$

(evaluate the i_2-sum, i_4-sum, $\dots$ and i_{2p-2}-sum with (2.4), then use (2.6))

$$= \frac{1}{(q)^{2p+1}_\infty} \sum_{i_1,i_3\dots,i_{2p-3}\geq 0} (-1)^{\sum\limits_{k=1}^{p-1} i_{2k-1}} q^{\sum\limits_{k=1}^{p-1}\left(\sum\limits_{j=1}^{k} i_{2j-1}\right)\left(1+\sum\limits_{j=1}^{k} i_{2j-1}\right) - \frac{1}{2}\sum\limits_{k=1}^{p-1} i_{2k-1}(i_{2k-1}+1) - \sum\limits_{k=1}^{p-1} i_{2k-1} \sum\limits_{j=1}^{k-1} i_{2j-1}}$$

$$\times \frac{(q)_{\sum\limits_{k=1}^{p-1} i_{2k-1}}}{\prod\limits_{k=1}^{p-1} (q)_{i_{2k-1}}} \quad \text{(simplify the product)}$$

$$
\begin{aligned}
&= \frac{1}{(q)_\infty^{2p+1}} \sum_{i_1,i_3\ldots,i_{2p-3}\geq 0} (-1)^{i_{2p-3}} \frac{q^{\sum\limits_{k=1}^{p-2} i_{2k-1}(1+i_{2k-1})+\frac{1}{2}i_{2p-3}(i_{2p-3}+1)}(q)_{i_{2p-3}}}{(q)_{i_1}\prod\limits_{k=2}^{p-1}(q)_{i_{2k-1}-i_{2k-3}}} \\
&\text{(let } i_3 \to i_3 - i_1,\ i_5 \to i_5 - i_3,\ \ldots,\ i_{2p-3} \to i_{2p-3} - i_{2p-5}\text{, then apply (4.5))} \\
&= \frac{1}{(q)_\infty^{2p+1}} \sum_{n\geq 0} q^{pn^2+(p-1)n}(1-q^{2n+1}) \quad \text{(apply (2.18))} \\
&= \frac{1}{(q)_\infty^{2p+1}} h_{2p} \quad \text{(let } n \to -n-1 \text{ in the second sum).}
\end{aligned}
$$

For $\Phi_{-3_1}(q)$, it suffices to prove

$$
S_{-3_1} := \sum_{a,b,c\geq 0} \frac{q^{a+b^2+c^2+ab+ac}}{(q)_a(q)_b(q)_c(q)_{a+b}(q)_{a+c}} = \frac{1}{(q)_\infty^3}.
$$

Thus,

$$
\begin{aligned}
S_{-3_1} &= \frac{1}{(q)_\infty^2} \sum_{a\geq 0} \frac{q^a}{(q)_a} \quad \text{(evaluate the } b\text{-sum and } c\text{-sum with (2.3))} \\
&= \frac{1}{(q)_\infty^3} \quad \text{(evaluate the } a\text{-sum with (2.1)).}
\end{aligned}
$$

For $\Phi_{-7_7}(q)$, it suffices to prove

$$
\begin{aligned}
S_{-7_7} := &\sum_{a,b,c,d,e,f,g\geq 0} (-1)^{e+f} \frac{q^{d^2+\frac{e(3e+1)}{2}+\frac{f(3f+1)}{2}+g^2+ab+ad+ae+bc+be+bf+cf+cg+a+b+c}}{(q)_a(q)_b(q)_c(q)_d(q)_e(q)_f(q)_g(q)_{a+d}(q)_{a+e}(q)_{b+e}(q)_{b+f}(q)_{c+f}(q)_{c+g}} \\
= &\frac{1}{(q)_\infty^5}.
\end{aligned}
$$

Thus,

$$
\begin{aligned}
S_{-7_7} &= \frac{1}{(q)_\infty^2} \sum_{a,b,c,e,f\geq 0} (-1)^{e+f} \frac{q^{\frac{e(3e+1)}{2}+\frac{f(3f+1)}{2}+ab+ae+bc+be+bf+cf+a+b+c}}{(q)_a(q)_b(q)_c(q)_e(q)_f(q)_{a+e}(q)_{b+e}(q)_{b+f}(q)_{c+f}} \\
&\text{(evaluate the } d\text{-sum and } g\text{-sum with (2.3))} \\
&= \frac{1}{(q)_\infty^4} \sum_{a,b,c,e,f\geq 0} (-1)^{e+f} \frac{q^{\frac{e(e+1)}{2}+\frac{f(f+1)}{2}+ab+bc+be+cf+a+b+c}}{(q)_a(q)_b(q)_c(q)_e(q)_f(q)_{a+e}(q)_{b+f}} \\
&\text{(apply Lemma 2.1 to the } e\text{-sum and } f\text{-sum with } n=3) \\
&= \frac{1}{(q)_\infty^5} \sum_{a,b,e,f\geq 0} (-1)^{e+f} \frac{q^{\frac{e(e+1)}{2}+\frac{f(f+1)}{2}+ab+be+a+b}}{(q)_a(q)_b(q)_e(q)_f(q)_{a+e}} \quad \text{(evaluate the } c\text{-sum with (2.1))} \\
&= \frac{1}{(q)_\infty^7} \sum_{e,f\geq 0} (-1)^{e+f} \frac{q^{\frac{e(e+1)}{2}+\frac{f(f+1)}{2}}}{(q)_e(q)_f} \quad \text{(evaluate the } b\text{-sum and } a\text{-sum with (2.1))} \\
&= \frac{1}{(q)_\infty^5} \quad \text{(evaluate the } e\text{-sum and } f\text{-sum with (2.2)).}
\end{aligned}
$$

For $\Phi_{-8_4}(q)$, it suffices to prove

$$
\begin{aligned}
S_{-8_4} &:= \sum_{a,b,c,d,e,f,g,h\geq 0} (-1)^g \frac{q^{\frac{g(5g+3)}{2}+2h^2+ab+ah+bc+bh+cd+cg+ch+de+dg+ef+eg+fg+a+b+c+d+e+f+h}}{(q)_a(q)_b(q)_c(q)_d(q)_e(q)_f(q)_g(q)_h(q)_{a+h}(q)_{b+h}(q)_{c+g}(q)_{c+h}(q)_{d+g}(q)_{e+g}(q)_{f+g}} \\
&= \frac{1}{(q)_\infty^8} h_4 h_5.
\end{aligned}
$$

Thus,

$$
\begin{aligned}
&S_{-8_4} \\
&= \frac{1}{(q)_\infty} \sum_{a,b,c,d,e,f,g,i,j\geq 0} (-1)^{g+j} \frac{q^{\frac{g(5g+3)}{2}+i(i+1)+\frac{j(j+1)}{2}+ij+ab+a(i+j)+bc+bi+cd+cg+de+dg+ef+eg+fg+a+b+c+d+e+f}}{(q)_a(q)_b(q)_c(q)_d(q)_e(q)_f(q)_g(q)_i(q)_j(q)_{b+i+j}(q)_{c+g}(q)_{c+i}(q)_{d+g}(q)_{e+g}(q)_{f+g}} \\
&\text{(apply Lemma 2.1 to the } h\text{-sum with } n=4) \\
&= \frac{1}{(q)_\infty^3} \sum_{c,d,e,f,g,i,j\geq 0} (-1)^{g+j} \frac{q^{\frac{g(5g+3)}{2}+i(i+1)+\frac{j(j+1)}{2}+ij+cd+cg+de+dg+ef+eg+fg+c+d+e+f}}{(q)_c(q)_d(q)_e(q)_f(q)_g(q)_i(q)_j(q)_{c+g}(q)_{d+g}(q)_{e+g}(q)_{f+g}} \\
&\text{(evaluate the } a\text{-sum and } b\text{-sum with (2.1))}
\end{aligned}
$$

$$= \frac{1}{(q)_\infty^3} \sum_{c,d,e,f,g,i,j\geq 0} (-1)^{g+j} \frac{q^{\frac{g(5g+3)}{2}+i(i+1)+\frac{j(j-1)}{2}-ij+cd+cg+de+dg+ef+eg+fg+c+d+e+f}}{(q)_c(q)_d(q)_e(q)_f(q)_g(q)_{i-j}(q)_j(q)_{c+g}(q)_{d+g}(q)_{e+g}(q)_{f+g}}$$

(shift $i \to i-j$)

$$= \frac{1}{(q)_\infty^3} \sum_{c,d,e,f,g,i\geq 0} (-1)^{g+i} \frac{q^{\frac{g(5g+3)}{2}+\frac{i(i+1)}{2}+cd+cg+de+dg+ef+eg+fg+c+d+e+f}}{(q)_c(q)_d(q)_e(q)_f(q)_g(q)_{c+g}(q)_{d+g}(q)_{e+g}(q)_{f+g}}$$

(evaluate the j-sum with (2.4), then apply (2.6))

$$= \frac{1}{(q)_\infty^4} \sum_{c,d,e,f,i,r,s,t\geq 0} (-1)^{r+t+i} \frac{q^{\frac{3r(r+1)}{2}+s(s+1)+\frac{t(t+1)}{2}+2rs+rt+st}}{(q)_c(q)_d(q)_e(q)_f(q)_r(q)_s(q)_t(q)_{c+r}(q)_{d+r+s}(q)_{e+r+s+t}}$$

$$\times q^{\frac{i(i+1)}{2}+cd+de+dr+ef+e(r+s)+f(r+s+t)+c+d+e+f}$$

(apply Lemma 2.1 to the g-sum with $n=5$)

$$= \frac{1}{(q)_\infty^8} \sum_{i,r,s,t\geq 0} (-1)^{r+t+i} \frac{q^{\frac{3r(r+1)}{2}+s(s+1)+\frac{t(t+1)}{2}+2rs+rt+st+\frac{i(i+1)}{2}}}{(q)_r(q)_s(q)_t}$$

(evaluate the f-sum, e-sum, d-sum and c-sum with (2.1))

$$= \frac{1}{(q)_\infty^8} \sum_{i,r,s,t\geq 0} (-1)^{r+t+i} \frac{q^{\frac{r(r+1)}{2}+s(s+1)+\frac{t(t+1)}{2}+st+\frac{i(i+1)}{2}}}{(q)_r(q)_{s-r}(q)_t} \quad \text{(shift } s \to s-r\text{)}$$

$$= \frac{1}{(q)_\infty^8} \sum_{i,s,t\geq 0} (-1)^{t+i} \frac{q^{s(s+1)+\frac{t(t+1)}{2}+st+\frac{i(i+1)}{2}}}{(q)_t} \quad \text{(evaluate the } r\text{-sum with (2.4))}$$

$$= \frac{1}{(q)_\infty^7} \sum_{i\geq 0} (-1)^i q^{\frac{i(i+1)}{2}} \sum_{s\geq 0} \frac{q^{s(s+1)}}{(q)_s}$$

(evaluate the t-sum with (2.2), then simplify)

$$= \frac{1}{(q)_\infty^8} h_4 h_5$$

(by (1.1), $q \to q^{5/2}$, $z = -q^{3/2}$ in (2.5) and the proof of (4.2)).

□

Acknowledgements

The second author would like to thank Stavros Garoufalidis for his talk on June 23, 2011 at the Institut Mathématiques de Jussieu and the subsequent correspondence, the organizers (in particular, Ling Long and Holly Swisher) of the conference "Applications of Automorphic Forms in Number Theory and Combinatorics", April 12–15, 2014 at LSU, Oliver Dasbach and Mustafa Hajij for their helpful comments and suggestions and George Andrews for his continued interest and encouragement. He would also like to thank the organizers (in particular, Adam Sikora) of the workshop "Low-dimensional topology and number theory", August 17-23, 2014 at Oberwolfach for the opportunity to present this work. Finally, both authors are very grateful to the referee for their extremely careful reading of our paper.

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