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Low-dimensional Topology and Number Theory

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17 August – 23 August 2014

Abstract. The workshop brought together topologists and number theorists with the intent of exploring the many tantalizing connections between these areas.

Mathematics Subject Classification (2010): 57xx, 11xx.

Introduction by the Organisers

The workshop Low-Dimensional Topology and Number Theory, organised by Paul E. Gunnels (Amherst), Walter Neumann (New York), Don Zagier (Bonn) and Adam S. Sikora (New York) was held August 17th – August 23rd, 2014. This meeting was a part of a long-standing tradition of collaboration of researchers in these areas. The preceding meeting under the same name took place in Oberwolfach two years ago. At the moment the topic of most active interaction between topologists and number theorists are quantum invariants of 3-manifolds and their asymptotics. This year's meeting showed significant progress in the field. The workshop was attended by many researchers from around the world, at different stages of their careers – from graduate students to some of the most established scientific leaders in their areas. The participants represented diverse backgrounds. There were 24 talks ranging intertwined with informal discussions.

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Workshop: Low-dimensional Topology and Number Theory

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Abstracts

Quantum Chern-Simons theory for $SL(n,\mathbb{C})$

Joergen Ellegaard Andersen

By applying geometric quantization w.r.t. a real polarization (depending on a complex structure on the surface) of the $SL(n,\mathbb{C})$ moduli space of flat connections on a closed genus g surface we obtain in joint work with Niels Gammelgaard a representation of the mapping class group Γ of the surface:

$$\rho^t_{SL(n,\mathbb{C})} : \Gamma \to \mathbb{P}\mathrm{Aut}(C^\infty(M, L^k))$$

where M is the moduli space of flat $SU(n)$-connections on the surface, L is the $SU(n)$ Chern-Simons line bundle on M and $2\pi k = \mathrm{Re}(t)$.

In joint work with Rinat Kashaev we have extended these representations to the Ptolemy groupoid and we have given a mathematical construction of the associated TQFT by providing the explicit charged tetrahedral partition function in terms of Faddeev's quantum dilogarithm Ψ_b. In case the surface is the torus T^2 the formula reads:

$$T_b : C^\infty(T^2, L^k) \otimes C^\infty(T^2, L^k) \to C^\infty(T^2, L^k) \otimes C^\infty(T^2, L^k)$$

given by $T_b =$

$$\Psi_b(u_1 - v_1 + v_2)\Psi_{\bar{b}}(\bar{u}_1 - \bar{v}_1 + \bar{v}_2)e^{2\pi i(v_1 u_2 + \bar{v}_1 \bar{u}_2)}$$

and $T_b(a, c) =$

$$e^{-\pi i c_b^2(4(a-c)+1)/b} e^{4\pi i c_b(c u_2 - a u_1)} e^{-4\pi i c_{\bar{b}}(\bar{c}\bar{u}_2 - \bar{a}\bar{u}_1)} T_b e^{-4\pi i c_b(a v_2 - c u_2)} e^{4\pi i c_{\bar{b}}(\bar{a}\bar{v}_2 - \bar{c}\bar{u}_2)}$$

where

$$\begin{aligned} u &= \frac{1}{2\pi b}\nabla_{\frac{\partial}{\partial v'}} - ibu' \\ \bar{u} &= i\bar{b}u' \\ \bar{v} &= -\frac{1}{2\pi \bar{b}}\nabla_{\frac{\partial}{\partial u'}} - i\bar{b}v' \\ v &= ibv' \end{aligned}$$

where (u', v') are log-coordinates on T^2 and ∇ is the connection in L^k with curvature $F_\nabla = -2\pi i du' dv'$.

Analytic properties of fiber knot invariants in Seifert spaces

Gaëtan Borot
(joint work with Bertrand Eynard)

1. Introduction

If G is a compact Lie group and R an irreducible representation of G, quantum invariants $V_R^G(K)$ for knots K in $\mathbb{S}_3$ are usually defined as Laurent polynomials in a variable q. Such a definition is not possible for knots in 3-manifolds M different from $\mathbb{S}_3$. We focus on one of the approaches to bypass this impossibility. If M is a rational homology 3-sphere, one can build the generating series of LMO invariants of M [11, 2], which is a graph-valued formal series in $\hbar$. And, if K is a knot in M, we also have the Kontsevich integral of K, which is another graph-valued formal series in $\hbar$, that generates finite type invariants of K. Then, the group G defines a weight system that assigns "values" to a graph, and we can end up defining the free energy of a closed 3-manifold $\ln Z^G(M) \in \hbar^{-2}\mathbb{Q}[[\hbar]]$, and invariants $V_R^G(K) \in \hbar^{-|R|}\mathbb{Q}[[\hbar]]$ for a knot K in M. When $M = \mathbb{S}_3$, the latter coincide with the $\hbar \to 0$ expansion of the quantum invariants $V_R^G(K)$ evaluated at $q = e^{\hbar/d_G}$, where $d_G = 1$ or 2 depending on the group G. Comparing to Witten's approach to quantum invariants, those formal series realize the perturbative expansion of Chern-Simons theory in M around the trivial flat connection. We will call them *perturbative invariants*.

There is a theory of large $N = \mathrm{rank}(G)$ transitions, which relates Chern-Simons theory on M and topological strings on a target space depending on M. It is very explicit at least for $\mathbb{S}_3$ (or lens spaces) : the string target space is the resolved conifold (modified by a rational framing). From this perspective, one considers the infinite series of Lie groups:

$$A_N = \mathrm{SU}(N+1), \qquad B_N = \mathrm{SO}(2N+1), \qquad C_N = \mathrm{Sp}(2N), \qquad D_N = \mathrm{SO}(2N)$$

and let $N \to \infty$ while $u = N\hbar$ is kept fixed. Then, studying the weight system of G, one can show that free energy of M can be decomposed as a formal series in $\hbar$:

$$\ln Z^G(M) = \sum_g N^{2-2g}\, F_g(M; u) \tag{1}$$

with $F_g^G(M; u) \in \mathbb{Q}[[u]]$. Here, $g \in \mathbb{N}$ for the A series, while $g \in \mathbb{N}/2$ for the BCD series. By a theorem of [10], for any rational homology sphere M, the so-called genus g free energy $F_g^A(M; u)$ has a non-zero radius of convergence independent of $g \in \mathbb{N}$. It is thus analytic in the neighborhood of $u = 0$, and one may wonder where are the singularities in the u complex plane, and what is their nature.

For a knot K from the large N perspective, it is clever to arrange the perturbative invariants differently. Firstly, we can extend the definition of V_R^G by linearity to any R in the representation ring of G. In particular, we can form the invariant $\overline{W}_k^G[K; m_1, \ldots, m_k](u)$ for the virtual representation associated to the power sum

$p_{m_1}\cdots p_{m_k}$. Secondly, we introduce its connected parts $W_k^G[K;m_1,\ldots,m_k](u)$, uniquely defined by the property:

$$\overline{W}_k^G[K;m_1,\ldots,m_k](u) = \sum_{J \text{ partition of } [1,k]} \prod_i W^G_{|J_i|}[K;m_j,\ j\in J_i](u) \tag{2}$$

The collection of $W_\ell^G[K][m_1,\ldots,m_\ell]$ for $\ell \leq k$ and $m_1,\ldots,m_\ell \geq 0$ contain the same information as the collection of $V_R^G(K)$ for highest weight representations with less than k rows. The advantage is that now, we have a decomposition:

$$W_k^G[K;m_1,\ldots,m_k](u) = \sum_g N^{2-2g-k}\, W^G_{g,k}[K;m_1,\ldots,m_k](u) \tag{3}$$

where $W^G_{g,k}[K;m_1,\ldots,m_k](u) \in \mathbb{Q}[[u]]$, and the range of g depends on G as before. We can also ask if it is possible to upgrade the definition of $W_{g,k}$'s from formal series of u to analytic functions of u, and if their singularities in the u-complex plane can be described. For knots in $\mathbb{S}_3$, we know that the genus g correlators are entire functions of e^u. For knots in other manifolds, one does expect singularities, but very few is known so far. Our work bring some insight for simple knots in the simplest manifolds which are not $\mathbb{S}_3$, namely fiber knots in Seifert spaces.

2. The matrix model for Seifert spaces

The Seifert spaces M we consider are $\mathbb{S}_1$-bundles over $\mathbb{S}_2$ with orbifold points $\mathbb{Z}_{a_1},\ldots,\mathbb{Z}_{a_r}$ [13]. The orbifold Euler characteristics $\chi = 2 - r + \sum_{i=1}^r 1/a_i$ will play an important role. The geometry also depends on another rational number, denoted σ. There is a finite list of Seifert spaces with $\chi > 0$, and they all appear as quotients of $\mathbb{S}_3$ by a finite group of isometries. Up to central extension by a cyclic group, these are the binary polyhedral groups, which have an ADE classification. The most famous example, $\mathbb{S}_3/E_8$, is the Poincaré integral homology sphere.

There are several ways to compute the LMO generating series of Seifert spaces [1, 12, 3, 4], and they all yield:

$$Z^G(M) = C^G(M)\int_{\mathbb{R}^N}\prod_{\alpha>0}\big[\mathrm{sh}(\alpha\cdot\mathbf{t}/2)\big]^{2-r}\prod_{i=1}^r \mathrm{sh}(\alpha\cdot\mathbf{t}/2a_i)\prod_{j=1}^N e^{-N\sigma t_j^2/2u}\mathrm{d}t_j \tag{4}$$

The product ranges over all positive roots of the Lie algebra of G, and the Cartan subalgebra has been identified with $\mathbb{R}^N$. We assume $\mathrm{sign}(u\sigma) > 0$ so that the integral converges, and $C^G(M)$ is a known constant. The invariants of the knot K_{a_i} along the exceptional fiber of order a_i can be computed as an expectation value with respect to the measure integrated in (4):

$$\overline{W}_k^G[K_{a_i};m_1,\ldots,m_k](u) = \Big\langle \prod_{j=1}^k \mathrm{Tr}(e^{\mathbf{t}m_j/a_i})\Big\rangle_G$$

This is conveniently repackaged in the functions:

$$\overline{w}_k^G(x_1,\ldots,x_k;u) = \Big\langle \prod_{j=1}^k \mathrm{Tr}\,\frac{x_j}{x_j - e^{\mathbf{t}/a}}\Big\rangle_G, \qquad a = \mathrm{lcm}(a_1,\ldots,a_r)$$

upon series expansion when $x_i \to 0$, and their corresponding connected parts $w_k(x_1, \ldots, x_k; u)$.

Our strategy is to study the large N asymptotics of this model, considering u not as a formal parameter, but as a fixed value. In the case $\chi > 0$ and u small enough, this model falls into the scope of the theory developed in [6], which establish existence of an *asymptotic* expansion of the form (1) for the free energy, and (3) for the correlators. Once this is guarantees, a general result of [5] shows that $W^G_{g,k}[K_{a_i}; m_1, \ldots, m_k](u)$ are computed by the topological recursion of Eynard and Orantin [9], for any a_i. This is a universal recursion on $2g-2+k$, whose initial data is $w^G_{0,1}$ and $w^G_{0,2}$, to which one should add $w^G_{1/2,1}$ for the BCD series. This initial data contain for instance the large N limit of the HOMFLY-PT and Kauffman invariants of the knots K_{a_i}. The main task of our work [7] is the computation of this initial data, and I will describe now some of our results on $w^G_{0,1}(x; u)$, which is called *the spectral curve.*

We find that $w^A_{0,1}(x; 2u) = w^{BCD}_{0,1}(x; u)$, and the function $w(x) = w^A_{0,1}(x; u)$ is characterized by a scalar, non-local Riemann-Hilbert problem with boundary conditions $\lim_{x\to 0} w(x) = 0$ and $\lim_{x\to\infty} w(x) = 1$. To find the solution, we need to understand the monodromy group of $w(x)$. It is in general infinite, but we can cut it down and surprisingly, finite Weyl groups come on the stage.

Theorem 1. [7] *Assume $\chi > 0$. Set $c = e^{-\chi u/2a}$ and $y(x) = -xe^{\chi u/a(w(x)-1/2)}$. There exists a non-zero $(v_j)_{j\in\mathbb{Z}_a}$ such that $Y(x) = \prod_{j\in\mathbb{Z}_a} y^{v_j}(e^{2i\pi j/a}x)$ is an algebraic function, i.e. $P_c(x, Y(x); c) = 0$ for some polynomial P_c. It defines a curve Σ, and there exists a finite Weyl group $\mathfrak{W}$ acting on the sheets of the covering $x : \Sigma \to \mathbb{C}$. We denote $d = \deg_x P_c$ and $d' = \deg_Y P_c$, and $h = \mathrm{genus}(\Sigma)$.*

Seifert	$(a_1, \ldots, a_r)$	d	d'	h	$\mathfrak{W}$
lens spaces	(a_1, a_2)	a	$a_1 + a_2$	$(a_1 - 1)(a_2 - 1)$	$A_{a_1+a_2-1}$
$\mathbb{S}_3/D_{p+2}$ (p even)	$(2, 2, p)$	$2p$	$p + 1$	0	A_p
$\mathbb{S}_3/D_{p+2}$ (p odd)	$(2, 2, p)$	$4p$	$2(p + 1)$	$2p + 1$	D_{p+1}
$\mathbb{S}_3/E_6$	$(2, 3, 3)$	8	8	5	D_4
$\mathbb{S}_3/E_7$	$(2, 3, 4)$	36	27	46	E_6
$\mathbb{S}_3/E_8$	$(2, 3, 5)$	540	270	1471	E_8

We can also describe explicitly the Newton polygon of P_c, and the coefficients on the boundary of the Newton polygon, which are monomials in c. The inner coefficients satisfy algebraic constraints in terms of c, and if these constraints were independent – a property we have not been able to check, given their complexity –, it would show P that depends on c algebraically. The fact that a always coincides with the Coxeter number of $\mathfrak{W}$ is quite mysterious. We can also show, for $\chi < 0$, that there does not exist $v \neq 0$ such that $Y(x)$ is an algebraic function. We think that the algebraicity of the spectral curve for $\chi > 0$ is the sign for existence of a nice dual geometry in topological strings, still to be identified.

3. Analyticity of perturbative knot invariants

The initial data for lens spaces is well-understood [8, 5]. We could also obtain it for the prismatic varieties $(2, 2, p)$ with p even. In particular, the spectral curve is rational and can be parametrized:

$$x(z) = z\,\frac{z^p - \kappa^2}{z^p\kappa^2 - 1}, \qquad y(z) = -\frac{(z^{p/2} - \kappa)(z^{p/2}\kappa + 1)}{(z^{p/2} + \kappa)(z^{p/2}\kappa - 1)}, \qquad \frac{2\kappa^{1+1/p}}{1 + \kappa^2} = e^{-u/4p^2}$$

with the choice of branch $0 < \kappa < 1$ for $u > 0$. We deduce a complete description of the analytic properties of the perturbative invariants of the fiber knots.

Theorem 2. [7] *Assume p even. Let $L = \mathbb{Q}(\kappa)[\kappa_2]$ where*

$$\kappa_2 = \sqrt{(1 + \kappa^{-2})((p + 1) - (p - 1)\kappa^2)}$$

All the A series, genus g perturbative invariants of the fiber knots K_2 and K_p in $\mathbb{S}_3/D_{p+2}$ belong to L.

For the BCD series, a similar result holds with u replaced by $u/2$ in the definition of L. It would be interesting to find a geometric meaning L. We actually propose:

Conjecture 3.1. *For any Seifert space with $\chi > 0$, there exists a finite degree extension $L(M)$ of $\mathbb{Q}(e^u)$, so that all A series genus g perturbative invariants of fiber knots in M belong to $L(M)$.*

References

[1] D. Bar-Natan and R. Lawrence, *A rational surgery formula for the LMO invariant*, Israel J. Math., **140** 29–60 (2004), math.GT/0007045.

[2] D. Bar-Natan, S. Garoufalidis, L. Rozansky, and D.P. Thurston, *The Aarhus integral of rational homology 3-spheres III: the relation with the Le-Murakami-Ohtsuki invariant*, Selecta Mathematica, **10** 305–324 (2004), math.QA/9808013.

[3] C. Beasley and E. Witten, *Non-abelian localization for Chern-Simons theory*, J. Diff. Geom **70** 183–323 (2005), hep-th/0503126.

[4] M. Blau and G. Thompson, *Chern-Simons theory on Seifert manifolds*, JHEP **33** (2013), hep-th/1306.3381.

[5] G. Borot, B. Eynard, and N. Orantin, *Abstract loop equations, topological recursion, and applications*, 2013, math-ph/1303.5808.

[6] G. Borot, A. Guionnet, and K. Kozlowski, *Large-N asymptotic expansion for mean field models with Coulomb gas interaction*, 2013, math-ph/1312.6664.

[7] G. Borot, B. Eynard, with an appendix by A. Weisse, *Root systems, spectral curves, and analysis of a Chern-Simons matrix model for Seifert fibered spaces*, math-ph/1407.4500

[8] A. Brini, B. Eynard, and M. Mariño, *Torus knots and mirror symmetry*, Annales Henri Poincaré (2012), hep-th/1105.2012.

[9] B. Eynard and N. Orantin, *Invariants of algebraic curves and topological expansion*, Commun. Num. Theor. Phys. **1** (2007), 347-452, math-ph/0702045.

[10] S. Garoufalidis, T.T.Q. Lê, and M. Mariño, *Analyticity of the free energy of a closed 3-manifold*, SIGMA **4** 080 (2008), math.GT/0809.2572.

[11] T.Q.T. Lê, J. Murakami and T. Ohtsuki, *On a universal perturbative invariant of 3-manifolds*, Topology **37** 3 539–574 (1998), q-alg.9512002.

[12] M. Mariño, *Chern-Simons theory, matrix integrals, and perturbative three-manifold invariants*, Commun. Math. Phys., **253** 25–49 (2004), hep-th/0207096.

[13] H. Seifert, *Topology of 3-dimensional fibered spaces*, Academic Press (1980).

Unramified Extensions of Imaginary Quadratic Fields

Nigel Boston

Throughout, K will denote an imaginary quadratic field and L/K an unramified Galois extension, Galois over $\mathbb{Q}$, and we denote $\mathrm{Gal}(L/K)$ by H and $\mathrm{Gal}(L/\mathbb{Q})$ by G. We let the maximal unramified extension of K be K^{ur}. We are interested in which H arise, and how often, as K varies.

We begin with the case of H abelian. From Class Field Theory the maximal unramified abelian extension of K has Galois group $Cl(K)$, the ideal class group of K, and so its maximal subextension of p-power degree has Galois group the Sylow-p subgroup, $Cl_p(K)$. Cohen and Lenstra [5] conjectured that the proportion of K with $Cl_p(K)$ isomorphic to a given finite abelian p-group H (p odd) is $\frac{1/|\mathrm{Aut}(H)|}{W_p}$ where $W_p = \prod(1-p^{-n})^{-1}$.

Note that there are only finitely many K with $Cl(K)$ of any given order, so only finitely many with $Cl(K)$ isomorphic to a given H. [9] notes that there are 93 K with class number 27, none of which have $Cl_3(K) \cong (\mathbb{Z}/3)^3$. Likewise, a given finite group H arises as $\mathrm{Gal}(K^{ur}/K)$ only finitely often. Secondly, the case $p = 2$ is different but [7] conjectures that the 4-rank behaves similarly (proven in [6]). Finally, the Cohen-Lenstra conjecture is equivalent to the tidy conjecture that, for any given finite abelian p-group H, the expected number of unramified H-extensions of K is 1.

As for more general H, we begin with p-groups. Let $K^{p,ur}$ denote the maximal unramified p-extension of K. For p odd, [2] conjectures that the proportion of K with $\mathrm{Gal}(K^{p,ur}/K)$ isomorphic to H is $\frac{1/|\mathrm{Aut}_\sigma(H)|}{W_p}$, for any finite Schur σ-group H (meaning that $d(H) = r(H)$ and H has an automorphism of order 2 acting by inversion on its finite abelianization H^{ab} [8]), where $\mathrm{Aut}_\sigma(H)$ is the centralizer of σ in $\mathrm{Aut}(H)$. For a given finite p-group H, this is equivalent to saying that the expected number of unramified H-extensions Galois over $\mathbb{Q}$ is 1 [4]. Without the Galois assumption it is $f_{a(H)}(p^{d(H)})$, where $a(H)$ is a certain invariant of H and f_n the nth Rogers-Szëgo polynomial.

As for completely general H, since the inertia subgroups of G have order 1 or 2, intersect H trivially, and generate G (Minkowski), we see that H embeds with index 2 in a group G generated by the involutions outside H. Equivalently, H has an automorphism σ of order 2 such that H is generated by the elements inverted by σ. We then say H has a GI-extension G.

The first thing to determine is which groups do not have a GI-extension. [3] showed that exactly 2 groups of order 64 do not and also found infinitely many such 2-groups. [1] proved that 2-generated 2-groups have at most 1 GI-extension. Another source of groups with no GI-extension are Frobenius groups such as $F(q,d) := \{x \mapsto ax + b \mid a, b \in \mathbb{F}_q, a^d = 1\}$, where Alberts proved that this has a GI-extension if and only if there exists g such that $p^g \equiv -1 \pmod{d}$. All instances of groups of order ≤ 100 with no GI-extension can be explained by extending this result or by p-groups coming from [3] or not being Schur. [10] found $\mathrm{Gal}(K^{ur}/K)$

for all K of discriminant ≥ -719 under GRH, allowing testing of natural heuristics for how often H is a quotient of this Galois group.

References

[1] N. Boston, *Embedding 2-groups in groups generated by involutions*, J.Algebra **300** (2006), 73–76.

[2] N. Boston, M.R. Bush, and F. Hajir, *Heuristics for p-class towers of imaginary quadratic fields*, preprint.

[3] N. Boston and C.R. Leedham-Green, *Counterexamples to a conjecture of Lemmermeyer*, Arch. Math. (Basel) **72** (1999), 177–179.

[4] N. Boston, D. Ross, and M.M. Wood, preprint.

[5] H. Cohen and H. W. Lenstra, Jr., *Heuristics on class groups of number fields*, pp. 33–62 in: Number theory, Noordwijkerhout 1983, LNM **1068**, Springer, Berlin, 1984.

[6] E. Fouvry and J. Klüners, *On the 4-rank of class groups of quadratic number fields*, Invent. Math **167** (2007), 455–513.

[7] F. Gerth, *The 4-class ranks of quadratic fields*, Invent. Math **77** (1984), 489–515.

[8] H. Koch and B.B. Venkov, *Über den p-Klassenkörperturm eines imaginär-quadratischen Zahlkörpers*, Soc. Math. France, Astérisque **24-25** (1975), 57–67.

[9] C. McLeman, *p-groups as class groups of quadratic imaginary number fields*, preprint.

[10] K. Yamamura, *Maximal unramified extensions of imaginary quadratic number fields of small conductors*, J. Theor. Nombres Bordeaux **6** (1994), 261–272.

Topological Consequences of Actions of 3-manifold Groups on the Reals

Steven Boyer

(joint work with Adam Clay)

The use of linear representations has long been a crucial ingredient in the study of 3-dimensional topology and geometry. More recently it has become apparent that the existence of a representation of the fundamental group of a 3-manifold with values in $\mathrm{Homeo}_+(\mathbb{R})$ has strong topological consequences for the manifold[1]. This talk discussed some of these consequences in the context of graph manifolds.

Three-manifold groups which admit non-trivial representations to $\mathrm{Homeo}_+(\mathbb{R})$ are left orderable, and conversely [BRW, Theorem 1.1]. Studying the existence of such representations through left orders is helpful in various ways. For instance, the topology defined by Sikora [Si] on the set of left orders is compact metric, which can be exploited. Also, the *non*-left orderability of a finitely generated group can be determined algorithmically, a fact which was used to produce the first examples of hyperbolic 3-manifolds which do not support co-oriented taut foliations. See [CD] (and compare with [RSS]).

Theorem 1. ([BC1]) *Let W be a graph manifold. The following statements are equivalent.*

(1) *There is a homomorphism $\rho : \pi_1(W) \to Homeo_+(\mathbb{R})$ with non-trivial image.*

[1]We assume that such representations are non-trivial in the sense that they have non-trivial image.

(2) $\pi_1(W)$ *is left orderable.*
(3) W *admits a co-oriented taut foliation.*

Conjecture 1 of [BGW] contends that an irreducible rational homology 3-sphere W is not an L-space if and only if its fundamental group is left orderable. Closed, orientable 3-manifolds which admit smooth co-oriented taut foliations are not L-spaces ([OSz, Theorem 1.4]) and Ozsváth and Szabó have asked whether the converse is true. Recently Juhász conjectured that it is. See [Ju, Conjecture 5].

While it is unknown whether manifolds admitting topological co-oriented taut foliations can be L-spaces, there is enough control in our constructions for us to obtain the following consequence of the theorem above. (See [BC2, Theorem 1.1] for a proof.)

Theorem 2. ([BC2]) *If a graph manifold rational homology* 3*-sphere has a left orderable fundamental group, then it is not an L-space.*

A co-dimension 1 foliation in a graph manifold W is called *horizontal* if it is transverse to the Seifert fibres in each piece of W. We can refine our results by restricting attention to foliations of this type. Let $\mathrm{sh}(1) : \mathbb{R} \to \mathbb{R}$ denote the homeomorphism $\mathrm{sh}(1)(x) = x + 1$.

Theorem 3. ([BC1]) *Let W be a graph manifold rational homology* 3*-sphere. The following statements are equivalent.*
(1) W *admits a co-oriented horizontal foliation.*
(2) $\pi_1(W)$ *admits a left order in which the class of any Seifert fibre in any piece of W is cofinal.*
(3) *There is a homomorphism $\rho : \pi_1(W) \to Homeo_+(\mathbb{R})$ such that the image of the class of any Seifert fibre in any piece of W is conjugate in $Homeo_+(\mathbb{R})$ to $sh(\pm 1)$.*

Here is another refinement. Call a co-oriented taut foliation *strongly rational* if up to isotopy it intersects each JSJ torus of W in a fibration by simple closed curves. Up to arranging that the Seifert structures on pieces homeomorphic to twisted I-bundles over the Klein bottle have orientable base orbifolds, a strongly rational co-oriented taut foliation is necessarily horizontal. One interest in considering strongly rational co-oriented taut foliations is that graph manifolds which admit them also admit smooth strongly rational co-oriented taut foliations.

It was shown in [BB] that a graph manifold integer homology 3-sphere admits a strongly rational co-oriented taut foliation if and only if it is neither the 3-sphere nor the Poincaré homology 3-sphere. For the general graph manifold rational homology 3-sphere we have the following result.

Theorem 4. ([BC1]) *Let W be a graph manifold rational homology* 3*-sphere. The following statements are equivalent.*
(1) W *admits a strongly rational co-oriented taut foliation.*
(2) $\pi_1(W)$ *admits a left order $\mathfrak{o}$ in which the class of any Seifert fibre in any piece of W is cofinal and there is an $\mathfrak{o}$-convex normal subgroup C of $\pi_1(W)$ such that $C \cap \pi_1(T) \cong \mathbb{Z}$ for each JSJ-torus T in W.*

(3) *There is a homomorphism* $\rho : \pi_1(W) \to Homeo_+(\mathbb{R})$ *such that the image of the class of a ny Seifert fibre in any of* W *is conjugate in* $Homeo_+(\mathbb{R})$ *to* $sh(\pm 1)$ *and* $ker(\rho|\pi_1(T)) \cong \mathbb{Z}$ *for each JSJ-torus* T *in* W.

The strategy for establishing these theorems is based on two technical results, a slope detection theorem and a gluing theorem. More precisely, it is possible to define four different methods of detecting a family of slopes on the boundary of a Seifert fibred manifold M: using representations, using left orders, using foliations, and using Heegaard-Floer homology. (It would also be possible to detect slopes in a fifth way, via contact structures, but we do not consider this in [BC1].)

Let N_t be a Seifert fibred space with base orbifold a 2-disk and two singular fibres with Seifert invariants $(t, 1)$ and $(t, t-1)$. Thus N_2 is the twisted I-bundle over the Klein bottle.

Here is a special case of the slope detection theorem.

Theorem 5. ([BC1]) *Let* M *be a Seifert manifold with base orbifold* $P(a_1, \dots, a_n)$ *or* $Q(a_1, \dots, a_n)$ *where* P *is a punctured* 2*-sphere and* Q *is a punctured projective plane. Let* $\emptyset \neq \partial M = T_1 \cup \ldots \cup T_r$ *be the decomposition of* ∂M *into its toral boundary components. Let* $[\alpha_j]$ *be a slope on* T_j *and set* $[\alpha_*] = ([\alpha_1], [\alpha_2], \dots, [\alpha_r])$. *The following statements are equivalent.*

(1) $[\alpha_*]$ *is detected by some co-oriented taut foliation on* M.

(2) $[\alpha_*]$ *is detected by some left order on* $\pi_1(M)$.

(3) *If no* $[\alpha_j]$ *is vertical in a Seifert piece incident to* T_j, $[\alpha_*]$ *is detected by some homomorphism* $\rho : \pi_1(M) \to \widetilde{Homeo}_+(S^1)$.

(4) *If* $[\alpha_*]$ *is rational, then there is an integer* $t \geq 2$ *such that if* W *is any manifold obtained by attaching* r *copies of* N_t *to* M *such that the rational longitude of* N_t *is identified with* $[\alpha_j]$ *for each* j, *then* W *is not an L-space.*

Next we have a special case of the gluing theorem.

Theorem 6. ([BC1]) *Let* W *be a graph manifold rational homology* 3*-sphere with JSJ pieces* $M_1, \dots, M_n$. *For each piece* M_i *and* m*-tuple of slopes* $[\alpha_*] = ([\alpha_1], [\alpha_2], ...$
$\dots, [\alpha_m])$, *one for each of the JSJ tori, let* $[\alpha_*^{(i)}]$ *be the sub-tuple of* $[\alpha_*]$ *corresponding to the boundary components of* M_i. *Then,*

(1) W *admits a co-oriented taut foliation if and only if there is an* m*-tuple of slopes* $[\alpha_*]$ *such that for each* i, $[\alpha_*^{(i)}]$ *is detected by some co-oriented taut foliation on* M_i.

(2) $\pi_1(W)$ *is left orderable if and only if there is an* m*-tuple of slopes* $[\alpha_*]$ *such that for each* i, $[\alpha_*^{(i)}]$ *is detected by some left order on* $\pi_1(M_i)$.

It is of great interest to prove an analogue of the gluing theorem in the context of L-spaces.

References

[BB] M. Boileau and S. Boyer, *Graph manifolds* $\mathbb{Z}$*-homology* 3*-spheres and taut foliations*, preprint 2013, arXiv:1303.5264.

[BC1] S. Boyer and A. Clay, *Foliations, orders, representations, L-spaces and graph manifolds*, preprint 2014, arXiv:1401.7726.
[BC2] ———, *Graph manifolds with left orderable fundamental groups are not L-spaces*, preprint 2014.
[BGW] S. Boyer, C. McA. Gordon, and L. Watson, *On L-spaces and left orderable fundamental groups*, Math. Ann. **356** (2013), 1213–1245.
[BRW] S. Boyer, D. Rolfsen, and B. Wiest, *Orderable 3-manifold groups*, Ann. Inst. Fourier **55** (2005), 243–288.
[CD] D. Calegari and N. Dunfield, *Laminations and groups of homeomorphisms of the circle*, Invent. Math. **152** (2003), 149–204.
[Ju] A. Juhász, *A survey of Heegaard Floer homology*, preprint 2013, arXiv:1310.3418.
[OSz] P. Ozsváth and Z. Szabó, *Holomorphic disks and genus bounds*, Geom. Topol. **8** (2004), 311–334 (electronic).
[RSS] R. Roberts, J. Shareshian and M. Stein, *Infinitely many hyperbolic 3-manifolds which contain no Reebless foliation*, J. Amer. Math. Soc. **16** (2003), 639–679.
[Si] A.S. Sikora, *Topology on the spaces of orderings of groups*, Bull. London Math. Soc. **36** (2004), 519–526.

Negative curves on surfaces

Ted Chinburg
(joint work with Matthew Stover)

This talk was about curves on surfaces with negative self intersection, which we will call negative curves. A well known problem concerning such curves is the *bounded negativity conjecture*. This states that if X is a smooth projective surface over $\mathbb{C}$, then there is a lower bound for the self intersection of any reduced irreducible curve on X.

For a survey of recent results related to the bounded negativity conjecture, see [1] and [2]. For example, in [1] it was shown that one cannot find a counterexample to the bounded negativity conjecture that involves only curves of bounded arithmetic genus. Further, it was also shown in [1] that if X is a Shimura surface uniformized by the product of upper half planes that is not a product of curves, one cannot find a counterexample to the bounded negativity conjecture using only totally geodesic curves. More recently, this was extended to all Shimura surfaces in [5, 6]. It follows from [4, Ex. V.1.10] that the bounded negativity conjecture does not hold over an algebraically closed field of positive characteristic.

The following more precise question was studied in [1]. Given integers m and g, what is the size of the set $\mathcal{C}_X(m,g)$ of curves C on X with arithmetic genus $g(C) = g$ for which $C^2 = m$? In [1] it was shown that for all $m < -1$ and $g \geq 0$, there is a complex projective surface for which $\mathcal{C}_X(m,g)$ is infinite. Question 4.3 of [1] asks if for each $g > 1$ there is always an X for which $\mathcal{C}_X(-1,g)$ is infinite.

In this paper we consider upper bounds on the size of the set

$$\mathcal{C}_X^-(g) = \bigcup_{m<0} \mathcal{C}_X(m,g)$$

of all negative curve of a given genus g, where X is a projective surface over an arbitrary field k. Let $b_1(X)$ denote the first betti number of X, which equals the dimension of the ℓ-adic cohomology group $H^1_{\text{ét}}(X, \mathbb{Q}_\ell)$ for $\ell \neq \text{char}(k)$. If k is the field of complex numbers, $b_1(X)$ is the dimension of Betti cohomology group $H^1(X, \mathbb{C})$, $b_1(X)/2$ equals the irregularity $q(X) = h^{0,1}(X)$ of X, and $q(X)$ also equals the dimensions of $H^1(X, \mathcal{O}_X)$ and $H^0(X, \Omega^1)$. Let $N(X)$ be the Néron Severi group of X, and let $\hat{\rho}(X)$ be the rank of $N(X)$ modulo torsion, i.e., the dimension of $N(X) \otimes_{\mathbb{Z}} \mathbb{Q}$ as a vector space over $\mathbb{Q}$. Note that $\hat{\rho}(X) \leq \rho(X)$, where $\rho(X)$ is the Picard number. We prove the following.

Theorem 1. *Let X be a smooth projective surface over a field k with first betti number $b_1(X)$ and let $\hat{\rho}(X) = \dim_{\mathbb{Q}}(N(X) \otimes_{\mathbb{Z}} \mathbb{Q})$. There is a universal constant $t > 1$ such that the set*

$$\bigcup_{g < b_1(X)/4} \mathcal{C}_X^-(g)$$

of all irreducible negative curves on X of arithmetic genus less than $b_1(X)/4$ has order less than $t^{\hat{\rho}(X)-1}$. In particular, this set is finite.

When k is an algebraically closed field of characteristic $p > 0$, this result is optimal. Indeed, this is closely related to the failure of the bounded negativity conjecture. Let X be the direct product $C \times C$, where C is a smooth absolutely irreducible curve of genus g defined over $\mathbb{F}_p$, and let D_n be the graph of $F^n : C \to C$, where F is the Frobenius automorphism. Then D_n is a reduced irreducible curve on X of arithmetic genus g, and [4, Ex. V.1.10] implies that $D_n^2 \to -\infty$ as $n \to \infty$. Note that $b_1(X) = 4g$. In particular, there are infinitely many distinct reduced irreducible curves on X of genus $g = b_1(X)/4$ with negative self-intersection. For the examples in [1] with $\mathcal{C}_X(m, g)$ infinite, $g \geq (b_1(X) - 2)/2$. We do not know if our result is optimal in characteristic zero.

To prove Theorem 1, we show that sufficiently large collections of negative curves on a surface must generate an effective ample divisor D with two irreducible components. Here, the number two is clearly optimal. More precisely, we prove the following.

Theorem 2. *There is a universal constant $t > 1$ with the following property. Let X be smooth projective surface over a field k, and let $\hat{\rho}(X) = \dim_{\mathbb{Q}}(N(X) \otimes_{\mathbb{Z}} \mathbb{Q})$. Suppose $\mathcal{F}$ is a set of irreducible curves on X with negative self-intersection. If $\mathcal{F}$ contains more than $t^{\hat{\rho}(X)-1}$ elements, then there are two curves $C_1, C_2 \in \mathcal{F}$ such that $aC_1 + bC_2$ is an ample divisor on X for some integers $0 < a, b \in \mathbb{Z}$.*

This theorem is proved by restating the result as a packing problem concerning N discs on the surface of the unit sphere in $\mathbb{R}^{\hat{\rho}(X)-1}$. We then bound the N for which this packing problem has a solution.

Theorem 1 is proved in the following way. For simplicity, we assume k has characteristic 0. If Theorem 1 were false, Theorem 2 would imply that there are two curves C_1 and C_2 on X of genus less than $b_1(X)/4$ such that $D = aC_1 + bC_2$ is ample for some $0 < a, b \in \mathbb{Z}$. By a Lefschetz theorem (see [3]), the induced map of étale fundamental groups

$$\pi_1(|D|, x) \to \pi_1(X, x)$$

at some geometric point $x \in D \cap X$ has image of finite index in $\pi_1(X)$. This implies that the natural morphism

$$\mathrm{Jac}(C_1^\sharp) \oplus \mathrm{Jac}(C_2^\sharp) \to \mathrm{Alb}(X)$$

is surjective, where $C_i^\sharp$ is the normalization of C_i and $\mathrm{Alb}(X)$ is the Albanese variety of X. Since

$$g(C_i) \geq g(C_i^\sharp) = \dim(\mathrm{Jac}(C_i^\sharp)),$$

this implies that

$$\max(g(C_1), g(C_2)) \geq \dim(\mathrm{Alb}(X))/2 = b_1(X)/4,$$

which is a contradiction.

We should note that it was shown in [7] that if the set $\mathcal{F}$ in Theorem 2 has more than $\rho(X)^2 + \rho(X) + 1$ elements, then there is a nonnegative integral linear combination of elements of elements of $\mathcal{F}$ that is ample. However, by the above arguments, this leads to replacing the genus bound $b_1(X)/4$ in Theorem 1 by the weaker bound $b_1(X)/(\rho(X)^2 + \rho(X) + 1)$. In particular, it is crucial for the proof of Theorem 2 that we reduce the number of curves from $\mathcal{F}$ involved in an ample divisor down to the minimum possible of two.

References

[1] T. Bauer, B. Harbourne, A. L. Knutsen, A. Küronya, S. Müller-Stach, X. Roulleau and T. Szemberg, *Negative curves on algebraic surfaces*, Duke Math. J. **162** (2013), no. 10, 1877-1894.

[2] T. Bauer, C. Bocci, S. Cooper, S. Di Rocco, M. Dumnicki, B. Harbourne, K. Jabbusch, A. L. Knutsen, A. Küronya, R. Miranda, J. Roé, H. Schenck, T. Szemberg and Z. Teitler, *Recent developments and open problems in linear series*, Contributions to algebraic geometry, (2012), EMS Ser. Congr. Rep., Eur. Math. Soc., Zürich, 93-140.

[3] J.-B. Bost, *Potential theory and Lefschetz theorems for arithmetic surfaces*, Ann. Sci. École Norm. Sup. (4) **32** (1999), no. 2, 241-312.

[4] R. Hartshorne, Algebraic geometry, Graduate Texts in Mathematics, No. 52, Springer-Verlag (1977).

[5] V. Koziarz and J. Maubon, *On the equidistribution of totally geodesic submanifolds in compact locally symmetric spaces and application to boundedness results for negative curves*, arXiv:1407.6561 (2014).

[6] M. Möller and D. Toledo, *Bounded negativity of self-intersection numbers of Shimura curves on Shimura surfaces*, arXiv:1407.5181 (2014).

[7] S. Müller-Stach, E. Viehweg and K. Zuo, *Relative proportionality for subvarieties of moduli spaces of $K3$ and abelian surfaces*, Pure Appl. Math. Q. **5** (2009), no. 3 part 2, 1161-1199.

Geometrically and diagrammatically maximal knots

Abhijit Champanerkar
(joint work with Ilya Kofman, Jessica Purcell)

In this paper, we prove the existence of families of hyperbolic knots and links that are, as precisely defined below, geometrically and diagrammatically maximal.

Volume density. For a hyperbolic knot or link K, let $\mathrm{vol}(K)$ denote the hyperbolic volume of $S^3 - K$ and $c(K)$ denote its crossing number. We call $\mathrm{vol}(K)/c(K)$ the *volume density* of K.

For any diagram of K, D. Thurston gave an upper bound for $\mathrm{vol}(K)$ by decomposing $S^3 - K$ into octahedra, placing one octahedron at each crossing, and pulling remaining vertices to $\pm\infty$. Any hyperbolic octahedron has volume bounded above by the volume of the regular ideal octahedron, $v_8 \approx 3.66386$. Hence

$$\frac{\mathrm{vol}(K)}{c(K)} \leq v_8. \tag{1}$$

Definition 1. *A sequence of knots or links K_n with $c(K_n) \to \infty$ is* geometrically maximal *if*

$$\lim_{n\to\infty} \frac{\mathrm{vol}(K_n)}{c(K_n)} = v_8.$$

Diagrammatic density. Similarly, for any non-split knot or link K, we call $2\pi \log \det(K)/c(K)$ its *diagrammatic density.* The following upper bound for the diagrammatic density comes from Kenyon's conjecture for planar graphs ([4]).

Conjecture 0.1. *If K is any knot or link,*

$$\frac{2\pi \log \det(K)}{c(K)} \leq v_8.$$

Definition 2. *A sequence of knots or links K_n with $c(K_n) \to \infty$ is* diagrammatically maximal *if*

$$\lim_{n\to\infty} \frac{2\pi \log \det(K_n)}{c(K_n)} = v_8.$$

Infinite weave. Let the infinite weave W be the infinite alternating link with the square lattice projection. We obtain a hyperbolic structure on $\mathbb{R}^3 - W$ by coning the square lattice to $\pm\infty$, which gives a tessellation by ideal hyperbolic octahedra. We obtain a complete hyperbolic structure by giving each of these ideal octahedra the structure of a regular ideal octahedron. See Figure 1. Therefore, a natural place to look for geometrically maximal knots is among those with geometry approaching $\mathbb{R}^3 - W$.

Definition 3. *Let $G(W)$ be the infinite square lattice. For any finite subgraph H, let ∂H be the set of vertices of H that share an edge with a vertex not in H. Let $|H|$ and $|\partial H|$ denote the number of vertices in the graph and in the set, respectively.*

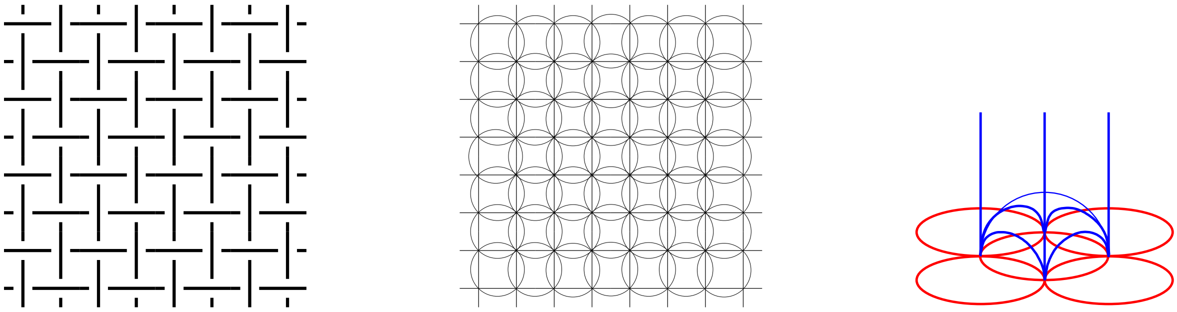

FIGURE 1. Infinite weave W, associated circle packing and top half of regular, ideal octahedron

An exhausting nested sequence of connected subgraphs, $\{H_n \subset G(W) : H_n \subset H_{n+1}, \cup_n H_n = G\}$, *is a* Følner sequence *for* $G(W)$ *if*

$$\lim_{n\to\infty} \frac{|\partial H_n|}{|H_n|} = 0.$$

Finally, we say H_n *is a* regular Følner sequence *for* $G(W)$ *if there is a sequence of convex sets* S_n *in* $\mathbb{R}^2$ *such that* $S_n \subset S_{n+1}$ *and each* S_n *contains a ball of radius* R_n*, with* $R_n \to \infty$ *as* $n \to \infty$*, and* $H_n = S_n \cap G(W)$.

For any link diagram K, let $G(K)$ denote the projection graph of the diagram. We show two strikingly similar ways to obtain geometrically and diagrammatically maximal links.

Theorem 1. *Let* K_n *be a sequence of links with prime, alternating, twist-reduced diagrams that contain no cycle of tangles, such that*

(1) *there are subgraphs* $G_n \subset G(K_n)$ *that form a regular Følner sequence for* $G(W)$*, and*
(2) $\lim\limits_{n\to\infty} |G_n|/c(K_n) = 1$.

Then K_n *is geometrically maximal:* $\lim\limits_{n\to\infty} \dfrac{\mathrm{vol}(K_n)}{c(K_n)} = v_8$.

Theorem 2. *Let* K_n *be any sequence of alternating links such that*

(1) *there are subgraphs* $G_n \subset G(K_n)$ *that form a Følner sequence for* $G(W)$*, and*
(2) $\lim\limits_{n\to\infty} |G_n|/c(K_n) = 1$.

Then K_n *is diagrammatically maximal:* $\lim\limits_{n\to\infty} \dfrac{2\pi \log \det(K_n)}{c(K_n)} = v_8$.

Ideas for the proof of Theorem 1 follow from unpublished results of Agol, (for example, mentioned in [2]). Theorem 1 applies to more general families of links.

Special case. We provide an explicit example of a family of knots and links satisfying the conditions of Theorems 1 and 2. A *weaving knot* $W(p,q)$ is the alternating knot or link with the same projection as the standard torus knot or

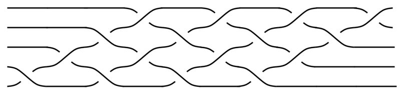

FIGURE 2. $W(5,4)$ is the closure of this braid.

link $T(p,q)$. For example see Figure 2. X.-S. Lin studied weaving knots to find knots with largest volume for fixed crossing number.

Another main result of this paper is to give asymptotically sharp, explicit volume bounds for $W(p,q)$ in terms of p and q alone. We use completely different methods than the ones used in the proof of Theorem 1. Using angle structures, we are also able to prove geometric convergence of the complements.

Theorem 3. (1) *If $p \geq 3$ and $q > 6$, then*

$$v_8\,(p-2)q\left(1-\frac{(2\pi)^2}{q^2}\right)^{3/2} \leq \operatorname{vol}(W(p,q)) \leq$$

(2) *As $p,q \to \infty$, $W(p,q)$ is geometrically maximal.*
(3) *As $p,q \to \infty$, $S^3 - W(p,q) \to \mathbb{R}^3 - W$ geometrically.*

Questions and Conjectures. We highlight some questions and conjectures arising out of our investigations.

Question 1. *Is any diagrammatically maximal sequence of knots geometrically maximal, and vice versa?*

Conjecture 0.2. *For any alternating hyperbolic link K,*

$$\operatorname{vol}(K) < 2\pi \log \det(K).$$

Remark 1. *Using SnapPy* [1], *we have verified Conjectures 0.1 and 0.2 for alternating knots up to 16 crossings and for weaving knots $W(p,q)$, for $3 \leq p \leq 50$ and $2 \leq q \leq 50$.*

The proof of Theorem 1 uses the following main ingredients: (1) Guts of checkerboard surfaces to get volumes bounds in terms of right-angled polyhedra, (2) Circle pattern rigidity to approach the circle pattern for the square grid, and (3) Volume decay to control the asymptotics of the lower volume bound. Theorem 2 follows from results on asymptotic enumeration of spanning trees for the square lattice.

REFERENCES

[1] Marc Culler, Nathan M. Dunfield, and Jeffrey R. Weeks, *SnapPy, a computer program for studying the geometry and topology of 3 -manifolds*, `http://snappy.computop.org`.

[2] Stavros Garoufalidis and Thang T. Q. Lê, *Asymptotics of the colored Jones function of a knot*, Geom. Topol. **15** (2011), no. 4, 2135–2180. MR 2860990

[3] Jim Hoste and Morwen Thistlethwaite, *Knotscape 1.01*, available at `http://www.math.utk.edu/~morwen/knotscape.html`.

[4] Richard Kenyon, *Tiling a rectangle with the fewest squares*, J. Combin. Theory Ser. A **76** (1996), no. 2, 272–291. MR 1416017 (98d:05040)

A Spectral Perspective on Neumann-Zagier: extended abstract

Tudor Dan Dimofte

(joint work with Roland van der Veen)

This talk is based on joint work with Roland van der Veen, which appeared in a preliminary form in `arxiv.org/abs/1403.5215`. It is also heavily inspired by work of Gaiotto, Moore, and Neitzke (GMN). Our main philosophy, adopted from GMN, is that coordinates for the moduli space of $PGL(K,\mathbb{C})$ flat connections on a d-manifold M can be obtained from coordinates on the moduli space of *abelian* $GL(1,\mathbb{C})$ flat connections on a K-fold cover of M. Many deep properties of coordinates on $PGL(K,\mathbb{C})$ moduli spaces then trivialize when viewed in terms of the cover. In the context of Higgs bundles on surfaces (closely related to flat connections), such covers are called "spectral covers," whence the title of the talk.

We applied the spectral cover philosophy to a famous old result of Neumann and Zagier about the symplectic properties of Thurston's gluing equations for hyperbolic 3-manifolds. These symplectic properties have been central to the quantization of hyperbolic structures (more generally, flat $PGL(2,\mathbb{C})$ or $PGL(K,\mathbb{C})$ connections) on 3-manifolds, among many other contexts. We wanted to find an intuitive, topological proof of the symplectic properties, which would immediately allow them to be generalized beyond the cusped hyperbolic manifolds that Neumann and Zagier had considered.

Let us recall what the gluing equations look like. An ideal hyperbolic tetrahedron Δ has a triple (z,z',z'') of shape parameters assigned to its edges, equal on opposite edges, and satisfying $zz'z''=-1$ around each vertex (as well as $z+z'^{-1}-1=0$, which will not play a role here). Suppose that $M=S^3\backslash K$ is a knot complement with an ideal triangulation $M=\cup_{i=1}^N\Delta_i$. Topologically, this means that M is tiled by truncated tetrahedra, in such a way that the small triangles at truncated vertices come together to tile the torus boundary $\partial M=T^2$. Thurston's gluing equations take the form

$$
\begin{aligned}
\ell^2 &= \pm z^A z'^{A'} \qquad =: \pm \textstyle\prod_{i=1}^N z_i^{A_i} z_i'^{A_i'} \\
m^2 &= \pm z^B z'^{B'} \\
c_j &= \pm z^{C_j} z'^{C_j'} = 1 \qquad (j=1,\dots,N)\,,
\end{aligned} \tag{1}
$$

where z_i,z_i',z_i'' are the triples of shapes for each tetrahedron, ℓ^2 and m^2 are square-eigenvalues of the hyperbolic or $PGL(2,\mathbb{C})$ holonomy around the meridian and longitude cycles of ∂M (expressed as products of shapes on the angles subtended by a meridian or longitude path), and the c_j are products of shapes around each internal edge I_j of the triangulation, forced to equal one in order for the hyperbolic structure to glue up consistently. Neumann and Zagier proved that the $(N+2)\times(2N)$ matrix of exponents of the gluing equations has rank $N+1$ and obeys

$$
\begin{pmatrix} A & A' \\ B & B' \\ C & C' \end{pmatrix}
\begin{pmatrix} 0 & I_{N\times N} \\ -I_{N\times N} & 0 \end{pmatrix}
\begin{pmatrix} A & A' \\ B & B' \\ C & C' \end{pmatrix}^T
=
\begin{pmatrix} 0 & 2 & 0_{2\times N} \\ -2 & 0 & \\ 0_{N\times 2} & & 0_{N\times N} \end{pmatrix}. \tag{2}
$$

This "symplectic" property can be usefully rephrased in terms of moduli spaces. The space of framed flat $PGL(2,\mathbb{C})$ connections on the boundary of a tetrahedron has the (local) form $\mathcal{P}_\Delta = \{z, z', z'' \in \mathbb{C}^* \,|\, zz'z'' = -1\} \simeq \mathbb{C}^* \times \mathbb{C}^*$, with Atiyah-Bott symplectic form $\Omega_\Delta = d\log z \wedge d\log z'$. The corresponding space of framed flat $PGL(2,\mathbb{C})$ connections on $T^2 = \partial M$ has the form $\mathcal{P}_{T^2} = \{\ell^2, m^2\} \simeq \mathbb{C}^* \times \mathbb{C}^*$, with $\Omega_{T^2} = \frac{1}{2} d\log \ell^2 \wedge d\log m^2$. Then the symplectic property implies that $\mathcal{P}_{T^2}$ is a finite quotient of the symplectic quotient $\left(\prod_{i=1}^N \mathcal{P}_{\Delta_i}\right)//(\mathbb{C}^*)^{N-1}$, where $N-1$ independent c_j are used as moment maps.

We re-prove (and generalize) the symplectic properties using double covers of boundaries of 3-manifolds. For the torus $T^2 = \partial M$, we consider a trivial, disconnected double cover $\Sigma_{T^2} \xrightarrow{\pi} T^2$ (so $\Sigma_{T^2} = T^2 \sqcup T^2$). The homology $H_1(\Sigma_{T^2}, \mathbb{Z}) = \langle \alpha^+, \alpha^-, \beta^+, \beta^- \rangle$ is generated by lifts of the A- and B-cycles of T^2 to the two sheets of the cover. The *odd* homology (odd under deck transformations) is generated by $\alpha := \alpha^+ - \alpha^-$ and $\beta := \beta^+ - \beta^-$, with intersection form $\langle \alpha, \beta \rangle = 2$.

Dually, we may consider the moduli space $\mathcal{P}_{T^2}^{\mathrm{ab}}$ of abelian $GL(1,\mathbb{C})$ flat connections $\mathcal{A}$ on Σ_{T^2} that are also odd under deck transformations (meaning roughly that the determinant of the push-forward connection $\pi_*(\mathcal{A})$ is trivial). Coordinates x_λ on this space are given by computing abelian holonomies around cycles $\lambda \in H_1^-(\Sigma_{T^2}, \mathbb{Z})$, and satisfy $x_{\lambda+\lambda'} = x_\lambda x_{\lambda'}$. Moreover, the Atiyah-Bott Poisson bracket is just the intersection form, $\{x_\lambda, x_{\lambda'}\} = \langle \lambda, \lambda' \rangle x_\lambda x_{\lambda'}$. Thus, $\mathcal{P}_{T^2}^{\mathrm{ab}}$ has exactly the same form as the $PGL(2,\mathbb{C})$ space $\mathcal{P}_{T^2}$ if we identify $(\ell^2, m^2) = (x_\alpha, x_\beta)$.

Similarly, for a tetrahedron, we consider a double cover $\Sigma_\Delta \to \partial\Delta$ that is branched over four points, one in the middle of each tetrahedron face. Topologically, $\partial\Delta \simeq S^2$ and $\Sigma_\Delta \simeq T^2$. The odd homology $H_1^-(\Sigma_\Delta, \mathbb{Z}) = H_1(\Sigma_\Delta, \mathbb{Z}) = \mathbb{Z}^2$ is generated by three cycles $\gamma, \gamma', \gamma''$ with $\gamma + \gamma' + \gamma'' = 0$ and intersection form $\langle \gamma, \gamma' \rangle = 1$. Roughly, the cycles label the edges of Δ. Correspondingly, the space of abelian flat connections $\mathcal{P}_\Delta^{\mathrm{ab}} = \{x_\gamma, x_{\gamma'}, x_{\gamma''} \,|\, x_\gamma x_{\gamma'} x_{\gamma''} = 1\}$ looks almost identical to $\mathcal{P}_\Delta$, aside from a sign in the relation $x_\gamma x_{\gamma'} x_{\gamma''} = 1$. (This sign can be fixed by considering certain twisted abelian flat connections instead.)

An elementary topological argument now shows that when gluing N tetrahedra Δ_i to form a knot complement M, the odd homology groups of double covers of the boundaries are related by a lattice symplectic quotient with respect to the intersection form,

$$(3) \qquad H_1^-(\Sigma_{T^2}, \mathbb{Z}) \simeq H_1^-(\sqcup_{i=1}^N \Sigma_{\Delta_i}, \mathbb{Z})//G := \ker\langle G, * \rangle|_{H_1^-(\sqcup_{i=1}^N \Sigma_{\Delta_i}, \mathbb{Z})} \big/ G\,,$$

where G is the subgroup of $H_1^-(\sqcup_{i=1}^N \Sigma_{\Delta_i}, \mathbb{Z})//G$ generated by elements $\mu_j = \sum_{i=1}^N (C_{ji}\gamma_i + C'_{ji}\gamma'_i)$, one for every internal edge of the triangulation, with C and C' as in (1). This implies that the abelian moduli spaces $\mathcal{P}_{T^2}^{\mathrm{ab}}$ and $\prod_{i=1}^N \mathcal{P}_{\Delta_i}^{\mathrm{ab}}$ are also related by a holomorphic symplectic quotient, with respect to the Atiyah-Bott symplectic structure. This sort of topological gluing argument extends easily to prove symplectic properties of gluing equations for more general triangulated manifolds.

To tie everything together, we define (following GMN) a non-abelianization map

$$\Phi : \begin{array}{ccc} \mathcal{P}^{\mathrm{ab}}_{T^2} & \xrightarrow{\sim} & \mathcal{P}_{T^2} \\ \mathcal{P}^{\mathrm{ab}}_{\Delta} & \xrightarrow{\sim} & \mathcal{P}_{\Delta}\,, \end{array} \tag{4}$$

which is a symplectomorphism relating (algebraically open subsets of) spaces of abelian flat connections on covers of boundaries and $PGL(2,\mathbb{C})$ flat connections on the boundaries themselves. The non-abelianization map involves pushing forward an abelian flat connection, then including unipotent modifications to extend the push-forward smoothly over branch points. (The map is also defined for boundaries of much more general triangulated manifolds.) The non-abelianization map commutes with symplectic reduction, and lets us translate simple statements about symplectic gluing of abelian connections on covers to statements about symplectic gluing of $PGL(2,\mathbb{C})$ flat connections on boundaries of the 3-manifolds we're interested in.

Topology versus geometry of hyperbolic 3-manifolds

Nathan Dunfield

(joint work with Jeff Brock)

By Mostow rigidity, the geometric properties of a hyperbolic 3-manifold are completely determined by its underlying topology. In this lecture, I will explore the extent to which certain geometric and topological properties can be varied independently. The particular questions discussed are motivated by the work of [BV, Le, ABBGNRS, BSV], and so have connections to deep conjectures concerning torsion growth, behavior of Ray-Singer torsion in towers of covers, automorphic forms, etc. A specific focus will be some very recent joint work with Jeff Brock exploring the relationship between the Thurston and harmonic norms discovered by Bergeron-Şengün-Venkatesh. Specifically, we've just shown:

Theorem (Brock-D. 2014) *There exist closed hyperbolic 3-manifolds M_n with $b_1 = 1$ so that for all n:*

(a) $R^1_{Th}\big(M_n\big) \geq C_0 \exp\big(C_1 \mathrm{Vol}(M_n)\big)$.
(b) $\mathrm{inj(M_n)} \geq \epsilon_0$ *and* $\mathrm{Vol}(M_n) \to \infty$.

References

[ABBGNRS] M. Abert, N. Bergeron, I. Biringer, T. Gelander, N. Nikolov, J. Raimbault, and I. Samet. On the growth of L^2-invariants for sequences of lattices in Lie groups. `arXiv:1210.2961`.

[BSV] N. Bergeron, M. H. Şengün, A. Aenkatesh Torsion homology growth and cycle complexity of arithmetic manifolds. Preprint 2014, 47 pages. `arXiv:1401.6989`.

[BV] N. Bergeron and A. Venkatesh. The asymptotic growth of torsion homology for arithmetic groups. Preprint 2010, 49 pages. `arXiv:1004.1083`.

[Le] T. T. Q. Le. Hyperbolic volume, Mahler measure, and homology growth. Talk at *Interactions Between Hyperbolic Geometry, Quantum Topology*, Columbia University, June 2009. http://www.math.columbia.edu/ volconf09/notes/leconf.pdf

State integrals, q-series and their evaluations

Stavros Garoufalidis

I will talk about three sources of q-series in quantum topology: (a) tails of colored Jones polynomials of alternating knots (b) the 3D index of Dimofte-Gaiotto-Gukov, proven to be a topological invariant of hyperbolic knots and (c) state integrals whose integrand is the quantum dilogarithm of Faddeev.

The story starts with the tail

$$\sum_{n=0}^{\infty}(-1)^n \frac{q^{n(3n+1)/2}}{(q)_n^3}$$

and its radial limits, guessed at fist by joint work with Zagier and now proven to be related to the asymptotics of the Kashaev invariant of the 4_1 knot at roots of unity. The state integral of 4_1 and its expressions in terms of q-series and $\tilde{q}$-series gives a companion series $G(q)$ to $g(q)$, explains why the 3D index of 4_1 is $G(q)$ and explains the relation between the radial asymptotics of $g(q)$ and the Kashaev invariant of 4_1. The same holds for 5_2 and the (-2,3,7)-pretzel knot. This is joint work with Rinat Kashaev and Don Zagier.

Quantum modular form from torus knots

Kazuhiro Hikami

(joint work with Jeremy Lovejoy)

Quantum modular form (QMF) was introduced by Zagier [12] as a function $f : \mathbb{Q} \to \mathbb{C}$ such that

$$h_\gamma(x) := f(x) - \chi(\gamma)\,(c\tau + d)^{-k} f\left(\frac{ax+b}{cx+d}\right), \quad \gamma \in \Gamma \subset SL(2;\mathbb{Z}),$$

has “some nice property of continuity or analyticity”. A typical example of QMF is the Kontsevich–Zagier series

$$F(q) := \sum_{n=0}^{\infty}(q)_n, \tag{1}$$

where $(x)_n := (x;q)_n = \prod_{j=1}^{n}(1 - xq^{j-1})$. A transformation formula was given in [11] for $\phi(z) := e^{\frac{\pi i}{12}z}F(e^{2\pi i z})$ and $z \in \mathbb{Q}$

$$\phi(z) + \frac{1}{(iz)^{\frac{3}{2}}}\phi(-1/z) = \frac{\sqrt{3i}}{2\pi}\int_0^{i\infty}\frac{\eta(w)}{(w-z)^{\frac{3}{2}}}dw.$$

In [2] studied is a strongly unimodal sequence, $0 < a_1 < a_2 < \cdots < a_k > a_{k+1} > a_{k+2} > \cdots > a_s > 0$. Let $u(m,n)$ be the number of such sequences with $a_1 + \cdots + a_s = n$ and $s - 2k + 1 = m$. A generating function $U(x;q) := \sum_{m,n} u(m,n) x^m q^n$ is identified with

$$U(x;q) = \sum_{n=0}^{\infty} (-xq)_n (-x^{-1}q)_n q^{n+1} = q + q^2 + (x + 1 + x^{-1})q^3 + \cdots \tag{2}$$

This is a mixed mock modular form satisfying [2, 1]

$$\psi(z) + \frac{1}{(-iz)^{\frac{3}{2}}} \psi(-1/z) = \frac{\sqrt{3i}}{2\pi} \int_0^{i\infty} \frac{\eta(w)}{(w+z)^{\frac{3}{2}}} dw + \frac{\eta(z)^2}{2\sqrt{i}} \int_0^{i\infty} \frac{\eta(w)^3}{(w+z)^{\frac{1}{2}}} dw$$

for $\psi(z) := e^{-\frac{\pi i}{12} z} U(-1; e^{2\pi i z})$ and $z \in \mathbb{Q} \cup \mathbb{H}$.

It was further shown [2] that for $\zeta_N = e^{\frac{2\pi i}{N}}$

$$F(\zeta_N^{-1}) = U(-1; \zeta_N). \tag{3}$$

This identity can be seen from quantum topology as follows. Let $J_N(K;q)$ be the N-colored Jones polynomial for knot K normalized to be $J_N(\text{unknot}; q) = 1$. We have nice q-hypergeometric expressions for some knots K in literature, *e.g.* [3, 8, 9]

$$J_N(T_{(2,3)}; q) = q^{1-N} \sum_{n=0}^{\infty} q^{-nN} (q^{1-N})_n,$$
$$J_N(T^*_{(2,3)}; q) = \sum_{n=0}^{\infty} q^n (q^{1-N})_n (q^{1+N})_n,$$

where $T_{(s,t)}$ is torus knot, and K^* denotes a mirror image of knot K. The identity (3) follows immediately from the well known fact that $J_N(K^*; q) = J_N(K; q^{-1})$.

Our motivation is to study a generalization of (3). On the one hand, a generalization of the Kontsevich–Zagier series is studied in [5]. We define for $1 \le m \le t$

$$F_t^{(m)}(q) := q^t \sum_{k_1, \ldots, k_t = 0}^{\infty} (q)_{k_t} \, q^{{k_1}^2 + \cdots + k_{t-1}^2 + k_m + \cdots k_{t-1}} \prod_{j=1}^{t-1} \begin{bmatrix} k_{j+1} + \delta_{j,m-1} \\ k_j \end{bmatrix}_q . \tag{4}$$

This coincides with the N-colored Jones polynomial for torus knot,

$$F_t^{(1)}(\zeta_N) = J_N(T_{(2,2t+1)}; \zeta_N),$$

and the following transformation formula [5, 6] proves the quantum modularity:

$$\phi_t^{(m)}(z) + \frac{1}{(iz)^{\frac{3}{2}}} \sum_{m'=1}^{t} \frac{2}{\sqrt{2t+1}} (-1)^{t+1+m+m'} \sin\left(\frac{2mm'}{2t+1}\pi\right) \phi_t^{(m')}(-1/z)$$
$$= \frac{\sqrt{(2t+1)i}}{2\pi} \int_0^{i\infty} \frac{\Phi_t^{(m)}(w)}{(w-z)^{\frac{3}{2}}} dw,$$

where $z \in \mathbb{Q}$ and $\phi_t^{(m)}(z) := e^{2\pi i z\left(-t+\frac{(2t+1-2m)^2}{8(2t+1)}\right)} F_t^{(m)}(e^{2\pi i z})$. A weight-1/2 vector modular form $\Phi_t^{(m)}(\tau)$ is

$$\Phi_t^{(m)}(\tau) := q^{\frac{(2t+1-2m)^2}{4(4t+2)}} \left(q^m, q^{2t+1-m}, q^{2t+1}; q^{2t+1}\right)_\infty,$$

where $(a_1, a_2, \cdots; q)_\infty = (a_1; q)_\infty (a_2; q)_\infty \cdots$. This corresponds to the Gordon–Andrews q-series (the Rogers–Ramanujan q-series for $t = 2$).

To find a dual pair to the function $F_t^{(m)}(q)$, we recall the Habiro cyclotomic expansion of the colored Jones polynomial for knot K [4],

$$J_N(K; q) = \sum_{n=0}^{\infty} C_n(K; q)\, (q^{1+N})_n\, (q^{1-N})_n.$$

This expansion is regarded as the Bailey pair, and an inverse formula is

$$C_n(K; q) = -q^{n+1} \sum_{\ell=1}^{n+1} \frac{(1-q^\ell)(1-q^{2\ell})}{(q)_{n+1-\ell}(q)_{n+1+\ell}} (-1)^\ell q^{\frac{1}{2}\ell(\ell-3)} J_\ell(K; q).$$

By use of an explicit form of the colored Jones polynomial for $T_{(2,2t+1)}$ [10], we obtain the Habiro expansion. Applying several identities of q-hypergeometric series, we define [7]

$$(5) \quad U_t^{(m)}(x; q) := q^{-t} \sum_{\substack{k_t \geq \cdots \geq k_1 \geq 0 \\ k_t \geq 1}} (-xq)_{k_t - 1} (-x^{-1} q)_{k_t - 1} q^{k_t} \times q^{{k_1}^2 + \cdots + {k_{t-1}}^2} \prod_{j=1}^{t-1} \frac{\left(q^{1-j+\sum_{i=1}^{j}(2k_i + \chi(m>i))}\right)_{k_{j+1}-k_j}}{(q)_{k_{j+1}-k_j}},$$

which satisfies $U_t^{(1)}(-q^N; q) = J_N(T^*_{(2,2t+1)}; q)$. Here we mean $\chi(X) = 1$ (resp. 0) when X is true (resp. false).

One of our main results [7] is

$$(6) \qquad F_t^{(m)}(\zeta_N^{-1}) = U_t^{(m)}(-1; \zeta_N).$$

We do not have a transformation formula of $U_t^{(m)}(-1; q)$ for $|q| < 1$ at this stage. Although, we expect that $U_t^{(m)}(-1; q)$ is a mixed mock modular form, as we have a Hecke-type formula [7] such as

$$U_t^{(m)}(-1; q) = -q^{-t} \sum_{n \geq 1} q^{(t+1)n^2} \frac{1+q^n}{1-q^n} \sum_{k=-n}^{n-1} (-1)^k q^{-\frac{2t+1}{2}k(k+1)+mk}.$$

References

[1] G.E. Andrews, R.C. Rhoades, and S.P. Zwegers, *Modularity of the concave composition generating function*, Alg. Number Theory, **8**, 2103–2139 (2013).

[2] J. Bryson, K. Ono, S. Pitman, and R.C. Rhoades, *Unimodal sequences and quantum and mock modular forms*, Proc. Natl. Acad. Sci. **109**, 16063–16067 (2012).

[3] K. Habiro, *On the colored Jones polynomial of some simple links*, RIMS Kokyuroku **1172**, 34–43 (2000)
[4] K. Habiro, *A unified Witten–Reshetikhin–Turaev invariant for integral homology sphere*, Invent. math. **171**, 1–81 (2008)
[5] K. Hikami, *q-series and L-functions related to half-derivatives of the Andrews–Gordon identity*, Ramanujan J. **11**, 175–197 (2006).
[6] K. Hikami and A.N. Kirillov, *Torus knot and minimal model*, Phys. Lett. B **575**, 343–348 (2003).
[7] K. Hikami and J. Lovejoy, in preparation.
[8] T.T.Q. Le, *Quantum invariants of 3-manifolds: Integrality, splitting, and perturbative expansion*, Topology Appl. **127**, 125–152 (2003).
[9] G. Masbaum, *Skein-theoretical derivation of some formulas of Habiro*, Algebraic & Geometric Topology **3**, 537–556 (2003).
[10] H.R. Morton, *The coloured Jones function and Alexander polynomial for torus knots*, Proc. Cambridge Philos. Soc. **117**, 129–135 (1995).
[11] D. Zagier, *Vassiliev invariants and a strange identity related to the Dedekind eta-function*, Topology **40**, 945–960 (2001).
[12] D. Zagier, *Quantum modular forms*, in "Quanta of Maths", Clay Math. Proc. **11**, 659–675, 2010.

Hyperbolic 3-manifolds of bounded volume and trace field degree

BoGwang Jeon

For a hyperbolic 3-manifold M, let $\rho : \pi_1(M) \longrightarrow SL_2\mathbb{C}$ be a faithful representation inducing the complete hyperbolic structure of M and $\mathbb{Q}\big(\mathrm{tr}(\pi_1 M)\big)$ be the trace field defined by
$$\mathbb{Q}(\mathrm{tr}\rho(r) : r \in \pi_1(M)).$$
By Mostow rigidity, both $\mathbb{Q}\big(\mathrm{tr}(\pi_1 M)\big)$ and the volume of M depend only on the undelying topology of M, and these are widely researched topics in the study of 3-manifolds. So the following question is very natural.

Question 1. For a given number $D > 0$, are there only finitely many hyperbolic 3-manifolds whose volume and degree of its trace field are bounded by D?

For hyperbolic 3-manifolds of bounded volume, Jorgensen and Thurston showed their structure can be greatly simplified as follows [1]:

Theorem 0.1. *For any $D > 0$, there exists a finite set of non-compact manifolds $M_1, ..., M_n$ such that all closed hyperbolic 3-manifolds of volume less than or equal to D are obtained by hyperbolic Dehn surgery on some M_i.*

Here Dehn filling is the topological action which attaches solid tori $D^2 \times S^1$ to the boundary tori of ∂M. Now applying the Jorgensen-Thurston theorem, to answer Question 1 it is enough to answer the following question.

Question 2. For an n-cusped manifold M and a constant $D > 0$, are there finitely many Dehn fillings of M whose trace fields have degree $\leq D$?

It is commonly believed that the answer to both questions is yes and this was proved for the 1-cusped case by Hodgson, but little was known for manifolds with $k \geq 2$ cusps in general. In this talk, we suggest a way to attack this question for more cusped cases. For instance, the following is one of the main theorems of our paper [3]:

Theorem 0.2. *Suppose that the answer is yes to* ***Question 2*** *for any s-cusped manifolds where $1 \leq s \leq k-1$. Let X be the deformation variety of k-cusped hyperbolic 3-manifold M. If X is simple, then the answer is yes to* ***Question 2*** *for M.*

To prove the theorem, we first employ the notion of height from number theory, which is the standard way of measuring the complexity of algebraic numbers, and define it for each Dehn filling of M. Specifically, we define it as the trace value of the core geodesic of a Dehn filling. It is a fundamental theorem in number theory that there are only finitely many algebraic numbers of bounded height and degree. Hence, in terms of height instead of degree, to get the affirmative answer to **Question 2**, it is enough to answer the following stronger question:

Question 3. For a k-cusped manifold M, is there a constant $D > 0$ such that, for any Dehn filling of M, its height is uniformly bounded by D?

According to Thurston's hyperbolic Dehn filling theory, each Dehn filled manifold of M corresponds to a point on the deformation variety (of hyperbolic structures on M) satisfying certain additional conditions regarding to its Dehn filling coefficients. (Let's call this a "Dehn filling point") By using the appropriate version of the deformation variety (precisely, the one having the holonomies of the longitude-meridian pairs as parameters), these conditions can be represented by a set of multiple equations defining an algebraic subgroup. So a Dehn filling point on the deformation variety becomes an intersection point between the deformation variety and an algebraic subgroup. Furthermore, using some elementary properties of height, it can be shown that if the height of a Dehn filling point is bounded, then the height of the corresponding Dehn filled manifold is also bounded. Thus, to answer **Question 3**, it is sufficient to prove the heights of intersection points (i.e. Dehn filling points) between the given algebraic varieties are uniformly bounded. As a result, the original problem in hyperbolic geometry is transformed into a problem in arithmetic geometry.

The height distribution of points on an algebraic variety is widely studied topic in arithmetic geometry and there are various theorems regarding to this theme. Among them, we use the one which is so called the Bounded Height Conjecture, originally formulated by E. Bombieri, D. Masser, U. Zannier, and proved by P. Habegger in [2].

Theorem 0.3. (Bounded Height Conjecture=Habegger's theorem) *Let $X \subset (\overline{\mathbb{Q}}^*)^n$ be an irreducible variety over $\overline{\mathbb{Q}}$. Then there is a Zariski open subset X^{oa} of X, which is the complement of the union of anomalous subvarieties of X, so that the* *height is bounded in the intersection of X^{oa} with the union of algebraic subgroups of dimension $\leq n - \dim X$.*

In fact, Habegger's theorem already tells us a lot about the uniform boundedness of heights of most Dehn filling points unless $X^{oa} \neq \emptyset$. In Theorem 0.2, the holonomy variety X being "simple" is an ideal assumption on X so that each subvariety X, $(X \backslash X^{oa})$ and $(X \backslash X^{oa}) \backslash (X \backslash X^{oa})^{oa}$ (and so on) contains only a finite number of anomalous subvarieties. As a result, we prove the conjecture by applying Habegger's theorem repeatedly, a finite number of times, to each of them and their anomalous subvarieties.

Although the holonomy variety being "simple" is defined from a purely algebro-geometrical (or number-theoretical) viewpoint, interestingly enough, it turns out that this definition gives a very nice structure from the hyperbolic geometric side as well, as the following theorem shows:

Theorem 0.4. [3] *Let M be a 2-cusped hyperbolic 3-manifold with rationally independent cusp shapes. If the deformation variety of X is not simple, then the two cusps of M are strongly geometrically isolated.*

If a hyperbolic 3-manifold M has strongly geometrically isolated cusps, then its deformation variety X is always non-simple so there's no obvious way to get the desired result from Habegger's theorem. However, in this case, it is known that its structure appears as a product of two less cusped manifolds, so we still get uniform boundedness of the heights of Dehn filling points by the induction step. As a consequence, combining with Theorem 0.2, when a 2-cusped manifold has rationally independent cusp shapes, then whether its deformation variety is simple or not, the height of each Dehn filling is uniformly bounded.

For the higher cusped cases in general, the non-simple phenomenon is poorly understood, but we think Theorem 0.4 can be further extended, so we formulate it as a conjecture:

Conjecture 1. *Let X be a k-cusped hyperbolic 3-manifold. If the deformation variety of X is not simple, then M has a set of cusps which are strongly geometrically isolated from the rest.*

This conjecture, together with Theorem 0.2, suggest the following seemingly plausible conjecture, which is the affirmative answer to **Question 3**:

Conjecture 2. (Bounded Height Conjecture in Hyperbolic 3-manifolds) *Let M be a k-cusped hyperbolic 3-manifold. Then the height of any Dehn filling of M is uniformly bounded.*

Even though we only deal with manifolds under certain restrictions, it is strongly believed that the above conjecture is true and this approach will eventually give us the complete positive answer to **Question 1**.

Remark. Recently, we announced a proof of **Conjecture 1** in [4]

References

[1] R. Benedetti, C. Petronio, *Lectures on Hyperbolic Geometry*, Springer (1992).
[2] P. Habegger, *On the bounded height conjecture*, Int. Math. Res. Not. (2009), 860-886.
[3] B. Jeon, *Hyperbolic 3-manifolds of bounded volume and trace field degree*, arxiv.org/abs/1305.0611.
[4] B. Jeon, *Hyperbolic 3-manifolds of bounded volume and trace field degree II*, arxiv.org/abs/1409.2069.

Calculations in Teichmüller TQFT

Rinat Kashaev
(joint work with Joergen Ellegaard Andersen)

During the last 3 years I and J.E. Andersen formulated a TQFT model based on the quantum Teichmüller theory [1, 2]. The model is based on a special function called Faddeev's quantum dilogarithm defined by the formula

$$\Phi_b(x) = \exp\left(\frac{1}{4}\int_C \frac{e^{-2ixz}}{\sinh(zb)\sinh(zb^{-1})}\frac{dz}{z}\right)$$

Where the contour C is given by $C = \mathbb{R} + i\epsilon$. The function is extended to the complex plane by analytic continuation.

The geometric input is given by a triangulated 3-manifold where each tetrahedron is provided by the structure of an ideal hyperbolic tetrahedron so that we have a 3-manifold with conical singularities along some of the edges of the triangulation along which the total dihedral angle is different from 2π. By calculations in particular simple examples we conjecture that the partition function of our model decays exponentially, the decay rate being given by the hyperbolic volume of the corresponding (underlying) 3-manifold with conical singularities. Namely the formula reads as follows:

$$|Z_b(X)| \sim e^{-\mathrm{vol}(X)/2\pi b^2} \quad b \to 0$$

References

[1] Jørgen Ellegaard Andersen and Rinat Kashaev. A TQFT from Quantum Teichmüller Theory. *Comm. Math. Phys.*, 330(3):887–934, 2014.
[2] Jørgen Ellegaard Andersen and Rinat Kashaev. A new formulation of the Teichmüller TQFT. arXiv:1305.4291, 2013.

Commensurability of hyperbolic Coxeter groups

Ruth Kellerhals
(joint work with Rafael Guglielmetti, Matthieu Jacquemet)

This work deals with the determination of the wide commensurability classes of a certain large family $\mathcal{P}$ of discrete groups of isometries of n-dimensional hyperbolic space $\mathbb{H}^n$. For $n > 2$ this family consists of all hyperbolic Coxeter n-pyramid groups of finite covolume. It is a finite set as shown by Tumarkin [11], [12] who listed them in 2004. For the basic notions of geometric Coxeter group theory, including Coxeter graphs, combinatorics, criteria for finite covolume (cofiniteness) and arithmeticity, we refer to Vinberg's seminal work as summarised in [14] and [15].

In the sequel we abbreviate the terminology and shall use the term *commensurable* for two groups $\Gamma_1, \Gamma_2 \subset \mathrm{Isom}(\mathbb{H}^n)$ if there exists an element $\gamma \in \mathrm{Isom}(\mathbb{H}^n)$ such that the intersection $\Gamma_1 \cap \gamma\Gamma_2\gamma^{-1}$ is of finite index in both Γ_1 and $\gamma\Gamma_2\gamma^{-1}$. In particular, any subgroup of finite index is commensurable to its supergroup. Furthermore, commensurability is an equivalence relation preserving properties such as cocompactness, cofiniteness and arithmeticity.

For $n = 2$ and $n = 3$, any discrete subgroup $\Gamma \subset \mathrm{PSL}_2(k)$ of orientation preserving hyperbolic isometries, where $k = \mathbb{R}$ or $k = \mathbb{C}$, gives rise to a subalgebra of the matrix group $\mathrm{M}_2(k)$, and is in fact a quaternion algebra. For *arithmetic* subgroups $\Gamma \subset \mathrm{PSL}_2(k)$, these quaternion algebras are defined over number fields, and their classification up to commensurability corresponds to the classification up to isomorphism of the quaternion algebras (see also [13]). In this way, Takeuchi [10] classified the arithmetic triangle groups while Maclachlan and Reid [7] determined the commensurability classes of all cocompact arithmetic Coxeter tetrahedral groups.

In [4] and [5], together with Johnson, Ratcliffe and Tschantz, we determined all subgroup relations and covolumes of hyperbolic Coxeter n-simplex groups and classified them up to commensurability. These are groups generated by the reflections in the hyperplanes bounding hyperbolic n-simplices whose dihedral angles are all of the form π/m for an integer $m \geq 2$ and which are of finite volume. When assuming $n > 2$, this family comprises finitely many examples, including some non-arithmetic ones. Notice that they exist in $\mathrm{Isom}(\mathbb{H}^n)$ for $n \leq 9$, only.

Let us return to the class $\mathcal{P}$ of Tumarkin's hyperbolic Coxeter n-pyramid groups of finite covolume. They are generated by $n + 2$ reflections in the hyperplanes bounding an n-dimensional Coxeter pyramid with an apex on the boundary $\partial\mathbb{H}^n$ at infinity whose horospherical neighborhood is a product of two simplices, each of dimension ≥ 2. A nice combinatorial-metrical feature of such a pyramid is that it relates to a hyperbolic truncated Coxeter simplex (for more details, see [3]). Observe that there is no classification of hyperbolic Coxeter groups with more than $n+2$ generators which are not cocompact but of finite volume. By Vinberg's arithmeticity criterion, one checks easily that there are non-arithmetic elements in $\mathcal{P}$. For example, the group $\Gamma_4 \subset \mathrm{Isom}(\mathbb{H}^{10})$ described by the Coxeter graph in Figure 1 is the top-dimensional non-arithmetic group in $\mathcal{P}$. Observe that the group $\Gamma_3 \subset \mathrm{Isom}(\mathbb{H}^{10})$ given by the same graph after replacement of the edge with weight 4 by an edge without weight (or equivalently by an edge with weight 3) is an arithmetic element in $\mathcal{P}$.

The classification results of Tumarkin show that $\mathcal{P}$ contains groups acting on $\mathbb{H}^n$ for $n \leq 17$, only.

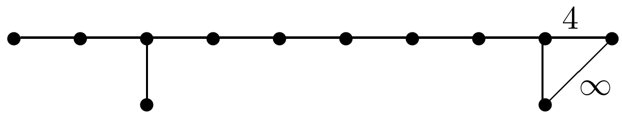

FIGURE 1. The non-arithmetic Coxeter pyramid group acting on $\mathbb{H}^{10}$

In fact, the group Γ_* given by the Coxeter graph in Figure 2 is the (single) top-dimensional group in $\mathcal{P}$. The orientation preserving subgroup Γ'_* of Γ_* is distinguished by the amazing fact that the quotient space $\mathbb{H}^n/\Gamma'_*$ built upon the 17-dimensional pyramid P_* has minimal volume among ALL orientable arithmetic hyperbolic n-orbifolds, and that it is as such unique. This result is due to Emery [1] who computed the minimal volume according to

$$\mathrm{vol}_{17}(\mathbb{H}^n/\Gamma'_*) = \mathrm{vol}_{17}(P_*) = \frac{691 \cdot 2617}{2^{38} \cdot 3^{10} \cdot 5^4 \cdot 7^2 \cdot 11 \cdot 13 \cdot 17}\,\zeta(9)$$

by a clever exploitation of Prasad's volume formula and other sophisticated tools.

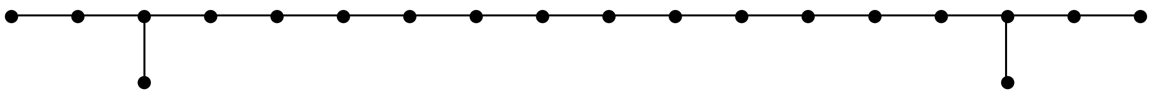

FIGURE 2. The single Coxeter pyramid group Γ_* acting on $\mathbb{H}^{17}$

Let us return to and discuss the commensurability classification of elements in $\mathcal{P}$. For the large subset of arithmetic groups in $\mathcal{P}$, we exploit the results of Maclachlan [7] about commensurability of discrete arithmetic hyperbolic groups in the special case of Coxeter groups in $\mathcal{P}$. The results in [7] show that the commensurability classes in $\mathcal{P}$ are in one-to-one correspondence with the isomorphism classes of quaternion algebras over certain number fields. We determined explicitly the classes and identified representatives in terms of certain hyperbolic Coxeter simplex groups whenever possible. Some representatives are related to Mcleod's Coxeter groups which appear as a subgroup of the automorphism group of the Lorentzian lattice $-3x_0^2 + x_1^2 + \cdots + x_n^2$ for certain n. All their covolumes were determined by Ratcliffe and Tschantz [9]. Therefore, by exploiting the volume results in [4] and [9], we can proceed in order to list the volumes of all arithmetic Coxeter pyramids and to find all subgroup relations.

In the case of the non-arithmetic elements in $\mathcal{P}$ which show up mainly in low dimensions, different ad hoc methods such as scissors congruences, glueings in the spirit of Gromov-Piatetski Shapiro [2] for groups such as Γ_l for $l = 2, 3, 4$, volume computations based on Schläfli's volume differential and ratio tests, are involved. This part of the work is also much inspired by [5].

References

[1] V. Emery, *Even unimodular Lorentzian lattices and hyperbolic volume*, J. Reine Angew. Math. **690** (2014), 173–177.
[2] M. Gromov, I. Piatetski-Shapiro, *Nonarithmetic groups in Lobachevsky spaces*, Inst. Hautes Études Sci. Publ. Math. **66** (1988), 93–103.
[3] M. Jacquemet, *The inradius of a hyperbolic truncated n-simplex*, Discrete Comput. Geom. **51** (2014), 997–1016.
[4] N. Johnson, R. Kellerhals, J. Ratcliffe, S. Tschantz, *The size of a hyperbolic Coxeter simplex*, Transform. Groups **4** (1999), 329–353.
[5] N. Johnson, R. Kellerhals, J. Ratcliffe, S. Tschantz, *Commensurability classes of hyperbolic Coxeter groups*, Linear Algebra Appl. **345** (2002), 119–147.
[6] C. Maclachlan, A. Reid, *Commensurability classes of arithmetic Kleinian groups and their Fuchsian subgroups*, Math. Proc. Cambridge Philos. Soc. **102** (1987), 251–257.
[7] C. Maclachlan, *Commensurability classes of discrete arithmetic hyperbolic groups*, Groups Geom. Dyn. **5** (2011), 767–785.
[8] J. Mcleod, *Hyperbolic reflection groups associated to the quadratic forms $-3x_0^2+x_1^2+\cdots+x_n^2$*, Geom. Dedicata **152** (2011), 1–16.
[9] J. Ratcliffe, S. Tschantz, *On volumes of hyperbolic Coxeter polytopes and quadratic forms*, Geom. Dedicata **163** (2013), 285–299.
[10] K. Takeuchi, *Commensurability classes of arithmetic triangle groups*, J. Fac. Sci. Univ. Tokyo Sect. IA Math. **24** (1977), 201–212.
[11] P. Tumarkin, *Hyperbolic Coxeter n-polytopes with $n+2$ facets*, arXiv:math/0301133, pp. 13, 2004.
[12] P. Tumarkin, *Hyperbolic Coxeter polytopes in H^n with $n+2$ hyperfacets*, Math. Notes **75** (2004), 848–854.
[13] M. Vignéras, *Arithmétique des algèbres de quaternions*, Lecture Notes in Mathematics vol. 800, Springer 1980.
[14] E. Vinberg, *Hyperbolic groups of reflections*, Russian Math. Surveys **40** (1985), 29–66.
[15] E. Vinberg, O. Shvartsman *Geometry II*, Encyclopedia Math. Sci. vol. 29, Springer 1993.

From isolated codes to isolated 3-manifolds

Matthias Kreck

1. Isolated codes

Codes mean linear error correcting codes over a finite field F_q, i.e. linear subspaces C of F_q^n. There are two fundamental invariants attached to codes, the dimension $d(C)$ and the minimal distance $\Delta(C)$, which is the minimal distance for the Hamming metric between two different elements of C. From these one obtains by dividing by n two rational numbers

$$R(C) := \frac{d(C)}{n}$$

and

$$\delta(C) := \frac{\Delta(C)}{n}.$$

They are a measure for the quality of C, one wants both to be large, since $\Delta(C) \geq 2k+1$ implies that one can correct k errors, and $d(C)$ large allows to send many different messages.

Manin in the eighties (for a actual account see [M-M]) has studied the subset P_q of the unit cube $[0,1] \times [0,1]$ given by the pairs $(\delta(C), R(C))$ coming from linear codes over F_q. He proved the following remarkable result:

Theorem 1. *There is a continuous decreasing function*

$$\alpha_q : [0,1] \longrightarrow [0,1]$$

such that all points below the graph are accumulation points of P_q and all points above are isolated points.

He calls codes above α_q **isolated codes**.

2. From manifolds to codes

V. Puppe [P] has attached to a closed manifold M with involution τ with isolated fixed points an self dual code $C(M,\tau)$ over F_2. Here self dual means that with respect to standard bilinear form on F_2^n this form vanishes identically on C and $\dim C = n/2$. Self dual codes are closely related to arithmetic since there is an attached unimodular lattice $L(C)$ in $\mathbb{R}^n$, and these lattices are not understood.

There is an obvious generalization to periodic maps ρ of order a prime p and not necessarily self dual codes over F_p. One actually obtains a family $C^k(M,\rho)$ of codes. Let F_0 be the set of isolated fixed points.

$$C(M,\rho) := image(i^* : H^k_{\mathbb{Z}/p}(M;F_p) \longrightarrow H^k_{\mathbb{Z}/p}(F_0;F_p)).$$

Here i is the inclusion. To interpret this as a code one has to identify $H^k_{\mathbb{Z}/p}(pt;F_p)$ with F^p. We do this by choosing the generator in H^1 given by the dual of the standard generator of π_1 and the generators in H^k by the powers of this, for $p = 2$ and the generators obtained by applying the Bockstein and powers.

3. Isolated manifolds

Although the following definition looks a bit ad hoc, the context above somehow suggests it.

Definition. *A closed smooth manifold is called* **isolated** *if for some p it has a $\mathbb{Z}/p$ action ρ such that*

$$C(M,\rho)$$

is isolated.

This definition is interesting in the light of a "conjecture" which says that a manifold "picked at random" is asymmetric which means has no effective action of a compact Lie group. Thus typically a manifold should be asymmetric and so not isolated.

4. Codes from manifolds and existence of isolated manifolds

In [K-P] we proved that all self dual codes are realized by an involution on a closed 3-manifold. The methods of the proof allow the following generalization:

Theorem 2. *Let C be a linear code over F_p. Then there is a closed smooth 4-manifold M with $\mathbb{Z}/p$-action ρ such that*

$$C^1(M,\rho) = C$$

if and only if the diagonal element $(1,...,1)$ is contained in C.

Using this one can show that most of the isolated codes come from smooth manifolds, which by definition then are isolated. Examples of isolated manifolds include the obvious candidates: Spheres, projective spaces, all surfaces, the 3-torus, the K_3-surface. But the question which manifolds are isolated is of course widely open.

References

[K-P] Kreck, M., Puppe, V.: *Involutions on 3-manifolds and self dula codes.* Homology, Homotopy Appl. 10 (2008), no. 2, 139148

[M-M] Manin,Y. and Marcolli, M.: *Errorcorrecting codes and phase transitions.* Math. Comput. Sci. 5 (2011), no. 2, 133170

[P] Puppe, V.: *Group Actions and Codes.* Can. J. Math. 53, 212 – 224, 2001.

On Kauffman Bracket Skein Modules at Root of Unity

Thang Lê

1. Definitions

1.1. **Skein modules.** Suppose M is an oriented compact 3-manifold. The Kauffman bracket skein module (J. Przytycki, V. Turaev) of M is defined by

$\mathcal{S}(M) = \mathbb{C}[t^{\pm 1}]$ – span of framed unoriented links in M/relations (1) & (2).

Here

$$L = tL_+ + t^{-1}L_- \tag{1}$$

$$L \sqcup U = -(t^2 + t^{-2})L, \tag{2}$$

where in (1), the links L, l_+, L_- are identical everywhere except in a small balls where they look like in Figure 1, and in (2) $L \sqcup U$ means the union of a framed link L and a trivial knot U which lies in a 3-ball disjoint from L.

By convention, the empty set is also considered a framed link.

For $0 \neq \xi \in \mathbb{C}$, define

$$\mathcal{S}_\xi(M) := \mathcal{S}(M)/(t - \xi).$$

Suppose Σ is an oriented compact surface, possibly with boundary. Define

$$\mathcal{S}(\Sigma) := \mathcal{S}(\Sigma \times [-1, 1]).$$

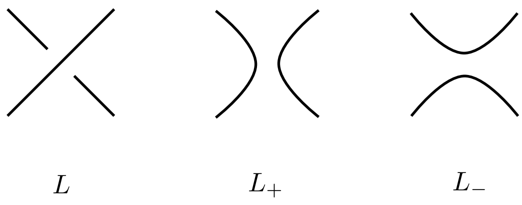

FIGURE 1. The links L, L_+, and L_- in (1).

Then $\mathcal{S}(\Sigma)$ has a $\mathbb{C}[t^{\pm 1}]$-algebra structure, where the product is given by:

$$L_1 L_2 = L_1 \text{ on top of } L_2,$$

for framed links L_1, L_2 in $\mathcal{S}(\Sigma \times [-1,1]$.

It might happen that $\Sigma_1 \times [-1,1] \cong \Sigma_2 \otimes [-1,1]$ with $\Sigma_1 \ncong \Sigma_2$. In that case, $\mathcal{S}(\Sigma_1)$ and $\mathcal{S}(\Sigma_2)$ are the same as $\mathbb{C}[t^{\pm 1}]$-modules, but the algebra structures may be different.

A non-trivial simple closed curve $K \subset \Sigma = \Sigma \times 0 \subset \Sigma \times [-1,1]$ is considered as a framed knot in $\Sigma \times [-1,1]$, where the framing at every point is given by the vector colinear with $[-1,1]$.

If $p(z) \in \mathbb{C}[z]$ and $K \subset \Sigma$ is a simple closed curve, one can define $p(K)$ by applying the polynomial p to the element K of the algebra $\mathcal{S}(\Sigma)$.

1.2. **Chebyshev polynomial.** Let $T_N(z) \in \mathbb{C}[z]$ be the Chebyshev polynomials of type 1 defined recursively by

$$T_0(z) = 2, T_1(z) = 1, T_n(z) = zT_{n-1}(z) - T_{n-2}(z), \ \forall n \geq 2.$$

2. Central elements

Definition 1. *A polynomial $p(z) \in \mathbb{C}[z]$ is* central *at $\xi \in \mathbb{C}^\times$ if for any oriented surface Σ and any knot K in Σ, $p(K)$ is central in the algebra $\mathcal{S}_\xi(\Sigma)$.*

Theorem 1 (T. Lê [Le1])**.** *A non-constant polynomial $p(z) \in \mathbb{C}[z]$ is central at $\xi \in \mathbb{C}^\times$ if and only if ξ is a root of unity and $p(z) \in \mathbb{C}[T_N(z)]$, i.e. p is a $\mathbb{C}$-polynomial in $T_N(z)$, where N is the order of ξ^2.*

This is an extension of a result of Bonahon and Wong [BW], which says if ξ is root of unity of order $2N$, then $T_N(z)$ is central. Bonahon and Wong's proof used representation theory of quantum Teichmüller spaces (Chekhov-Goncharov, Kashaev), although the formulation of the theorem does not involve quantum Teichmüller spaces. Our proof uses only elementary skein techniques.

3. Positive basis

Now we assume that the ground ring is $R = \mathbb{Z}[t^{\pm 1}]$ (instead of $\mathbb{C}[t^{\pm 1}]$). It is known that $\mathcal{S}(\Sigma)$ is a free $\mathbb{Z}[t^{\pm 1}]$-module with basis the set of all links in Σ without trivial components, including the empty link, see [PS].

One can group components of links which are parallel and get the following parametrization of the above mentioned basis.

An *integer lamination* of Σ is an unordered collection $\mu = \{(a_i, C_i)\}_{i=1}^k$, where

- each $a_i \in \mathbb{Z}_{>0}$.
- each C_i is an essential simple closed curve
- no two C_i intersect, no two C_i are parallel.

Define

$$b(\mu) = \prod_{i=1}^k C_i^{a_i}.$$

Then $\{b(\mu) \mid \mu \text{ integer laminations}\}$ is a $\mathbb{Z}[t^{\pm 1}]$-basis of $\mathcal{S}(\Sigma)$.

Suppose $\mathcal{P} = \{p_i\}_{i=0}^\infty$, where $p_i \in \mathbb{Z}[z]$ with leading term z^i. Define

$$b_{\mathcal{P}}(\mu) = \prod_{i=1}^k p_{a_i}(C_i).$$

Then $b_{\mathcal{P}} = \{b_{\mathcal{P}}(\mu) \mid \mu \text{ integer laminations}\}$ is a R-basis of $\mathcal{S}(\Sigma)$.

Let $\mathcal{T} = \{T_i\}_{i=0}^\infty$, the collection of Chebyshev polynomials.

Theorem 2 (D.Thurston [Th]). *$b_{\mathcal{T}}$ is a positive basis of the algebra $\mathcal{S}_1(\Sigma)$.*

Here for a $\mathbb{Z}$-algebra A a $\mathbb{Z}$-module basis $\{e(i) \mid i \in I\}$ is a positive basis if

$$e_i e_j = \sum_{k \in I} c_{ij}^k \, e(k),$$

with the structure constants $c_{ij}^k \in \mathbb{Z}_{\geq 0}$. For general t, Thurston gave the following conjecture.

Conjecture 3.1. *$b_{\mathcal{T}}$ is a positive basis for $\mathcal{S}(\Sigma)$. (The structure constants are polynomials in t, t^{-1} with non-negative coefficients.)*

We have the following result which complements Thurston's one and support the above conjecture.

Theorem 3 (T. Lê [Le2]). *(a) Suppose $b_{\mathcal{P}}$, with $\mathcal{P} = \{p_i\}_{i=0}^\infty$, is a positive basis for $\mathcal{S}_1(\Sigma)$, then each p_i is a $\mathbb{Z}_{\geq 0}$-linear combination of $T_n(z)$.*

(b) Suppose $b_{\mathcal{P}}$, with $\mathcal{P} = \{p_i\}_{i=0}^\infty$, is a positive basis for $\mathcal{S}(\Sigma)$, then each p_i is a $\mathbb{Z}_{\geq 0}[t^{\pm 1}]$-linear combination of $T_n(z)$.

References

[BW] F. Bonahon and H. Wong, *Representations of the Kauffman skein algebra I: invariants and miraculous cancellations*, preprint arXiv:1206.1638, 2012.

[Le1] T. T. Q. Lê, *On Kauffman Bracket Skein Modules at Root of Unity*, preprint arXiv:1312.3705, 2013.

[Le2] T. T. Q. Lê, to appear.

[PS] J. Przytycki and A. Sikora, *On the skein algebras and $Sl_2(\mathbb{C})$-character varieties*, Topology **39** (2000), 115–148.

[Th] D. Thurston *A positive basis for surface skein algebras*, preprint arXiv:1310.1959, 2013.

The twisted L^2-torsion function and its application to 3-manifolds

Wolfgang Lück

The talk is about an ongoing project joint with Stefan Friedl.

Let G be a group, $G \to \overline{X} \to X$ be a G-covering over a finite CW-complex X and $\phi\colon G \to \mathbb{Z}$ be a group homomorphism. If G is residually finite and $\overline{X}$ is L^2-acyclic, i.e., all L^2-Betti numbers $b_n^{(2)}(\overline{X}, \mathcal{N}(G))$ vanish, we can assign to it a function

$$\rho^{(2)}(\overline{X}, \mathcal{N}(G); \phi)\colon (0,\infty) \to \mathbb{R}$$

which is essentially the L^2-torsion of $\overline{X}$ twisted with the 1-dimensional real representation $\mathbb{R}$ on which $g \in G$ acts by multiplication with $t^{\phi(g)}$. (Actually this function is only well-defined up to adding $k \cdot \ln(t)$ for some $k \in \mathbb{Z}$). If $G = \pi_1(X)$ and $\overline{X}$ is the universal covering $\widetilde{X}$, then we abbreviate $\rho^{(2)}(\widetilde{X}; \phi) := \rho^{(2)}(\widetilde{X}, \mathcal{N}(\pi_1(X)); \phi)$. See [5, 4, 3]. For basics about L^2-invariants we refer to [7].

We present some basic properties such as homotopy invariance, sum formula, product formula or more generally a formula for fibrations with L^2-acyclic fiber, passage to finite covering, scaling ϕ, Poincaré duality, and compute it for S^1-spaces with appropriate S^1-action and mapping tori T_f for ϕ the canonicial homomorphism $\pi_1(T_f) \to \pi_1(S^1) = \mathbb{Z}$.

Then we pass to 3-manifolds and compute it for graph manifolds and 3-manifolds which fiber over S^1. We show that for a knot $K \subseteq S^3$ with knot complement $X(K)$ and $\phi \in H^1(X(K);\mathbb{Z}) \cong \mathbb{Z}$ a generator that $\rho^{(2)}(\widetilde{X(K)}, \phi)$ detects the trivial knot, see [1, 8].

A function ρ is asymptotically monomial if for some constants C_0 and C_∞ the limits $\lim_{t\to 0}\bigl(\rho(t) - C_0 \cdot \ln(t)\bigr)$ and $\lim_{t\to\infty}\bigl(\rho(t) - C_\infty \cdot \ln(t)\bigr)$ exists. In this case we define the degree $\deg(\rho)$ to be $C_\infty - C_0$. Denote by $x_M(\phi)$ the Thurston norm of $\phi \in H^1(X;\mathbb{Z})$.

Our main the theorem is

Theorem 1. *Let M be a compact connected orientable irreducible 3-manifold with infinite fundamental group π and empty or incompressible torus boundary. Consider $\phi \in H^1(X;\mathbb{Z})$. Then*

$$\deg\bigl(\rho^{(2)}(\widetilde{M}; \phi)\bigr) = -x_M(\phi).$$

We can actually generalize it to other coverings than the universal covering.

Theorem 2. *Let M be a compact connected orientable irreducible 3-manifold with infinite fundamental group π and empty or incompressible torus boundary which is not a closed graph manifold.*

Then there is a virtually finitely generated free abelian group Γ, and a factorization $\pi_1(M) \xrightarrow{\alpha} \Gamma \xrightarrow{\beta} H_1(M)_f := H_1(M)/\operatorname{tors}(H_1(M))$ of the canonical projection into epimorphisms, an element $m \in H_1(M)_f$, an integer $k \geq 1$ such that the following holds:

For any group homomorphism $\phi\colon H_1(\pi)_f := H_1(\pi)/\operatorname{tors}(H_1(\pi)) \to \mathbb{Z}$ and any factorization of $\alpha\colon \pi \to \Gamma$ into group homomorphisms $\pi \xrightarrow{\mu} G \xrightarrow{\nu} \Gamma$ for a residually *finite group G, there exists real numbers constants $D_2 \geq 0$ and $D_4 \geq 0$ such that for the G-covering $\overline{M} \to M$ associated to μ we get*

$$\frac{\phi(m)}{k} \cdot \ln(t) - D_2 \leq \rho^{(2)}(\overline{M}, \mathcal{N}(G); \phi \circ \beta \circ \nu)(t) \leq \frac{\phi(m)}{k} \cdot \ln(t) \quad \textit{for } t \leq 1;$$

and

$$\left(-x_M(\phi) + \frac{\phi(m)}{k}\right) \cdot \ln(t) - D_4 \leq \rho^{(2)}(\overline{M}, \mathcal{N}(G); \phi \circ \nu)(t)$$
$$\leq \left(-x_M(\phi) + \frac{\phi(m)}{k}\right) \cdot \ln(t) \quad \textit{for } t \geq 1.$$

In particular $\rho^{(2)}(\overline{M}, \mathcal{N}(G); \phi \circ \nu)$ is asymptotically monomial and satisfies

$$\deg\big(\rho^{(2)}(\overline{M}, \mathcal{N}(G); \phi \circ \nu)\big) = -x_M(\phi).$$

We use this to show for the higher order Alexander polynomial of Cochrane and Harvey, see [2, 6], that their degree coincides with the Thurston norm in the situation of the last theorem provided that G is torsionfree elementary amenable and residually finite. Previously only an inequality was known.

References

[1] F. Ben Aribi. The L^2-Alexander invariant detects the unknot. *C. R. Math. Acad. Sci. Paris*, 351(5-6):215–219, 2013.

[2] T. D. Cochran. Noncommutative knot theory. *Algebr. Geom. Topol.*, 4:347–398, 2004.

[3] J. Dubois, S. Friedl, and W. Lück. The L^2–Alexander torsion of 3-manifolds. in preparation, 2014.

[4] J. Dubois, S. Friedl, and W. Lück. Three flavors of twisted knot invariants. in preparation, 2014.

[5] J. Dubois and C. Wegner. L^2-Alexander invariant for torus knots. *C. R. Math. Acad. Sci. Paris*, 348(21-22):1185–1189, 2010.

[6] S. L. Harvey. Higher-order polynomial invariants of 3-manifolds giving lower bounds for the Thurston norm. *Topology*, 44(5):895–945, 2005.

[7] W. Lück. *L^2-Invariants: Theory and Applications to Geometry and K-Theory*, volume 44 of *Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics]*. Springer-Verlag, Berlin, 2002.

[8] W. Lück and T. Schick. L^2-torsion of hyperbolic manifolds of finite volume. *GAFA*, 9:518–567, 1999.

Dimension and character formulas for modular TQFT representations of mapping class groups in the natural characteristic

Gregor Masbaum

The Witten-Reshetikhin-Turaev quantum invariants of 3-manifolds fit into a Topological Quantum Field Theory (TQFT) in the sense of Atiyah and Segal. Here we consider SO(3)-TQFT at the p-th root of unity, where $p \geq 5$ is an odd integer. The dimension of the SO(3)-TQFT vector space associated to a surface of genus $g \geq 1$ with one boundary component labelled by an even integer $2c$ (where $0 \leq 2c \leq p-3$) is given by the following variant of the Verlinde formula:

$$D_g^{(2c)}(p) = \left(\frac{p}{4}\right)^{g-1} \sum_{j=1}^{(p-1)/2} \left(\sin \frac{\pi j(2c+1)}{p}\right) \left(\sin \frac{\pi j}{p}\right)^{1-2g} . \tag{1}$$

This (complex) vector space carries a linear representation of an appropriate central extension of the mapping class group $\Gamma_{g,1}$ of the genus g surface with one boundary component. In the special case $c = 0$, this representation factors through the corresponding central extension of the mapping class group Γ_g of the closed surface. Note that the right hand side of (1) simplifies for $c = 0$.

Let us now assume $p \geq 5$ is an odd prime. Let ζ_p be a primitive p-th root of unity. We denote the corresponding ring of cyclotomic integers by $\mathbb{Z}[\zeta_p]$. The theory of Integral SO(3)-TQFT developed in [1, 2] shows that our TQFT-vector space contains a natural free $\mathbb{Z}[\zeta_p]$-module (a.k.a. $\mathbb{Z}[\zeta_p]$-lattice) preserved by the action of the mapping class group. In [3], this $\mathbb{Z}[\zeta_p]$-module is denoted by $\mathcal{S}_p(\Sigma_g(2c))$. In this abstract, however, we will denote this module simply by $\mathcal{S}$. But notice that $\mathcal{S}$ depends on the genus g, the boundary label $2c$, and the order p of the root of unity. The rank of $\mathcal{S}$ is given again by (1) and $\mathcal{S}$ spans the TQFT-vector space over the complex numbers. We refer to $\mathcal{S}$ as the Integral SO(3)-TQFT representation of the mapping class group.

For every ideal $I \subset \mathbb{Z}[\zeta_p]$, we have an induced representation on the $\mathbb{Z}[\zeta_p]/I$-module $\mathcal{S}/I\mathcal{S}$. In this way, we can get modular representations of the mapping class group, namely when $\mathbb{Z}[\zeta_p]/I$ is a finite field of characteristic ℓ. If $\ell \neq p$, we speak of modular TQFT representations of mapping class groups in unequal characteristic, while the characteristic $\ell = p$ is called the natural characteristic.

Of course, $\mathcal{S}/I\mathcal{S}$ is only a first approximation to the Integral TQFT representation, and one may also consider the mapping class group representation on the higher quotients $\mathcal{S}/I^N\mathcal{S}$ for $N \geq 2$. But here we will only consider $\mathcal{S}/I\mathcal{S}$ and only in the case when $\mathbb{Z}[\zeta_p]/I$ is a finite field. Thus, $\mathcal{S}/I\mathcal{S}$ is indeed a modular representation, that is, a representation of a group on a vector space over a finite field.

The unequal characteristic case was exploited in my joint work with Alan Reid [5] where we used these modular representations to show that all finite groups are involved in the mapping class group Γ_g (for any fixed genus g.) I talked about this result last time [6].

I would now like to report on my joint work with Pat Gilmer [3], where we study irreducibility properties of the mapping class group representations $\mathcal{S}/I\mathcal{S}$. These can be deduced fairly easily if one knows the sub $\mathbb{Z}[\zeta_p]$-algebra of the endomorphism algebra $\mathrm{End}_{\mathbb{Z}[\zeta_p]}(\mathcal{S})$ generated by the matrices by which mapping class group elements act on $\mathcal{S}$. (Note that $\mathrm{End}_{\mathbb{Z}[\zeta_p]}(\mathcal{S})$ is isomorphic to the matrix algebra

$$M(D_g^{(2c)}(p), \mathbb{Z}[\zeta_p]) \ ,$$

since $\mathcal{S}$ is a free module of rank $D_g^{(2c)}(p)$.) Therefore we have computed this subalgebra in [3]. I will not give details here, but our result implies for example that in characteristic $\ell \neq p$ our modular TQFT representations over $\mathbb{F}_\ell$ of mapping class groups are always irreducible. Here, unequal characteristic can be understood to include the case of characteristic zero as well. Indeed, our result also implies that the original SO(3)-TQFT representation over the complex numbers is irreducible. This generalizes a result of Justin Roberts [9] who proved irreducibility over the complex numbers of the mapping class group representations. Actually Roberts considered SU(2)-TQFT rather than SO(3)-TQFT and only the case of closed surfaces, and he asked what happens for surfaces with boundary.

Let us now look at our modular TQFT representations of mapping class groups in the natural characteristic $\ell = p$. Then a new phenomenon appears and the representations are no longer irreducible. Here are some details of this, following [3]. In the natural characteristic case we have $I = (1 - \zeta_p)$ and $\mathbb{Z}[\zeta_p]/I$ is the finite field $\mathbb{F}_p$. Let us denote $\mathcal{S}/I\mathcal{S}$ by F (and recall that F depends on p, g, and $2c$, which we have suppressed from the notation). This F is an $\mathbb{F}_p$-vector of dimension given by the Verlinde formula (1). It turns out [2] that F carries a linear representation of the mapping class group Γ_g itself (in particular, no central extension is needed) and that the Johnson kernel (but not the Torelli subgroup) of Γ_g acts trivially on F. As already mentioned, this representation is usually not irreducible. The situation is described precisely as follows.

Theorem 1. *If $g \leq 1$ or if $(g, c) = (2, 0)$, then F is an irreducible representation of Γ_g. Otherwise, F has a composition series with exactly two irreducible factors: there is a unique irreducible subrepresentation F^{odd} of dimension denoted by $\mathfrak{o}_g^{(2c)}(p)$, and the quotient F/F^{odd}, of dimension denoted by $\mathfrak{e}_g^{(2c)}(p)$, is again irreducible. Moreover, the dimensions of these irreducible factors can be computed from the Verlinde formula (1) for*

$$D_g^{(2c)}(p) = \mathfrak{e}_g^{(2c)}(p) + \mathfrak{o}_g^{(2c)}(p)$$

and the following new kind of Verlinde formula for

$$\delta_g^{(2c)}(p) = \mathfrak{e}_g^{(2c)}(p) - \mathfrak{o}_g^{(2c)}(p) \ :$$

$$(2) \qquad (-1)^c \delta_g^{(2c)}(p) = \frac{4^{1-g}}{p} \sum_{j=1}^{(p-1)/2} \left(\sin \frac{\pi j(2c+1)}{p} \right) \left(\sin \frac{\pi j}{p} \right) \left(\cos \frac{\pi j}{p} \right)^{-2g} .$$

Notice the similarity of the formulas for $D_g^{(2c)}(p)$ and $\delta_g^{(2c)}(p)$: if one substitutes $p/\sin^2(\pi j/p)$ for $1/\cos^2(\pi j/p)$ in the right hand side of (2), one gets the right hand side of (1).

The reader may wonder about the notations F^{odd}, $\mathfrak{e}_g^{(2c)}(p)$, $\mathfrak{o}_g^{(2c)}(p)$ in the above statements, but I cannot explain the reason for these notations here for lack of space. See §2 of [3] for explanations, where one can also find more discussion of this result and of formula (2).

I concluded this talk with an application of the above to representation theory. One can show [7] that the irreducible Γ_g-representations F^{odd} and F/F^{odd} factor through the finite symplectic group $\mathrm{Sp}(2g, \mathbb{F}_p)$. A natural question then is exactly which irreducible representations of $\mathrm{Sp}(2g, \mathbb{F}_p)$ arise in this way? Pat Gilmer and I observed in §8 of [3] that for $p = 5$ the four representations we obtain have the same dimensions as the irreducible representations of $\mathrm{Sp}(2g, \mathbb{F}_p)$ with fundamental highest weight ω_j $(g-3 \le j \le g)$ considered by Gow [4]. But at that time we could not prove that our representations had those highest weights, i.e. that our representations were indeed isomorphic to those considered by Gow. In more recent work [8], we have now proved this, and for all odd primes p we have now computed the highest weights of the irreducible $\mathrm{Sp}(2g, \mathbb{F}_p)$-representations obtained as as above, i.e. obtained as first approximation to the Integral $SO(3)$-TQFT representations of mapping class groups in the natural characteristic. In particular, we have explicit dimension and character formulas for the irreducible $\mathrm{Sp}(2g, \mathbb{F}_p)$-representations with these highest weights. This may be interesting because the highest weights we obtain are not fundamental weights when $p > 5$. We are told that dimension and character formulas for such highest weights are probably not known, at least not in general. Details of this will be given in [8].

References

[1] P. M. Gilmer. *Integrality for TQFTs*, Duke Math J, **125** (2004), no. 2, 389–413

[2] P. M. Gilmer, G. Masbaum. *Integral lattices in TQFT*, Ann. Sci. Ecole Norm. Sup. **40**, (2007), 815–844

[3] P. M. Gilmer, G. Masbaum. *Irreducible factors of modular representations of mapping class groups arising in Integral TQFT*, Quantum Topol. 5 (2014), 225–258

[4] R. Gow. *Construction of $p-1$ irreducible modules with fundamental highest weight for the symplectic group in characteristic p*, J. London Math. Soc. **58** (1998) 619–632.

[5] G. Masbaum, A. W. Reid. *All finite groups are involved in the Mapping Class Group*, Geometry and Topology **16** (2012), 1393-1411

[6] G. Masbaum (joint with Alan W. Reid). *All finite groups are involved in the Mapping Class Group*, Oberwolfach Reports 42/2012, pages 2577 - 2578.

[7] G. Masbaum. *(in preparation)*

[8] P. M. Gilmer, G. Masbaum. *(in preparation)*

[9] J. D. Roberts. *Irreducibility of some quantum representations of mapping class groups*, J. Knot Theory Ramifications **10**, 763-767 (2001).

Arithmetic of universality in simple Lie algebras

Ruben L. Mkrtchyan

We report on the development of "universal" approach in the theory of simple Lie algebras and their applications, which reveals an unexpected relations with number theory. The origin is in Vogel's study of algebras of diagrams in knots theory [1], Deligne's series of Lie algebras [2], 't Hooft's 1/N [3] expansion in gauge theories, $N \to -N$ duality of SO(N)/Sp(N) gauge theories [4] and others.

Consider Vogel's universal expression for dimensions of simple Lie algebras:

$$dim\,\mathfrak{g} \quad = \quad \frac{(\alpha-2t)(\beta-2t)(\gamma-2t)}{\alpha\beta\gamma},\ t=\alpha+\beta+\gamma \tag{1}$$

where projective (Vogel's, universal) parameters α, β, γ define a point on a Vogel's plane, which is the projective plane factorized w.r.t. the all permutations of three projective parameters. Expression (1) gives dimensions of simple Lie (super)algebras when evaluated at special points at Vogel's plane, given in a Table 1, [1]. Such an expressions we shall call universal. Dimensions of many other irreps of simple Lie algebras can be represented in a similar universal form [5]. One can try to "universalize" (unify) the (part of the) theory of simple Lie algebras and groups, and their applications, expressing it on the language of Vogel's parameters.

Table 1. Vogel's parameters for simple Lie (super)algebras

Algebra/Parameters	α	β	γ	$t=\alpha+\beta+\gamma$
SL(N)	-2	2	N	N
SO(N)/Sp(-N)	-2	4	N-4	N-2
Exc(n)	-2	n+4	2n+4	3n+6
$D_{2,1,\lambda}$	α	β	γ	0

Here for SL(N) and SO(N) N is positive integer, for Sp(-N) N is negative even integer, for exceptional line Exc(n) n=-1,-2/3,0,1,2,4,8 for $A_2, G_2, D_4, F_4, E_6, E_7, E_8$ respectively. $N \to -N$ transformation corresponds to transposition $\alpha \leftrightarrow \beta$.

In [6] we present a universal expressions for generating function for eigenvalues of higher Casimir operators on adjoint representation. In [7] we derive universal expression for character of adjoint representation on Weyl line $x\rho$, where x is an arbitrary parameter, ρ is Weyl vector in root space, half the sum of positive roots. This expression appears to be very useful in further developments:

$$f(x) = \chi_{ad}(x\rho) = \frac{\sinh(x\frac{\alpha-2t}{4})}{\sinh(x\frac{\alpha}{4})}\frac{\sinh(x\frac{\beta-2t}{4})}{\sinh(x\frac{\beta}{4})}\frac{\sinh(x\frac{\gamma-2t}{4})}{\sinh(x\frac{\gamma}{4})} \tag{2}$$

Similar universal characters can be obtained for all representations with universal dimension formula.

For points from Table 1 this expression is regular at finite x plane since it is a finite sum of exponents. One can turn problem upside down and seek the points in Vogel's plane for which this expression is regular at finite x. It appears [8] that universal parameters of such points are expressed in terms of three integers k, n, m , which have to satisfy one of seven Diophantine equations of the type knm =second order polynomial over k, n, m. E.g. one of these equations is $knm = kn + nm + km + 3n + 3k + 3m + 5$. For $k, n, m \neq -1$ this equation and corresponding expressions of Vogel's parameters in terms of k, m, n can be written in more memorable form: $\frac{2}{k+1} + \frac{2}{n+1} + \frac{2}{m+1} = 1$, $\alpha = \frac{2t}{k+1}, \beta = \frac{2t}{n+1}, \gamma = \frac{2t}{m+1}$.

There are two types of solutions of this and other six equations. One is the series, i.e. depends on parameter (integer or continuous), and other an isolated ones. Series solutions are:

$$\begin{aligned} SL(N):(k,n,m) &= (-N-1,N-1,1),\ (\alpha,\beta,\gamma)=(2,-2,N) \\ D_{2,1,\lambda}:(k,n,m) &= (-1,-1,-1),\ \alpha+\beta+\gamma=0 \end{aligned}$$

There are 15 isolated solutions (up to symmetries on permutations of parameters), particularly (k,n,m)=(4,2,-31) and (3,2,-13), corresponding to E_8 and E_6, respectively. Beside these solutions, there are other solutions e.g. (k,n,m)=(41,6,2) denoted Y_{47} with $(\alpha,\beta,\gamma)=(1,6,14)$ with dimension -492. It is unclear, to what objects these type solutions correspond to, but they have features similar to others, particularly they give integers (of both signs) when substituted into dimension formulae.

Complete set of all seven Diophantine equations is given in [8], namely (after shift of variable so that r.h.s. becomes linear): $knm=l_i, i=1,2...7, l_1=4k+2m+2n+8, l_2=2k+4n+2m+10, l_3=4k+4n+2m+12, l_4=4m+4n+4k+16, l_5=4m+4n+k+8, l_6=m+2n+4k+6, l_7=2m+2n+2k+9$.

Complete set of solutions [8] includes as isolated solutions all exceptional groups, algebra $E_{7\frac{1}{2}}$ [9], two yet unidentified algebras X_1, X_2 of dimensions 156, 99, and 47 objects $Y_i, i=1,2,...47$ of negative dimensions. Series solutions contain, besides classical series (including symplectic Sp(N) with odd N, i.e. half integer rank), also 3d line $\alpha+\beta+2\gamma=0$, and 0d line $\alpha+2\beta+2\gamma=0$. Further study of their properties is presented in [13]. Particularly, the simple transformation [5] $\alpha'=\alpha, \beta'=\gamma-\beta, \gamma'=\beta$ gives a universally characterized subgroup of initial group. This explains an appearance of 3d and 0d lines - former appears in this way from line $D_{2,1,\lambda}$, and 0d line appear in the same way from 3d line.

P. Deligne [12] suggested that universal characters satisfy usual character relations for decomposition of product of representations at all values of universal parameters. He checked that up to some extent, particularly he carry on complete check for SL(N) line. Some additional cases with an arbitrary universal parameters have been checked in [13].

Another universalization result is universal representation of many quantities (perturbative partition function, central charge, unknot Wilson average) of Chern-Simons theory on 3d sphere [7]. The simple universal integral representation for complete (i.e. including both perturbative and nonperturbative contributions) partition function is found in [10], which reveals some other relations to number theory:

$$-\ln(Z)=(dim/2)\ln(\delta/t)+\int_0^\infty \frac{dx}{x}\frac{f(x/\delta)-f(x/t)}{(e^x-1)} \tag{3}$$

where δ in minimal normalization (when square of long root is 2, i.e. $\alpha=-2$) is usual shifted coupling of Chern-Simons theory. The non-perturbative part of Chern-Simons' partition function gives the universal expression for invariant volume of compact simple Lie groups [10]. Both quantities can be represented [11] as a ratio of quadruple Barnes' gamma functions. Moreover, the partition functions of Chern-Simons theory on classical and exceptional lines can be represented [13] as ratio of triple an double sine functions. The product representation of multiple sine functions gives a Gopakumar-Vafa representation of partition function (plus non-perturbative terms), hence corresponding invariants of manifolds after geometrical transition. Recurrent relations of Barnes' gamma-functions lead [11] to level-rank duality of SU(N) Chern-Simons theory. As discussed in [11, 10], volume function is not an analytical function on CP^2 but rather the fiber bundle of analytical functions with structure group S_3 of permutations of universal parameters and transition functions which in SU(N) case are given by Kinkelin's functional equation on Barnes' G-function. Analog of Ooguri-Vafa [14] expansion of volume for exceptional line is calculated [13].

References

[1] P. Vogel *Algebraic structures on modules of diagrams.* Preprint (1995), www.math.jussieu.fr/∼vogel/diagrams.pdf, J. Pure Appl. Algebra **215** (2011), no. 6, 1292–1339.

[2] P. Deligne *La série exceptionnelle des groupes de Lie.* C. R. Acad. Sci. Paris, Série I **322** (1996), 321-326.

[3] G. 't Hooft *A planar diagram theory for strong interactions.* Nucl.Phys. **B72** (1974), 461-473.

[4] R.L. Mkrtchyan *The equivalence of $Sp(2N)$ and $SO(-2N)$ gauge theories.* Phys. Lett. **105B** (1981), 174–176.

[5] J.M. Landsberg and L. Manivel *A universal dimension formula for complex simple Lie algebras.* Adv. Math. **201** (2006), 379-407.

[6] R.L. Mkrtchyan, A.N. Sergeev, A.P. Veselov *Casimir eigenvalues for universal Lie algebra.* arXiv:1105.0115 (2011).

[7] R.L. Mkrtchyan, A.P. Veselov, *Universality in Chern-Simons theories*, arXiv:1203.0766, JHEP08 (2012) 153.

[8] R.L. Mkrtchyan, *On a map of Vogel's plane*, arxiv:1209.5709.

[9] J. M. Landsberg and L. Manivel *The sextonions and $E_{7\frac{1}{2}}$* Adv. Math. (2006), 201(1): 143-179, MR2204753.

[10] R.L. Mkrtchyan,*Nonperturbative universal Chern-Simons theory*, arXiv:1302.1507, JHEP09(2013)054.

[11] R.L. Mkrtchyan, *Universal Chern-Simons partition functions as quadruple Barnes' gamma-functions*, arXiv:1309.2450, JHEP10(2013)190.

[12] P. Deligne, 2013, unpublished.

[13] R.L. Mkrtchyan, 2013-2014, unpublished.

[14] H. Ooguri and C. Vafa, *Worldsheet derivation of a large N duality.* Nucl. Phys. **B641** (2003), hep-th/0205297.

The colored Jones polynomial, the Chern-Simons invariant, and the Reidemeister torsion of a cable knot

Hitoshi Murakami

Let $J_N(K;q) \in \mathbb{Z}[q,q^{-1}]$ be the N-colored Jones polynomial of a knot K in the three-dimensional sphere S^3, associated with the irreducible N-dimensional representation of the Lie algebra $\mathfrak{sl}_2(\mathbb{C})$ [6, 8]. The following conjecture (Volume Conjecture) is well known.

Conjecture 0.1 ([7, 10]). *For any knot, the following would hold:*

$$\lim_{N\to\infty} \frac{\log\left|J_N\bigl(K;\exp(2\pi\sqrt{-1}/N)\bigr)\right|}{N} = \frac{\mathrm{Vol}(S^3\setminus K)}{2\pi},$$

where Vol *denotes the simplicial volume, that is, the sum of the hyperbolic volumes of the hyperbolic pieces in* $S^3 \setminus K$ *after the JSJ decomposition* [4, 5].

More generally, we are interested in the asymptotic behavior of the colored Jones polynomial evaluated at $\exp((2\pi\sqrt{-1}+u)/N)$ for a complex number u. Y. Yokota and I proved the following formula

Theorem 1 ([11, 9]). *Let E be the figure-eight knot and u be a real number with $0 < u < \log((3+\sqrt{5})/2)$. Then we have*

$$J_N\bigl(E;\exp((2\pi\sqrt{-1}+u)/N)\bigr) \underset{N\to\infty}{\sim} \frac{\sqrt{-\pi}}{2\sinh(u/2)} \exp\left(\frac{N}{2\pi\sqrt{-1}+u}S(u)\right)\left(\frac{N}{2\pi\sqrt{-1}+u}\right)^{1/2}\mathbb{T}(u)^{1/2},$$

where $S(u) - u\left(d\,S(u)/d\,u + \pi\sqrt{-1}\right)/2$ is the Chern–Simons invariant associated with a representation $\rho_u\colon \pi_1(S^3\setminus E) \to \mathrm{SL}(2;\mathbb{C})$ sending the meridian to $\begin{pmatrix} e^{u/2} & * \\ 0 & e^{-u/2}\end{pmatrix}$, and $\mathbb{T}(u)$ is the Reidemeister torsion twisted by ρ_u.

It is conjectured that a similar formula holds for any hyperbolic knot [2, 1].

On the other hand, for a torus knot, which is not hyperbolic (in fact there is no hyperbolic pieces in its complement, and so its simplicial volume is 0), we have the following result. Let $\Delta(K;t)$ denote the Alexander polynomial of a knot K.

Theorem 2 ([3]). *Let $T(2,2a+1)$ be the torus knot of type $(2,2a+1)$ with positive integer a. If ξ is a complex number with $\mathrm{Im}\,\xi > 0$ and $\mathrm{Re}\,\xi < 0$, then we have*

$$J_N\bigl(T(2,2a+1);\exp(\xi/N)\bigr) \underset{N\to\infty}{\sim} \frac{1}{\Delta\bigl(T(2,2a+1);\exp\xi\bigr)} + \frac{\sqrt{-\pi}}{2\sinh(\xi/2)}\left(\frac{N}{\xi}\right)^{1/2} \sum_{k=1}^{\lfloor (2a+1)|\xi|/\pi\rfloor} (-1)^{k+1}\exp\left(\frac{N}{\xi}S_k(\xi)\right)\mathbb{T}_k^{1/2},$$

where $S_k(u)-u\left(d\,S_k(u)/d\,u+\pi\sqrt{-1}\right)/2$ *is the Chern–Simons invariant associated with a representation* $\rho_{u,k}\colon \pi_1(S^3\setminus T(2,2a+1))\to \mathrm{SL}(2;\mathbb{C})$ *sending the meridian to* $\begin{pmatrix} e^{u/2} & * \\ 0 & e^{-u/2}\end{pmatrix}$*, and* $\mathbb{T}_k$ *is the Reidemeister torsion twisted by* $\rho_{u,k}$*.*

Note that any non-Abelian representation is given by ρ_k for some k, and the Alexander polynomial can be regarded as the Reidemeister torsion associated with an Abelian representation.

In this talk I show the following theorem.

Theorem 3. *Let* $T(2,2a+1)^{(2,2b+1)}$ *be the* $(2,2b+1)$*-cable of* $T(2,2a+1)$ *with* $a>0$ *and* $2b+1-4(2a+1)>0$*. We decompose* $S^3\setminus T(2,2a+1)^{(2,2b+1)}$ *into* $U:=S^3\setminus \operatorname{Int} N\left(T(2,2a+1)\right)$ *and* $V:=(D^2\times S^1)\setminus P^{(2,2b+1)}$*, where* $\operatorname{Int} N(X)$ *is the open tubular neighborhood of* X *in* S^3 *and* $P^{(2,2b+1)}$ *is a knot in the solid torus* $D^2\times S^1$ *that is wrapped twice along* S^1 *and is twisted* $2b+1$ *times around* S^1*. If* ξ *is a complex number with* $\operatorname{Im}\xi>0$ *and* $\operatorname{Re}\xi<0$*, then we have*

$$
\begin{aligned}
&J_N\left(T(2,2a+1)^{(2,2b+1)};\exp(\xi/N)\right)\\
\underset{N\to\infty}{\sim}\;&\frac{1}{\Delta\left(T(2,2a+1)^{(2,2b+1)};\exp\xi\right)}\\
&+\frac{\sqrt{-\pi}}{2\sinh(\xi/2)}\left(\frac{N}{\xi}\right)^{1/2}\sum_{j}(-1)^{j}\exp\left(\frac{N}{\xi}S_{1,j}(\xi)\right)\mathbb{T}_{1,j}^{1/2}\\
&+\frac{\sqrt{-\pi}}{2\sinh(\xi/2)}\left(\frac{N}{\xi}\right)^{1/2}\sum_{k}(-1)^{k+1}\exp\left(\frac{N}{\xi}S_{2,k}(\xi)\right)\mathbb{T}_{2,k}^{1/2}\\
&+\frac{\pi}{\sqrt{2}\sinh(\xi/2)}\left(\frac{N}{\xi}\right)\sum_{l,m}(-1)^{l+m}\exp\left(\frac{N}{\xi}S_{3,l,m}(\xi)\right)\mathbb{T}_{3,l,m}^{1/2},
\end{aligned}
$$

where

(1) $S_{1,j}(\xi)$ *determines the Chern–Simons invariant associated with a representation* $\rho_{1,j}(\xi)$ *such that* $\rho_{1,j}(\xi)\big|_{\pi_1(U)}$ *is Abelian and* $\rho_{1,j}(\xi)\big|_{\pi_1(V)}$ *is non-Abelian, and* $\mathbb{T}_{1,j}$ *is the Reidemeister torsion twisted by* $\rho_{1,j}(\xi)$*,*

(2) $S_{2,k}(\xi)$ *determines the Chern–Simons invariant associated with a representation* $\rho_{2,k}(\xi)$ *such that* $\rho_{2,k}(\xi)\big|_{\pi_1(U)}$ *is non-Abelian and* $\rho_{2,k}(\xi)\big|_{\pi_1(V)}$ *is Abelian, and* $\mathbb{T}_{2,k}$ *is the Reidemeister torsion twisted by* $\rho_{2,k}(\xi)$*, and*

(3) $S_{3,l,m}(\xi)$ *determines the Chern–Simons invariant associated with a representation* $\rho_{3,l,m}(\xi)$ *such that both* $\rho_{3,l,m}(\xi)\big|_{\pi_1(U)}$ *and* $\rho_{3,l,m}(\xi)\big|_{\pi_1(V)}$ *are non-Abelian, and* $\mathbb{T}_{3,l,m}$ *is the Reidemeister torsion twisted by* $\rho_{3,l,m}(\xi)$*,.*

Note that the Alexander polynomial can be regarded as the Reidemeister torsion associated with a representation $\rho_0(\xi)$ *such that both* $\rho_0(\xi)\big|_{\pi_1(U)}$ *and* $\rho_0(\xi)\big|_{\pi_1(V)}$ *are Abelian.*

References

[1] T. Dimofte and S. Gukov, *Quantum field theory and the volume conjecture*, Interactions between hyperbolic geometry, quantum topology and number theory, Contemp. Math., vol. 541, Amer. Math. Soc., Providence, RI, 2011, pp. 41–67. MR 2796627 (2012c:58037)

[2] S. Gukov and H. Murakami, $SL(2,\mathbf{C})$ *Chern-Simons theory and the asymptotic behavior of the colored Jones polynomial*, Modular Forms and String Duality (N. Yui, H. Verrill, and C.F. Doran, eds.), Fields Inst. Comm., vol. 54, Amer. Math. Soc. and Fields Inst., 2008, pp. 261–278.

[3] K. Hikami and H. Murakami, *Representations and the colored Jones polynomial of a torus knot*, Chern-Simons gauge theory: 20 years after, AMS/IP Stud. Adv. Math., vol. 50, Amer. Math. Soc., Providence, RI, 2011, pp. 153–171. MR 2809451 (2012f:57028)

[4] W. H. Jaco and P. B. Shalen, *Seifert fibered spaces in 3-manifolds*, Mem. Amer. Math. Soc. **21** (1979), no. 220, viii+192. MR 81c:57010

[5] K. Johannson, *Homotopy equivalences of 3-manifolds with boundaries*, Lecture Notes in Mathematics, vol. 761, Springer, Berlin, 1979. MR 82c:57005

[6] V. F. R. Jones, *A polynomial invariant for knots via von Neumann algebras*, Bull. Amer. Math. Soc. (N.S.) **12** (1985), no. 1, 103–111. MR 86e:57006

[7] R. M. Kashaev, *The hyperbolic volume of knots from the quantum dilogarithm*, Lett. Math. Phys. **39** (1997), no. 3, 269–275. MR 98b:57012

[8] A. N. Kirillov and N. Yu. Reshetikhin, *Representations of the algebra* $U_q(\mathrm{sl}(2))$, *q-orthogonal polynomials and invariants of links*, Infinite-dimensional Lie algebras and groups (Luminy-Marseille, 1988), Adv. Ser. Math. Phys., vol. 7, World Sci. Publishing, Teaneck, NJ, 1989, pp. 285–339. MR MR1026957 (90m:17022)

[9] H. Murakami, *The coloured Jones polynomial, the Chern-Simons invariant, and the Reidemeister torsion of the figure-eight knot*, J. Topol. **6** (2013), no. 1, 193–216. MR 3029425

[10] H. Murakami and J. Murakami, *The colored Jones polynomials and the simplicial volume of a knot*, Acta Math. **186** (2001), no. 1, 85–104. MR 2002b:57005

[11] H. Murakami and Y. Yokota, *The colored Jones polynomials of the figure-eight knot and its Dehn surgery spaces*, J. Reine Angew. Math. **607** (2007), 47–68. MR MR2338120

Idèle theory for 3-manifolds

Hirofumi Niibo
(joint work with Jun Ueki)

The purpose of this report is, following the sprit of arithmetic topology ([Mo1], [Mo2], [Mo3]), to study an idèle theoretic form of class field theory for 3-manifolds. We note that idèlic class field theory for 3-manifolds was firstly studied by A. Sikora ([Si1], [Si2]). Our approach is different from his and elementary.

1. Local class field theory for tori

Let k be a number field. For a finite prime $\mathfrak{p}$, let $v_{\mathfrak{p}}$ be the corresponding additive valuation of k, and $k_{\mathfrak{p}}$ be the local field obtained as the completion of a number field k. Let $\mathcal{O}_{\mathfrak{p}}$ be the valuation ring and let $\mathbb{F}_{\mathfrak{p}}$ be the residue field $\mathcal{O}_{\mathfrak{p}}/\mathfrak{p}$, a finite extension of $\mathbb{F}_p = \mathbb{Z}/p\mathbb{Z}$. We denote by $U_{\mathfrak{p}}$ the unit group $\mathcal{O}_{\mathfrak{p}}^{\times}$. We note that $U_{\mathfrak{p}} = \mathrm{Ker}(v_{\mathfrak{p}})$ and so we have the following split exact sequence

$$0 \to U_{\mathfrak{p}} \to k_{\mathfrak{p}}^{\times} \to \mathbb{Z} \to 0. \tag{1.1}$$

Let $k_{\mathfrak{p}}^{\mathrm{ab}}$ be the maximal Abelian extension of $k_{\mathfrak{p}}$. When $k_{\mathfrak{p}}$ is non-archimedean, we denote by $k_{\mathfrak{p}}^{\mathrm{ur}}$ the maximal unramified extension of $k_{\mathfrak{p}}$. Note that the Galois group $\mathrm{Gal}(k_{\mathfrak{p}}^{\mathrm{ur}}/k_{\mathfrak{p}})$ is identified with $\mathrm{Gal}(\bar{\mathbb{F}}_{\mathfrak{p}}/\mathbb{F}_{\mathfrak{p}}) \cong \hat{\mathbb{Z}}$, where $\hat{\mathbb{Z}}$ denotes the profinite completion of $\mathbb{Z}$. A main part of classical local class field theory for the local field $k_{\mathfrak{p}}$ is stated as follows.

Theorem 1 (Classical local class field theory)**.** *There is a canonical homomorphism, called the local reciprocity homomorphism, $\rho_{k_{\mathfrak{p}}} : k_{\mathfrak{p}}^{\times} \to \mathrm{Gal}(k_{\mathfrak{p}}^{\mathrm{ab}}/k_{\mathfrak{p}})$ which satisfies the following commutative diagrams with exact horizontal sequences:*

$$\begin{tikzcd}
0 \arrow[r] & U_{\mathfrak{p}} \arrow[r] \arrow[d, "\rho_{k_{\mathfrak{p}}}|_{U_{\mathfrak{p}}}"] & k_{\mathfrak{p}}^{\times} \arrow[r, "v_{\mathfrak{p}}"] \arrow[d, "\rho_{k_{\mathfrak{p}}}"] & \mathbb{Z} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathrm{Gal}(k_{\mathfrak{p}}^{\mathrm{ab}}/k_{\mathfrak{p}}^{\mathrm{ur}}) \arrow[r] & \mathrm{Gal}(k_{\mathfrak{p}}^{\mathrm{ab}}/k_{\mathfrak{p}}) \arrow[r] & \mathrm{Gal}(\bar{\mathbb{F}}_{\mathfrak{p}}/\mathbb{F}_{\mathfrak{p}}) \arrow[r] & 0.
\end{tikzcd}$$

Then we present a topological analogue of local class field theory for 2-dimensional tori based on the following analogies.

$$(1.2)\qquad \begin{array}{ccc}
\text{tubular neighborhood of } K & & \mathfrak{p}\text{-adic integers} \\
V_K & \longleftrightarrow & \mathrm{Spec}(\mathcal{O}_{\mathfrak{p}}) \\
\text{boundary of } V_K & & \mathfrak{p}\text{-adic field} \\
\partial V_K \simeq V_K \setminus K & \longleftrightarrow & \mathrm{Spec}(k_{\mathfrak{p}}) = \mathrm{Spec}(\mathcal{O}_{\mathfrak{p}}) \setminus \mathrm{Spec}(\mathcal{O}_{\mathfrak{p}}/\mathfrak{p})
\end{array}$$

Let K be a fixed knot in an orientable 3-manifold and let V_K be a tubular neighborhood of K. Let $T_K = \partial V_K$ be the boundary of V_K. According to (1.2), T_K and V_K are regarded as analogues of a $\mathfrak{p}$-adic local field $k_{\mathfrak{p}}$ and the integer ring $\mathcal{O}_{\mathfrak{p}}$. Let m and l be a meridian and a longitude on T_K, respectively. The inclusion $T_K \hookrightarrow V_K$ induces the homomorphism $v_K : H_1(T_K) \to H_1(V_K) = \mathbb{Z}[l]$ whose kernel is $\mathbb{Z}[m]$. Thus we have the exact sequence

$$(1.3)\qquad 0 \longrightarrow \mathbb{Z}[m] \longrightarrow H_1(T_K) \longrightarrow \mathbb{Z}[l] \longrightarrow 0$$

which may be regarded as an analogue of the exact sequence (1.1).

Let T_K^{ab} be the maximal Abelian covering of T_K. Since $V_K \setminus K$ is homotopy equivalent to the torus T_K, (unramified) coverings of T_K correspond to ramified covering of V_K along K. Let T_K^{ur} be the maximal covering of T_K which comes from the maximal (unramified) covering of V_K. Then we have the following theorem which is regarded as an analogy of Theorem 1.

Theorem 2 (Local class field theory for tori [N])**.** *There is a canonical isomorphism $\rho_{T_K} : H_1(T_K) \to \mathrm{Gal}(T_K^{\mathrm{ab}}/T_K)$ which satisfies following commutative diagram with exact horizontal sequences:*

$$\begin{tikzcd}
0 \arrow[r] & \mathbb{Z}[m] \arrow[r] \arrow[d, "\rho_{T_K}|_{\mathbb{Z}[m]}"] & H_1(T_K) \arrow[r, "v_K"] \arrow[d, "\rho_{T_K}"] & \mathbb{Z}[l] \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathrm{Gal}(T_K^{\mathrm{ab}}/T_K^{\mathrm{ur}}) \arrow[r] & \mathrm{Gal}(T_K^{\mathrm{ab}}/T_K) \arrow[r] & \mathrm{Gal}(T_K^{\mathrm{ur}}/T_K) \arrow[r] & 0.
\end{tikzcd}$$

2. Idèlic global local class field theory for 3-manifolds

Now, let k be a number field. We define the idèle group I_k of k by the following restricted product of $k_{\mathfrak{p}}^{\times}$'s with respect to $U_{\mathfrak{p}}$'s over all primes $\mathfrak{p}$ of k:

$$I_k := \left\{ (a_{\mathfrak{p}})_{\mathfrak{p}} \in \prod_{\mathfrak{p}\text{: prime}} k_{\mathfrak{p}}^{\times} \mid v_{\mathfrak{p}}(a_{\mathfrak{p}}) = 0 \text{ for almost all finite prime } \mathfrak{p} \right\}.$$

Since $k^{\times}$ is embedded into I_k diagonally, we let P_k be the image of $k^{\times}$ in I_k and call it the group of principal idèles. We then define the idèle class group of k by $C_k := I_k/k^{\times}$.

Let F/k be an Abelian extension, and be $N_{F/k} : C_F \to C_k$ the norm map. Then we equip the idele class group C_k with a topology by declaring the cosets $aN_{F/k}(C_F)$ to be a basis of neighbourhoods of $a \in C_k$, where F/k varies over all finite Abelian extensions of k. We call this topology the norm topology of C_k.

Theorem 3 (Classical global class field theory)**.** *There is a canonical homomorphism, called the global reciprocity map, $\rho_k : C_k \to \mathrm{Gal}(k^{\mathrm{ab}}/k)$ which has the following properties:*

1. *For any finite Abelian extension F/k, ρ_k induces the isomorphism $C_k/N_{F/k}(C_F) \cong \mathrm{Gal}(F/k)$ where $N_{F/k}$ denotes the norm map on the idèle groups.*
2. *The map $F \mapsto N_{F/k}(C_F)$ is a 1:1 correspondence between the finite abelian extensions F/k and the open subgroups of finite index in C_k. The field F/k corresponding to the subgroup N of C_k satisfies $\mathrm{Gal}(F/k) \cong C_k/N$.*

Then we construct a topological analogue of global class field theory for 3-manifolds. For a certain given a link $\mathcal{K}$ with at most countably many components in a compact oriented connected 3-manifold M, we introduce the *idèle group* $I_{(M;\mathcal{K})}$ as a restricted product of $H_1(\partial V_K;\mathbb{Z})$ over all K in $\mathcal{K}$, and getting local reciprocity map ρ_{T_K}'s together over all K in $\mathcal{K}$, we define the homomorphism

$$\varphi_{(M;\mathcal{K})} : I_{(M;\mathcal{K})} \to \mathrm{Gal}(M;\mathcal{K})^{\mathrm{ab}} := \varprojlim_{L} \mathrm{Gal}(X_L^{\mathrm{ab}}/X_L)$$

where L runs over all finite subsets of $\mathcal{K}$, $X_L := M \setminus L$ and X_L^{ab} is the maximal Abelian covering of X_L. The homomorphism $\varphi_{(M;\mathcal{K})}$ factors through the *idèle class group* $C_{(M;\mathcal{K})} := I_{(M;\mathcal{K})}/P_{(M;\mathcal{K})}$ with the *principal idèle group* $P_{(M;\mathcal{K})}$, and hence we obtain an analogue of the global reciprocity homomorphism $\rho_{(M;\mathcal{K})} : C_{(M;\mathcal{K})} \to \mathrm{Gal}(M;\mathcal{K})^{\mathrm{ab}}$.

Let $h : N \to M$ be an Abelian cover ramified over $L \subset \mathcal{K}$. We define the norm map $h_{(N/M)} : C_{(N;h^{-1}(\mathcal{K}))} \to C_{(M;\mathcal{K})}$. Then we equip the idele class group $C_{(M;\mathcal{K})}$ with a topology by declaring the cosets $a + h_{N/M}(C_{N;h^{-1}(\mathcal{K})})$ to be a basis of neighbourhoods of $a \in C_{(M;\mathcal{K})}$, where $h : N \to M$ varies over all finite Abelian cover ramified over $L \subset \mathcal{K}$. We call this topology the norm topology of $C_{(M;\mathcal{K})}$. Then our main result is stated as follows. ([N], [NU])

Theorem 4 (Idèlic global class field theory for a 3-manifold $(M;\mathcal{K})$ [NU])**.** *Let M be a rational homology sphere, i.e. $H_i(M;\mathbb{Q}) \cong H_i(S^3;\mathbb{Q})$.*

(1). *$\phi_{(M;\mathcal{K})}$ factors through $\rho_{(M;\mathcal{K})} : C_{(M;\mathcal{K})} \to \mathrm{Gal}(M;\mathcal{K})^{\mathrm{ab}}$ such that for any finite Abelian cover $h : N \to M$ ramified over a finite subset of $\mathcal{K}$, $\rho_{(M;\mathcal{K})}$ induces an isomorphism $C_{(M;\mathcal{K})}/h_*(C_{(N;h^{-1}(\mathcal{K}))}) \cong \mathrm{Gal}(N/M)$.*

(2). *The map $(h : N \to M) \mapsto h_{N/M}(C_N)$ is a 1:1 correspondence between the finite Abelian cover ramified over $L \subset \mathcal{K}$ and the open subgroups of finite index in $C_{(M;\mathcal{K})}$. The ramified cover $h : N \to M$ corresponding to the subgroup H of $C_{(M;\mathcal{K})}$ satisfies $\mathrm{Gal}(N/M) \cong C_{(M;\mathcal{K})}/H$.*

This result may be regarded as an analogue of the fundamental theorem in global class field theory for number fields ([Ne]).

References

[Mo1] M. Morishita, On certain analogies between knots and primes, J. Reine Angew. Math., **550** (2002), 141-167.

[Mo2] M. Morishita, Analogies between knots and primes, 3-manifolds and number rings, SUGAKU EXPOSITIONS, **23**, No.1, (2010), 1-30.

[Mo3] M. Morishita, Knots and primes. An introduction to arithmetic topology. Universitext. Springer, London, 2012.

[Ne] J. Neukirch, Algebraic Number Theory, Grundlehren Der Mathematischen Wissenschaften. Volume 322. Springer-Verlag, 1999.

[N] H. Niibo, Idèlic calss field theory for 3-manifolds, to be published in Kyushu Journal of Mathematics

[NU] H. Niibo, J. Ueki, Idèlic class field theory over rational homology 3-spheres, preprint.

[Si1] A. Sikora, Idelic Topology, Notes for personal use, unpublished note.

[Si2] A. Sikora, Slides from a talk at the workshop "Low dimensional topology and number theory III", March, 2011, Fukuoka.

q-hypergeometric series, alternating knots and identities

Robert Osburn

(joint work with Adam Keilthy (TCD))

Two of the most important results in the theory of q-series are the classical Rogers-Ramanujan identities which state that

$$\sum_{n\geq 0} \frac{q^{n^2+sn}}{(q)_n} = \frac{1}{(q^{1+s};q^5)_\infty (q^{4-s};q^5)_\infty} \tag{1}$$

where $s = 0$ or 1 and

$$(a)_n = (a;q)_n = \prod_{k=1}^{n}(1-aq^{k-1}),$$

valid for $n \in \mathbb{N} \cup \{\infty\}$. In 1974, Andrews [1] obtained a generalization of (1) to odd moduli, namely for all $k \geq 2$, $1 \leq i \leq k$,

$$\begin{aligned}(2)\qquad \sum_{n_1,n_2,\dots,n_{k-1}\geq 0} & \frac{q^{N_1^2+N_2^2+\cdots+N_{k-1}^2+N_i+N_{i+1}+\cdots+N_{k-1}}}{(q)_{n_1}(q)_{n_2}\cdots(q)_{n_{k-1}}} \\ &= \frac{(q^i;q^{2k+1})_\infty(q^{2k+1-i};q^{2k+1})_\infty(q^{2k+1};q^{2k+1})_\infty}{(q)_\infty}\end{aligned}$$

where $N_j = n_j + n_{j+1} + \cdots + n_{k-1}$. There has been recent interest in the appearance of these (and similar) identities in knot theory. For example, Hikami [10] considered (1) from the perspective of the colored Jones polynomial of torus knots while Armond and Dasbach [3] gave a skein-theoretic proof of (2). For similar identities related to false theta series, see [9] and for other connections between q-series and quantum invariants of knots, see [4]–[6], [8], [11] and [13].

We consider recent work in [7] whereby the q-multisums $\Phi_K(q)$ and $\Phi_{-K}(q)$ were associated to a given alternating knot K and its mirror $-K$. The q-multisum $\Phi_K(q)$ occurs as the 0-limit (or "tail") of the colored Jones polynomial of K (see Theorem 1.10 in [7]). In Appendix D of [7], Garoufalidis and Lê (with Zagier) conjectured evaluations of $\Phi_K(q)$ for 21 knots and of $\Phi_{-K}(q)$ for 22 knots in terms of modular forms and false theta series and state "every such guess is a q-series identity whose proof is unknown to us". Before stating these conjectures, we recall some notation from [7]. For a positive integer b, we define

$$h_b = h_b(q) = \sum_{n\in\mathbb{Z}} \epsilon_b(n) q^{\frac{bn(n+1)}{2}-n}$$

where

$$\epsilon_b(n) = \begin{cases} (-1)^n & \text{if } b \text{ is odd,} \\ 1 & \text{if } b \text{ is even and } n \geq 0, \\ -1 & \text{if } b \text{ is even and } n < 0. \end{cases}$$

Note that $h_1(q) = 0$, $h_2(q) = 1$ and $h_3(q) = (q)_\infty$. For an integers p, a and b, let K_p denote the pth twist knot obtained by $-1/p$ surgery on the Whitehead link and $T(a,b)$ the left-handed (a,b) torus knot. The 43 conjectures from [7] are as follows:

Here, we have corrected the entries for 6_1, 7_3, 8_1, 8_4, 8_5, K_p, $p<0$ (and their mirrors) and 7_5 in Appendix D of [7]. Note that a conjectural evaluation for $\Phi_{8_5}(q)$ is not currently known. Three of these Rogers-Ramanujan type identities, namely

$$(3)\qquad \Phi_{3_1}(q) = h_3, \quad \Phi_{4_1}(q) = h_3 \quad \text{and} \quad \Phi_{6_3}(q) = h_3^2$$

K	$\Phi_K(q)$	$\Phi_{-K}(q)$
3_1	h_3	1
4_1	h_3	h_3
5_1	h_5	1
5_2	h_4	h_3
6_1	h_5	h_3
6_2	h_3h_4	h_3
6_3	h_3^2	h_3^2
7_1	h_7	1
7_2	h_6	h_3
7_3	h_5	h_4
7_4	h_4^2	h_3
7_5	h_3h_4	h_4
7_6	h_3h_4	h_3^2
7_7	h_3^3	h_3^2
8_1	h_7	h_3
8_2	h_3h_6	h_3
8_3	h_5	h_5
8_4	h_3	h_4h_5
8_5	?	h_3
K_p, $p > 0$	h_{2p}	h_3
K_p, $p < 0$	$h_{2\lvert p\rvert+1}$	h_3
$T(2,p)$, $p > 0$	h_{2p+1}	1

Table 2.

have been proven by Andrews [2]. Motivated by his work (and in conjunction with (3)), the purpose of the talk given on August 19, 2014 at Oberwolfach was to highlight the role of q-series techniques in proving identities arising from knot theory. In particular, we discussed the following main result from [12]:

Theorem 1. *The identities in Table 2 are true.*

References

[1] G.E. Andrews, *An analytic generalization of the Rogers-Ramanujan identities for odd moduli*, Proc. Nat. Acad. Sci. U.S.A **71** (1974), 4082–4085.

[2] G.E. Andrews, *Knots and q-series*, Proceedings of Ramanujan 125, University of Florida, November 5-7, 2012, to appear.

[3] C. Armond, O. Dasbach, *Rogers-Ramanujan type identities and the head and tail of the colored Jones polynomial*, preprint available at `http://arxiv.org/abs/1106.3948`

[4] K. Bringmann, K. Hikami, J. Lovejoy, *On the modularity of the unified WRT invariants of certain Seifert manifolds*, Adv. in Appl. Math. **46** (2011), no. 1-4, 86–93.

[5] S. Garoufalidis, *Quantum knot invariants*, preprint available at `http://arxiv.org/abs/1201.3314`

[6] S. Garoufalidis, R. Kashaev, *From state integrals to q-series*, Math. Res. Lett., to appear.

[7] S. Garoufalidis, T. Lê, *Nahm sums, stability and the colored Jones polynomial*, Research in the Mathematical Sciences, to appear.

[8] S. Garoufalidis, T. Vuong, *Alternating knots, planar graphs and q-series*, Ramanujan J., to appear.
[9] M. Hajij, *The tail of a quantum spin network*, preprint available at `http://arxiv.org/abs/1308.2369`
[10] K. Hikami, *Volume conjecture and asymptotic expansion of q-series*, Experiment. Math. **12** (2003), no. 3, 319–337.
[11] K. Hikami, *Hecke type formula for unified Witten-Reshetikhin-Turaev invariants as higher-order mock theta functions*, Int. Math. Res. Not. IMRN 2007, no. 7, Art. ID rnm 022, 32pp.
[12] A. Keilthy, R. Osburn, *Rogers-Ramanujan type identities for alternating knots*, preprint available at `http://arxiv.org/abs/1407.3482`
[13] R. Lawrence, D. Zagier, *Modular forms and quantum invariants of 3-manifolds*, Asian J. Math. **3** (1999), no. 1, 93–107.

Homology and volume of hyperbolic 3-orbifolds, and enumeration of arithmetic groups

Peter B. Shalen

If $\mathcal{O}$ is a maximal order in a quaternion algebra over a number field k, then the group of units $\Gamma_{\mathcal{O}}$ of $\mathcal{O}$ is a natural object of study from the number-theoretical point of view. If k has exactly one complex place and B is ramified at all the real places of k then $\Gamma_{\mathcal{O}}$ is isomorphic in a natural way to a lattice in $PGL(2, \mathbb{C})$. I will refer to lattices that arise up to commensurability from this construction as arithmetic lattices.

A theorem of Borel's [4] asserts that for any positive real number V, there are at most finitely many arithmetic lattices of covolume at most V. Determining all of these for a given V is in principle algorithmically possible for a given V thanks to work by Chinburg and Friedman [5], but appears to be impractical except for very small values of V, say $V = 0.41$. (The smallest covolume of a lattice in $PGL(2, \mathbb{C})$ is about 0.39.) The step that makes computation impractical is obtaining a good upper bound, in the case of a lattice Γ that is maximal in its commensurability class, for the order of a certain elementary 2-group which is a quotient of Γ. The order of such an elementary 2-group is trivially bounded by 2^d where $d = \dim H_1(\Gamma, \mathbb{Z}/2\mathbb{Z})$.

In the case of a torsion-free lattice, Γ not necessarily arithmetic, joint work of mine with Marc Culler and others [1], [6], [7], gives good bounds on the dimension of $H_1(\Gamma, \mathbb{Z}/2\mathbb{Z})$ in the presence of a suitable bound on the volume of Γ. The results are stated in terms of hyperbolic 3-manifolds; if Γ is a torsion-free lattice in $PGL(2, \mathbb{C})$ then $M = \mathbb{H}^3/\Gamma$ is an orientable hyperbolic 3-manifold, the volume of M is the covolume of Γ, and we have $H_1(M, \mathbb{Z}/2\mathbb{Z}) \cong H_1(\Gamma, \mathbb{Z}/2\mathbb{Z})$.

Given an orientable hyperbolic 3-manifold M, let us set $d = \dim H_1(M, \mathbb{Z}/2\mathbb{Z})$ and let v denote the volume of M. It was shown in [1], [6], and [7], respectively, that

- if $v \leq 1.22$ then $d \leq 3$;
- if $v \leq 3.08$ then $d \leq 5$; and
- if $v \leq 3.44$ then $d \leq 7$.

These results cannot be applied directly to maximal arithmetic lattices, because they typically have torsion. When Γ has torsion, $O = \mathbb{H}^3/\Gamma$ is an orientable hyperbolic 3-orbifold, the volume of O is the covolume of Γ, and we have $H_1(O, \mathbb{Z}/2\mathbb{Z}) \cong H_1(\Gamma, \mathbb{Z}/2\mathbb{Z})$. The purpose of my talk was to describe work in progress concerned with finding results qualitatively similar to the ones given in [1], [6], and [7], which apply to the orbifold case and will be of practical use in enumerating arithmetic lattices with covolume subject to certain bounds.

I described the following results as "probable" in my talk because I had (and have) not yet checked all the details of the proofs, but I believe the essential arguments are in place.

Probable Theorem 1. *Let $O = \mathbb{H}^3/\Gamma$ be an orientable hyperbolic 3-orbifold. Suppose that every finite subgroup of Γ is cyclic, and that no subgroup of Γ is a hyperbolic triangle group. If O has volume at most 1.72, then $\dim H_1(O, \mathbb{Z}/2\mathbb{Z}) \leq 18$.*

Topologically, O is a 3-manifold (the "underlying manifold") labeled with a singular set, which is a graph; each node of the graph is labeled with a finite non-cyclic subgroup of $PGL(2, \mathbb{C})$ which is the stabilizer of a point of $\mathbb{H}^3$ which maps to the node under the quotient projection $\mathbb{H}^3 \to O$, and similarly each (arc or simple closed curve) component of the complement of the set of nodes within the singular set is labeled with a finite cyclic group. The hypothesis that every finite subgroup of Γ is cyclic is equivalent to the condition that the singular set of O has no nodes.

The assumption that Γ contains no triangle groups is harmless for the projected application to arithmetic lattices, because it turns out to be a fairly easy matter from the number-theoretic viewpoint to classify arithmetic lattices with volume subject to a prescribed bound that contain triangle groups. On the other hand, the assumption that the finite subgroups of Γ are cyclic is a serious one, because maximal arithmetic lattices typically contain dihedral groups. Thus the projected application will depend on relaxing this assumption.

Probable Theorem 1 would follow formally from two probable propositions:

Probable Proposition 1. *Let $O = \mathbb{H}^3/\Gamma$ be an orientable hyperbolic 3-orbifold, and let N denote the underlying manifold of O. Suppose that no subgroup of Γ is a hyperbolic triangle group. If O has volume at most 3.44, then $\dim H_1(N, \mathbb{Z}/2\mathbb{Z}) \leq 17$.*

Probable Proposition 2. *Let $O = \mathbb{H}^3/\Gamma$ be an orientable hyperbolic 3-orbifold. Suppose that every finite subgroup of Γ is cyclic. Then O has a two-sheeted orbifold cover O' such that the underlying manifold N' of O' satisfies $\dim H_1(N', \mathbb{Z}/2\mathbb{Z}) \geq \dim H_1(O, \mathbb{Z}/2\mathbb{Z}) + 1$.*

The probable proof of Propable Proposition 2 is an elementary application of Smith Theory.

In the special case where N is hyperbolic, Probable Proposition 1 can be deduced from the result that I quoted above from [7]. When N is not hyperbolic, if $\dim H_1(N, \mathbb{Z}/2\mathbb{Z}) \geq 2$, there always exists an essential sphere or torus in M by Perelman's Geometrization Theorem [3]. Such a sphere or torus gives rise to an incompressible suborbifold of O. The results of [2], which are stated for manifolds but are easily adapted to orbifolds, give lower volumes for the volume of O in terms of data involving incompressible suborbifolds of O. These estimates are used in the probable proof of Probable Proposition 1. The details are rather involved.

References

[1] Ian Agol, Marc Culler, and Peter B. Shalen. Dehn surgery, homology and hyperbolic volume. *Algebr. Geom. Topol.*, 6:2297–2312, 2006.

[2] Ian Agol, Peter A. Storm, and William P. Thurston. Lower bounds on volumes of hyperbolic Haken 3-manifolds. *J. Amer. Math. Soc.*, 20(4):1053–1077 (electronic), 2007. With an appendix by Nathan Dunfield.

[3] Laurent Bessières, Gérard Besson, Sylvain Maillot, Michel Boileau, and Joan Porti. *Geometrisation of 3-manifolds*, volume 13 of *EMS Tracts in Mathematics*. European Mathematical Society (EMS), Zürich, 2010.

[4] A. Borel. Commensurability classes and volumes of hyperbolic 3-manifolds. *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4)*, 8(1):1–33, 1981.

[5] Ted Chinburg and Eduardo Friedman. The smallest arithmetic hyperbolic three-orbifold. *Invent. Math.*, 86(3):507–527, 1986.

[6] Marc Culler and Peter B. Shalen. Singular surfaces, mod 2 homology, and hyperbolic volume, II. *Topology Appl.*, 158(1):118–131, 2011.

[7] Marc Culler and Peter B. Shalen. 4-free groups and hyperbolic geometry. *J. Topol.*, 5(1):81–136, 2012.

A MOY state-sum for colored HOMFLY

Roland van der Veen

1. Introduction

The colored HOMFLY polynomial is the most important quantum knot invariant. Through these invariants low dimensional topology interacts with an incredible diversity of other fields of mathematics and physics, including number theory, special functions, integrable systems, contact geometry, hyperbolic geometry, representation theory, Gromov-Witten invariants combinatorics and string theory. The colored HOMFLY unifies all quantum invariants coming from quantum groups of type A_N and as such includes the colored Jones and the Alexander polynomial. Direct formulas for the colored HOMFLY are out of reach but many mysterious structural properties are being uncovered.

We restrict ourselves here to knots and links colored by anti-symmetric powers of the standard representation. For such colors (representations) restricted to $U_q sl_N$ Murakami-Ohtsuki-Yamada develped a graphical technique generalizing the Kauffman bracket [MOY]. In this work we improve extend the technique to deal with the unified two-variable HOMFLY version directly. The resulting state sum is expected to have many applications, for example it gives direct insight into the behaviour of the maximal degree in q as a function of the weight of the representations.

2. MOY graphs

In this section we briefly recall the main definitions necessary to state our new state sum for MOY graphs. This leads to a state sum for knots and links as well because it was shown in [MOY] that a link diagram can be expanded as a sum of MOY graphs.

A *MOY graph* (Γ, γ) is a planar oriented trivalent graph Γ with no sources or sinks, together with a flow $\gamma : E(\Gamma) \to \mathbb{N}$. By a *flow* we mean an assignment of numbers to each edge so that the sum of the outgoing equals the sum of the incoming edges at each vertex. MOY graphs can be evaluated in terms of their cycles. By $\mathcal{C}$ we denote the set of *cycles.* A cycle C is a subset of $E(\Gamma)$ such that at each vertex either 2 or 0 edges of C appear and each component of C has a consistent orientation. By definition each component of a cycle has either clockwise (-1) or counter-clockwise (+1) orientation. We define the rotation number $\mathrm{rot}(C)$ to be the sum of the orientation numbers of its components. We call a MOY graph *positive* if all its non-empty cycles have positive rotation number. Finally we define an *intersection number* $\langle A, B\rangle$ on $\mathcal{C}$ as follows. At a vertex v the three adjacent edges by v_l, v_m and v_r. Here v_m is oriented opposite to the other two and if we approach v from v_m then v_l is on the left. The intersection number is defined as

$$\langle A, B\rangle = \frac{1}{4}(\#\{v \in V(\Gamma)|v_l \in A, v_r \in B\} - \#\{v \in V(\Gamma)|v_l \in B, v_r \in A\})$$

3. A new generalization of the binomial coeffient

Our new state sum is constructed from the following building blocks that are generalizations of symmetric q-binomial coefficients that may be of independent interest.

For a finite sequence $r = (r_1, \ldots, r_k)$ and a positive integer N we define the generalized binomial as:

$$\begin{bmatrix} N \\ r \end{bmatrix}(q) = \sum_{x_1 < \ldots < x_k} q^{\sum j r_j x_j}$$

Here the sum is to be taken over all $x_j \in \{-\frac{N-1}{2}, -\frac{N-3}{2}, \ldots \frac{N-1}{2}\}$

Lemma 1. *There exists a unique element* $\begin{bmatrix} * \\ r \end{bmatrix}(a, q) \in \mathbb{Q}(q^{\frac{1}{2}})\mathbb{Z}[a^{\pm\frac{1}{2}}]$ *such that for all* r, N

$$\begin{bmatrix} * \\ r \end{bmatrix}(q^N, q) = \begin{bmatrix} N \\ r \end{bmatrix}(q)$$

Note that the case $r = (1, 1, 1, 1, ...)$ corresponds to the symmetric q-binomial coefficient. It would be interesting to identify these binomial coefficients in terms of symmetric functions such as Schur functions or Macdonald polynomials.

4. State sum

With this preparation we can finally present our new state sum for MOY graph evaluations. By expanding each crossing in terms of MOY graphs the same also holds for the colored HOMFLY of any knot or link.

Theorem 1. *For every positive MOY graph* (Γ, γ) *we have*

$$\langle \Gamma, \gamma \rangle (a, q) = \sum_{\{C=(C_1,C_2,\dots)\in \mathcal{C} \mid \gamma_C = \gamma\}} q^{\sum_{i<j}\langle C_i, C_j\rangle} \begin{bmatrix} * \\ \text{rot } C \end{bmatrix} (a, q)$$

Our state sum should also be compared with the generating series developed in [GV].

References

[GV] S. Garoufalidis, R. van der Veen, *A generating series for Murakami-Ohtsuki-Yamada graph evaluations*, Proceedings international conference on quantum topology Nha Trang, Vietnam (2013), Arxiv 1312.2117

[MOY] H. Murakami, T. Ohtsuki, S. Yamada, *HOMFLY polynomial via an invariant of colored plane graphs* enseign. Math. 44 (1998) p. 325–360.

Coordinates for representations of 3-manifold groups

Christian Zickert

(joint work with Stavros Garoufalidis, Matthias Goerner, Dylan Thurston)

Let M be a compact 3-manifold and G a simple complex Lie group. It is well known that the set $\mathcal{R}$ of representations of $\pi_1(M)$ in G is an algebraic variety. Often one is only interested in the set of representations up to conjugation (i.e. flat connections), but the natural quotient is not in general a variety. The *character variety* is defined as $\mathrm{Spec}(O_{\mathcal{R}}^G)$, and captures much of the information about representations. We define here a different approach which allows for effective computation of representations in $\mathrm{SL}(n, \mathbb{C})$ and $\mathrm{PGL}(n, \mathbb{C})$.

Definition 1. *Let H be a subgroup of G. A representation $\pi_1(M) \to G$ is a (G, H)-representation if each peripheral subgroup maps to a conjugate of H.*

Important examples are $(\mathrm{SL}(n, \mathbb{C}), N)$ and $(\mathrm{PGL}(n, \mathbb{C}), B)$, where N and B are unipotent upper triangular and upper triangular matrices, respectively. Let $\widehat{\widetilde{M}}$ be the space obtained from the universal cover of M by collapsing each boundary component to a point, and let $V(\widehat{\widetilde{M}})$ be the set of ideal points.

Definition 2. *A* decoration *of a (G, H)-representation $\rho \colon \pi_1(M) \to G$ is a ρ-equivariant map*

$$D \colon V(\widehat{\widetilde{M}}) \to G/H.$$

Remark 1. *If D is a decoration of ρ, then gD is a decoration of $g\rho g^{-1}$, so we shall only consider decorations up to left multiplication.*

Note that if M has an ideal triangulation $\mathcal{T}$, a decoration assigns in an equivariant fashion a coset gH to each vertex of each simplex of M. We say that a decoration is *generic* if for each simplex, the cosets are in general position.

Theorem 1 (See [3, 2])**.** *For each ideal triangulation of M the set of generically decorated* $(\mathrm{SL}(n,\mathbb{C}),N)$*-representations is a variety cut out by explicit homogeneous equations of degree 2. The forgetful map to the set of* $(\mathrm{SL}(n,\mathbb{C}),N)$*-representations is given by explicit formulas.*

The above theorem gives rise to a very efficient way of *exact* computation of $(\mathrm{SL}(n,\mathbb{C}),N)$-representations. For a database see `curve.unhyperbolic.org`. The coordinates are inspired by coordinates on higher Teichmüller spaces due to Fock and Goncharov [1].

References

[1] V. Fock, A. Goncharov, *Moduli spaces of local systems and higher Teichmüller theory*, Publ. Math. Inst. Hautes Études Sci. **103** (2006), 1–211.

[2] S. Garoufalidis, M. Goerner, C. Zickert, *Gluing equations for* PGL(n, C)*-representations of 3-manifolds*, ArXiv:math.GT/1207.6711, 2012.

[3] S. Garoufalidis, D. Thurston, C. Zickert, *The complex volume of* SL(n, C)*-representations of 3-manifolds*, ArXiv:math.GT/1111.2828, 2011.

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