Fields Institute - Quantum Invariants of Knots and Modularity

Summer, 2019

Contents

Acknowledgements

Dear Robert, Hans and Will,

It was a pleasure working with you this summer and I really enjoyed the experience of working with young mathematicians from around the world. I look forward to learning more about the Colored Jones polynomial in my masters program!
Yours,
Colin

——— *** ———

Robert, Hans, and Will,

Thank you for a fantastic research experience at Fields this summer. I was inspired by your passion for mathematics, dedication to the project, and belief in our abilities. I learned lots of math, and more importantly how to work in a team to try solve a research problem. Your guidance was beyond any expectation I had for the summer, and I am lucky to have had an opportunity to work with such mathematicians during my undergraduate career. I look forward to continuing our work together and staying in touch.

Best wishes,
Beckham

——— *** ———

Hans, Robert, and Will,

Thank you for taking the time to supervise our project. It was an awesome experience, and I'm grateful for having met each of you. Hopefully we will be in touch in the future.

Aaron Tronsgard

——— *** ———

Dear Robert, Hans, and Will,

I want to thank Robert, Hans and Wills for giving the lectures, supervising the project and organizing the entertaining activities. Compared to other supervisors, I'm amazed by how much effort you guys put in and how much enthusiasm you guys have for the project. I feel comfortable to say that all three of you inspired me to become a better mathematician. The progress I have made during the summer is way beyond my own expectation. And such an achievement would be impossible without the patient guidance from all three of you.

Sincerely,
Shaoyang

7/2/2019 - Introduction and Project Outline

Today we'll begin by giving a brief overview of the project. The motivation will be a simple polynomial identity, which is called a *duality* result. There will be an *F-function* and a *U-function*. Next, we'll mention how this identity yields a completely different perspective, one that comes from knot theory. Finally, we'll describe the goals for the project and some of the big picture.

A duality result

Definition. *A finite sequence of integers* $\{a_i\}_{i=1}^s$ *is* ***strongly unimodal of size n*** *if*

$$0 < a_1 < a_2 < \ldots < a_k > a_{k+1} > \ldots > a_s > 0$$

for some k *and* $a_1 + a_2 + \ldots + a_n = n$.

Definition. *Let* $U(n)$ *denote the number of strongly unimodal sequences of size* n.

Definition. *The rank of a strongly unimodal sequence* $\{a_i\}$ *is defined to be* $s - 2k + 1$.

The rank of a sequence counts the number of terms after the peak minus the number of terms before the peak.

Example

- We have $U(5) = 6$, given by

$$\{5\}, \{1, 4\}, \{4, 1\}, \{1, 3, 1\}, \{2, 3\}, \{3, 2\}$$

The ranks of these are $0, -1, 1, 0, -1, 1$, respectively.

We will form a two-variable generating function that keeps track of the number of series of size n with a particular rank.

Definition. *Define*

$$U(x, q) = \sum_{\substack{n \geq 1 \\ m \in \mathbb{Z}}} U(m, n) x^m q^n$$

where $U(m, n)$ *is the number of series of size* n *and rank* m.

Proposition. *This series has generating function*

$$U(x, q) = \sum_{n \geq 0} (-xq)_n (-x^{-1}q)_n q^{n+1}$$

where

$$(a)_n = (a, q)_n = \prod_{k=0}^{n-1} (1 - aq^k)$$

Note that the Pochhammer symbol is defined

$$(q)_n = \prod_{k=0}^{n-1}(1 - qq^k) = \prod_{k=1}^{n}(1 - q^n)$$

We have the following interesting duality result.

Theorem (Bryson, Ono, Pitmam, Rhoades [2012]). *We have*

$$F(\zeta_N^{-1}) = U(-1, \zeta_N)$$

where $\zeta_N = e^{2\pi i/N}$ *is the* N*th root of unity and*

$$F(q) = \sum_{n\geq 0}(q)_n$$

is the Kontsevich-Zagier 'strange' series.

The original proof involved technical polynomial identities.

Some remarks about this Kontsevich-Zagier series:

1. $F(q)$ only converges at roots of unity. If $q = \zeta_N$, then the sum terminates after the $(N-1)$th summand.

2. $F(q)$ does not converge in the ring of formal power series $\mathbb{Z}[[q]]$, as the constant term in each summand is 1.

3. $F(1-q)$ converges in $\mathbb{Z}[[q]]$. This is because $(1-q, 1-q)_n = O(q^n)$ (the term of least power increases with each summand, as the constant terms are 0. Then coefficients of each term are a finite sum).

The first remark implies that the above duality is an equality of polynomials. We will see a similar phenomenon appear with knot invariants. By the third remark, we can make the following definition.

Definition. *The* ***Fishburn numbers*** *are given by generating function*

$$\begin{aligned} F(1-q) &= \sum_{n\geq 0}(1-q, 1-q)_n = \sum_{n\geq 0}\xi(n)q^n \\ &= 1 + q + 2q^2 + 5q^3 + 15q^4 + 53q^5 + 217q^6 + 1014q^7 \\ &\quad + 5335q^8 + 31240q^9 + 201608q^{10} + 1422074q^{11} \\ &\quad + 10886503q^{12} + 89903100q^{13} + 796713190q^{14} + \dots \end{aligned}$$

We are interested in finding congruences for this series. To do so, compute the prime factorization of these coefficients. For example, $\xi(4), \xi(9), \xi(14), \xi(19)$ are 0 modulo 5. Similarly, $\xi(3), \xi(8), \xi(13), \xi(18)$ are 0 modulo 5. And $\xi(6), \xi(13), \xi(20)$ are 0 modulo 7. More generally, we have the following proposition.

Theorem (Andrew, Sellers [2016]). *For all $n \in \mathbb{N}$, we have*

$$\xi(5n+3) \equiv \xi(5n+4) \equiv 0 \ mod\ 5$$
$$\xi(7n+6) \equiv 0 \ mod\ 7$$
$$\xi(11n+8) \equiv \xi(11n+9) \equiv \xi(11n+10) \equiv 0 \ mod\ 11$$

Moreover, there are similar congruences for half of the primes.[1]

For the ordinary partition function, it was proven that there are these congruences for only three primes. So it is remarkable that these hold for such a large class of numbers. We also have another, stronger result.

Theorem (Ahlgren, Kim, Lovejoy [2018]). *Let $p \geq 5$ be a prime. If $j \in T_{a,b,\xi}(p^r)$, then*

$$\xi(p^r n + j) \equiv 0 \ mod\ p^r$$

for all $n \in \mathbb{N}$.

The set $T_{a,b,\xi}(p^r)$ will be defined later.

Returning to the duality statement, the proof originally given is a bit ad hoc. Professor Osborn became interested in developing a broader perspective to understand why these results may be true.

Knot theory

The goal will be to construct knot invariants. In particular, given a knot K, we can associate an infinite sequence of polynomials $\{J_N(K,q)\}_{n\in\mathbb{N}}$ where $J_N(K,q) \in \mathbb{Z}[q^{\pm 1}]$ is the *colored Jones polynomial.*

Theorem (Habiro [2008]). *For any knot K, we can write*

$$J_N(K,q) = \sum_{n\geq 0} C_n(K,q)(q^{1+N})_n(q^{1-N})_n$$

This is the ***cyclotomic expansion****, and $C_n(K,q) \in \mathbb{Z}[q^{\pm 1}]$ are the* ***cyclotomic coefficients****, which are independent of N.*

Let K^* denote the mirror image of K (which is obtained diagramatically from K by switching all of the crossings).

[1] We will later describe for which primes this holds.

Example

- (Masbaum [2003]) If K is the trefoil, then the cyclotomic expansion[a] of K^* is

$$J_N(K^*, q) = \sum_{n\geq 0} q^n (q^{1+N})_n (q^{1-N})_n$$

A related expansion, which we will term the ***non-cyclotomic expansion***, of $J_N(K, q)$ was computed for

$$J_N(K, q) = q^{1-N} \sum_{n\geq 0} q^{-nN} (q^{1-N})_n$$

There is no clear substitution to go from $J_N(K^*, q)$ to $J_N(K, q)$. In short, a common theme will be that the shape of the colored Jones polynomial is significant.

[a] We'll eventually perform a similar knot-theoretic computation ourselves.

This example easily recovers the above duality. Recall

$$F(q) = \sum_{n\geq 0} (q)_n$$
$$U(-1, q) = \sum_{n\geq 0} (q)_n^2 q^{n+1}$$

F agrees with the Nth colored Jones polynomial for K up to multiplication by ξ_N when specialized to roots of unity:

$$\begin{aligned} J_N(K, \xi_N^{-1}) &= \xi_N^{N-1} \sum_{n\geq 0} (\xi_N^{N-1})_n \\ &= \xi_N^{-1} F(\xi_N^{-1}) \end{aligned}$$

U agrees with the Nth colored Jones polynomial for K^* when specialized to roots of unity:

$$\begin{aligned} J_N(K^*, \xi_N) &= \sum_{n\geq 0} \xi_N^n (\xi_N)_n (\xi_N)_n \\ &= \xi_N^{-1} \sum_{n\geq 0} (\xi_N)_n^2 \xi_N^{n+1} \\ &= \xi_N^{-1} U(-1, \xi_N) \end{aligned}$$

For all knots K, we have that

$$J_N(K, q^{-1}) = J_N(K^*, q)$$

Thus we have

$$\xi_N^{-1} F(\xi_N^{-1}) = J_N(K, \xi_N^{-1}) = J_N(K^*, \xi_N) = \xi_N^{-1} U(-1, \xi_N)$$

which recovers the duality.

In general, the cyclotomic expansion of the colored Jones polynomial corresponds to the U function, and the non-cyclotomic expansion corresponds to the F function.

Project Goals

Consider the torus knot $T(3,4)$. For this knot, the non-cylotomic expansion has been computed for

$$J_N(T(3,4),q) = q^{3(1-N)}\sum_{n\geq 0}(q^{1-N})_n q^{-2Nn}\big(q^N T(n,N,q)+T(n+1,N,q)\big)$$

where

$$T(n,N,q) = \sum_{k=0}^{\lfloor (n-1)/2\rfloor} q^{2k(k+1-N)}\begin{bmatrix} n \\ 2k+1\end{bmatrix}_q$$

where

$$\begin{bmatrix} n \\ k\end{bmatrix}_q = \frac{(q)_n}{(q)_k(q)_{n-k}}$$

is the q-binomial coefficient.

We can use this to define a corresponding $\mathcal{F}$ function that agrees with $J_N(T(3,4),q)$ on roots of unity. To do so, consider the series

$$\mathcal{F}(q) = \sum_{n\geq 0}(q)_n\big(T(n,0,q)+T(n+1,0,q)\big)$$

Note that we can take $N=0$, as $q^N=1$ anyway when $q=\zeta_N$. Then

$$\zeta_N^3\mathcal{F}(\zeta_N) = J_N(T(3,4),\xi_N)$$

We must

1. Consider the q-series
$$\mathcal{F}(1-q) = \sum_{n\geq 0}\gamma(n)q^n$$
Compute some terms. Do the coefficients satisfy congruences similar to those of the Fishburn numbers? What about for prime powers?

2. More generally, consider the family of torus knots $T(3,2^t)$, where $t\geq 2$. The non-cylotomic expansion of the colored Jones polynomial has been computed. We can similarly define $\mathcal{F}_t(q)$ such that
$$q^*\mathcal{F}_t(\xi_N) = J_N(T(3,2^t),\xi_N)$$
for some suitable power $*$. Consider
$$\mathcal{F}_t(1-q) = \sum_{n\geq 0}\gamma_t(n)q^n$$
Do these coefficients satisfy similar congruences? What about for prime powers?

We are interested in congruences in particular, as these are often symptomatic of deeper modular properties.

In general, given a knot K, we can consider the schematic

$$\begin{tikzcd}
 & \underbrace{J_N(K^*,q)}_{\text{cyclotomic}} \arrow[r, squiggly, "\substack{x=-1\\ q=\zeta_N}"] & U(x,q) \arrow[r, squiggly] & \text{`mock modular form'} \arrow[dd, dashed, leftrightarrow, "q\mapsto q^{-1}"] \\
K \arrow[ur] \arrow[dr] & & & \\
 & \underbrace{J_N(K,q)}_{\text{non-cyclotomic}} \arrow[r, squiggly, "q=\zeta_N"] & F(q) \arrow[r, squiggly] & \text{`quantum modular form'}
\end{tikzcd}$$

There are also deep connections to modular forms, which are reflected in the right side of the diagram. The specialized $F(q)$ is a *quantum modular form*, and the specialized U function is a *mock modular form*. The idea is that there should be some duality, which has been worked out in particular for the knots $T(2,3)$ and $T(2,2k+1)$, but not in general.

The second half of the project will be the U side of the matter, which requires knot theory. The goal will be to compute the cyclotomic expansions of the colored Jones polynomial for certain families of knots.

More generally, we have only discussed torus knots, but one can also ask questions about hyperbolic knots and satellite knots.

Summary

Given a knot, we can look at the colored Jones polynomials. We can construct a F function which agrees with the Nth Jones polynomial on the Nth root of unity. Congruences in the coefficients of the expansion of $F(1-q)$ indicate that F may have modular properties. If we can manage to obtain a cyclotomic expression for the colored Jones polynomial for the mirror image of the knot,[a] we may be able to discover duality properties with some other analogous U series.

[a] Via knot-theoretic computation techniques

7/4/2019 - Knot Theory, Operator Invariants, and the Colored Jones Polynomial

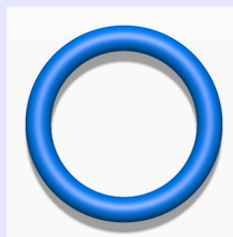
Introduction to knot theory

Definition. *A **knot** is an embedding $K : S^1 \hookrightarrow S^3$ of the circle into the 3-sphere.*

Definition. *Two knots are **equivalent** if there is a diffeomorphism $h : S^3 \to S^3$ carrying one knot to another.*

Example

- The ***unknot*** is the inclusion $S^1 \hookrightarrow \mathbb{R}^3 \cup \{\infty\} \simeq S^3$.



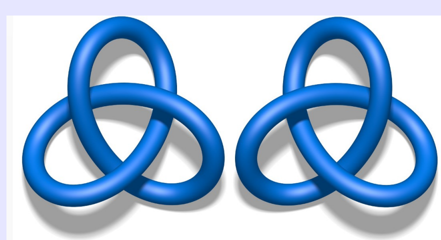
The main problem in knot theory is that of *classification*, namely listing the knot types and determining how to distinguish between them. Knots may also be different, depending on whether or not we keep track of

1. the orientation of the *knot*, which is indicated by an arrow along the strand of the knot.
2. the orientation of the *ambient space* S^3, in which case we require the ambient diffeomorphisms of S^3 to be orientation-preserving.

If we keep track of the orientation of S^3 and demand that the diffeomorphisms of S^3 be orientation preserving, then two knots are equivalent in the sense of this definition if and only if there exists an ambient isotopy between them.

Examples

- The knot below on the left is the ***left-handed trefoil***, and the knot below on the right is the ***right-handed trefoil***.



Definition. *For an oriented knot, there are **left-handed (negative)** and **right-handed (positive)** crossings:*

Reversing the orientation of the knot does not change whether a crossing is right-handed or left-handed. Reflecting the knot, which corresponds to changing every overcrossing to an undercrossing and every undercrossing to an overcrossing, does change whether a crossing is right-handed or left-handed.

When considering these two types of orientation, any knot K has potentially four distinct versions. These are $K, -K, K^*$, and $-K^*$, where $-K$ indicates K with opposite orientation and K^* indicates the reflection of K.

Remarks

- The first example of a knot K with K distinct from $-K$ is due to Trotter [1964]. It turns out that almost every knot is distinct from its reverse. There are few invariants that can distinguish between a knot and its reverse.
- On the other hand, the mirror image of a knot is often distinct from a knot. Knots which are equivalent to their mirror image are ***ampichiral***.

Definition. *A **link** is an embedding of several disjoint circles*

$$L : \bigsqcup_{i=1}^{m} S^1 \hookrightarrow S^3$$

*up to diffeomorphism. In such a case, L has m **components**.*

The first definitions and intuition for knots carries over similarly to links, except now there are 2^m possible orientations on L, where m is the number of components.

The main problem in knot/link theory is to classify all knots/links. Mathematicians constructing *invariants* to distinguish knots, and constructing *diffeomorphisms* to show knots are the same.

Definition. *A knot/link **invariant** is an object or quantity associated to each knot/link that takes the same value on two equivalent knots/links.*

There are many ways to define invariants. The most common approach is combinatorically, by examining projections of the link to the surface. There are a few necessary definitions and verifications. We'll proceed by discussing the case of knots, but the situation for links is entirely the same.

Definition. *Suppose $K : S^1 \hookrightarrow S^3$ is a knot, with $0 \notin imK$. A* ***projection*** *p of K is the tetravalent graph on S^2 obtained by projecting from $S^3 \setminus \{0\}$ radially to the equator S^2.*

By adjusting K up to diffeomorphism, we can assume that $p \circ K : S^1 \to S^2$ is a regular immersion, which has at worst double points (in general any nontrivial knot will result in a projection which cannot be thus adjusted to obtain an embedding).

Definition. *A* ***knot diagram*** *is a decorated graph obtained from a knot projection by indicating over/under crossings.*

Examples of Basic Invariants

- The ***crossing number*** of a knot is given by the number

$$c(K) = \min\{\text{crossing points of } D : \text{diagrams } D \text{ for } K\}$$

  For example, the crossing number of the unknot is 0, and the crossing number of the trefoil is 3.

- The ***knot group*** G_K is the fundamental group of the complement $S^3 \setminus \tau(K)$, where $\tau(K)$ is a regular neighborhood of K. The knot group is a very powerful invariant, but it is hard to compute.

- There is a knot polynomial called the *Alexander polynomial*, which will be introduced later.

It is important to keep in mind that there is a distinction between the representations of knots as diagrams and the knots themselves.

Philosophy of Invariant Construction

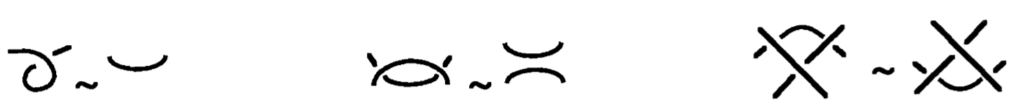
It can be difficult to understand knots as embeddings into S^3. Thus, often one will consider a *representation* of a knot, such as a diagram. One then characterizes the different such representations that correspond to the same knot.

Then, an invariant is constructed from this representation of the knot. It is shown that the invariant indeed respects the different representations which correspond to the same knot.[a]

[a] We are about to proceed with this project for knot diagrams, and will later do so with braids.

The following theorem then characterizes the diagrams which describe the same knot.

Theorem. *Two knot diagrams represent the same knot if and only if they are related by a sequence of* ***Reidemeister moves****:*



Type I Type II Type III

It's clear that if one diagram can be obtained from the other by a sequence of such moves, then the associated knots are equivalent. However, it is not obvious that if two knots are equivalent, their diagrams differ by a sequence of such moves.

The Jones Polynomial

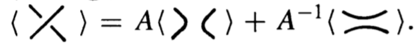
Our project is primarily concerned with an invariant called the *Jones polynomial.* We will define the Jones polynomial in the most mathematically efficient way, although this definition is ahistorical.

Lemma. *There exists a unique function* $\langle\cdot\rangle : \{\textit{link diagrams}\} \to \mathbb{Z}[A^{-1}, A]$ *satisfying*

(i) $\langle \bigcirc \rangle = 1,$

(ii) $\langle D \sqcup \bigcirc \rangle = (-A^{-2} - A^{2})\langle D \rangle,$

(iii)



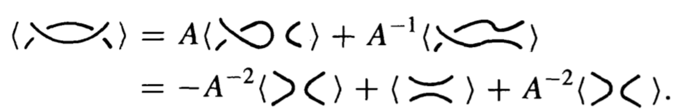
The function $\langle\cdot\rangle$ *is the* ***Kauffman bracket****.*

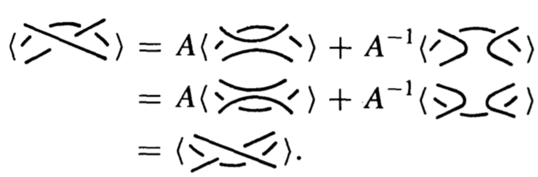
The first type of smoothing in (iii) is often called an A smoothing of the crossing, and the second type of smoothing is often called a B smoothing of the crossing.

Lemma. *The Kauffman bracket is invariant under Reidemester II and Reidemester III. We have*

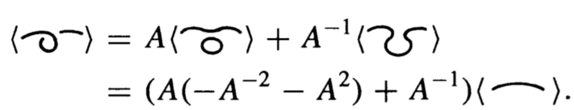
This means that the bracket is not a knot invariant, although it is close to being one.

Proof. We have the following verification of invariance under Reidemeister II and Reidemeister III:





The second part of the lemma follows from



□

By resolving all of the crossings in a diagram and then considering the resulting unknotted link, we obtain the following computationally useful *state sum model* for the bracket. Number the crossings

$1, 2, \ldots, n$ arbitrarily. Choose a complete smoothing for the knot, which is a binary string of length n, where each coordinate describes how to resolve the corresponding crossing in the diagram. Each such binary string results in an unlink, which is termed a *state*.

Theorem (State sum model)**.** *For a state S, let $a(S)$ and $b(S)$ be the number of A and B smoothings, respectively. Let $|S|$ be the number of components of S. We have the formula*

$$\langle K \rangle = \sum_{\textit{states } S} A^{a(S)-b(S)}(-A^2 - A^{-2})^{|S|-1}$$

The numbers $a(S)$ and $b(S)$ are easy to determine, but it is not straightforward to determine the number of resulting components in the unlink S algorithmically. A useful observation is that given a smoothing of the whole knot, adjusting the resolution of a single crossing results in a smoothing with either one more or one fewer components.[2]

Definition. *Given an oriented knot/link L, the **Jones polynomial** is the polynomial*

$$V_L(t) = [(-1)^{\omega(L)} A^{3\omega(t)} \langle L \rangle]_{A^2 = t^{1/2}}$$

*where $\omega(L)$ is the **writhe** of L, defined by*

$$\omega(L) = \sum_{\textit{crossings } x} \epsilon_x$$

with

$$\epsilon_x = \begin{cases} 1 & x \textit{ is right-handed} \\ 0 & x \textit{ is left-handed} \end{cases}$$

Proposition. *The Jones polynomial is a knot/link invariant.*

Proof. The proof is not difficult. The bracket is already invariant under Reidemeister II and Reidemeister III. $V_L(t)$ is defined so that performing a Reidemeister I move on L changes the writhe apppropriately, precisely accounting for the added factor present in $\langle L \rangle$. □

Note that the bracket $\langle D \rangle$ is completely independent of the orientation of D, while the Jones polynomial thus defined correspondingly adjusts for this. The Jones polynomial accounts for orientation in the leading term, which is significant, as it is one of the first easily computed invariants which is capable of distinguishing a knot from its mirror. Namely, if $V_L(t)$ is not a symmetric polynomial, then L is distinct from its mirror L^*.

$V_L(t)$ is a Laurent polynomial in $\mathbb{Z}[t, t^{-1}]$ if L is a knot or a link with an odd number of components. Otherwise, $t^{1/2}V_L(t)$ is a Laurent polynomial in $\mathbb{Z}[t, t^{-1}]$ if L has an even number of components. The following is a fantastic application of the Jones polynomial to alternating knots.

[2] For knots, there is a clever way to compute $|S|$ using linear algebra over $\mathbb{Z}/2\mathbb{Z}$. Zulli defined the *trip matrix* T, for which the nullity of

$$T + \begin{pmatrix} 1 & & & \\ & 0 & & \\ & & \cdots & \\ & & & 1 \end{pmatrix}$$

gives the number of components in the unlink, where the zeroes are present at crossings with smoothing A and the ones present at crossings with smoothing B.

Application of the Jones Polynomial to Alternating Knots

Definition. *A knot/link is **alternating** if the crossings alternate from passing over to passing under as each component is traversed.*

Note that the knot $T(3,4)$ we are considering in our project is not alternating.

Definition. *A **nugatory (removable)** crossing is one that can be removed by rotating part of the knot/link in space.*

Definition. *A knot/link diagram is **reduced** if it does not have any nugatory crossings.*

Definition. *A knot K is **prime** if it is not the connected sum*[a] *of two nontrivial knots.*

We have the following classical theorem as an application of the Jones polynomial.

Theorem. *Suppose K is a connected diagram of an oriented knot/link L with n crossings. Then*

1. *span$(V_K(t))$, which is the integer equal to the maximum degree present in $V_L(t)$ minus the minimum degree present in $V_K(t)$, is less than or equal to n.*
2. *if K is reduced and alternating, then span$(V_K(t)) = n$.*
3. *if K is prime and not alternating as a diagram, then span$(V_K(t)) < n$.*

Hence a reduced alternating diagram is minimal with respect to crossing number. The theorem also implies that the Jones polynomial distinguishes alternating knots if the crossing number is known.

The crossing number of a knot is a simple concept, but it is often elusive. The above theorem indicates that it is easy to compute the crossing number for alternating knots.

Conjecture. *The crossing number is additive*

$$c(K\#J) = c(K) + c(J)$$

The inequality $c(K\#J) \leq c(K) + c(J)$ is clear.

Proposition. *The Jones polynomial is multiplicative with respect to connected sum, namely*

$$V_{L\#K}(t) = V_L(t)V_K(t)$$

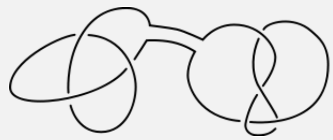
Then properties of polynomial multiplication imply the following corollary.

Corollary. *The conjecture holds for the class of alternating knots.*

Corollary. *Any two reduced alternating diagrams for a link have the same writhe.*

These sorts of results provide techniques to achieve a classification of alternating knots.[b]

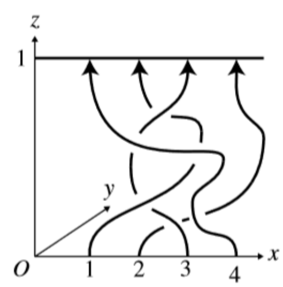
[a]The sum of two knots is given by the expected, intuitive operation



[b]The first and second Tait conjectures follow. Tait's third conjecture, the flype conjecture, says that any two reduced alternating diagrams for a link are related by a sequence of flype moves. It was settled by other techniques around the same time. In Kauffman's words, 'the three Tait conjectures provide the methods to make the classification of alternating knots trivial.'

Braids and their invariants

Definition. *A **n-braid** consists of n strands in the cube I^3 connecting points in $I^2 \times \{1\}$ to those in $I^2 \times \{0\}$ monotonically.*[3]



Each n-braid is associated to a permutation on n letters, determined by which points are connected to which other points by the strands. Denote by σ_i the braid which swaps the i and $(i+1)$th strand. It is visually evident that every braid can be written as the product of these elementary braids via isotopy.

Lemma. *The n-stranded braids up to isotopy form the **braid group** $\mathfrak{B}_n$ under the obvious composition given by 'stacking' two braids. The identity is the braid with no crossings, and the inverse of a braid is obtained by reflection.*

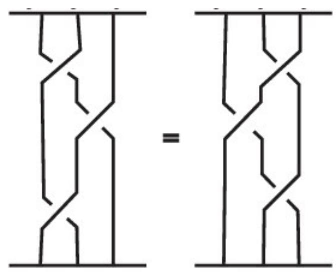
In the language of braid words, the inverse of $\sigma_{i_1} \dots \sigma_{i_\ell}$ is $\sigma_{i_\ell}^{-1} \dots \sigma_{i_1}^{-1}$, where σ_i^{-1} swaps the i and $(i+1)$th strands in the opposite order that σ_i does. Reidemeister II implies that these are indeed inverse, and the associativity of multiplication then implies the claim.

Lemma (Artin). *$\mathfrak{B}_n$ is generated by $\{\sigma_1, \dots, \sigma_{n-1}\}$ with relations*

1. *$\sigma_i \sigma_j = \sigma_j \sigma_i$ when $|i-j| > 1$ (far commutativity)*

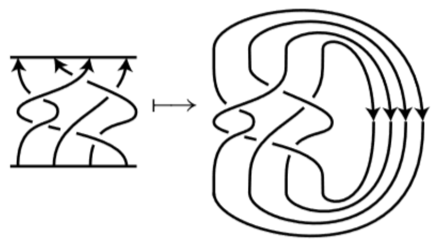
2. *$\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}$.*

[3]Meaning precisely one strand is attached to each point.



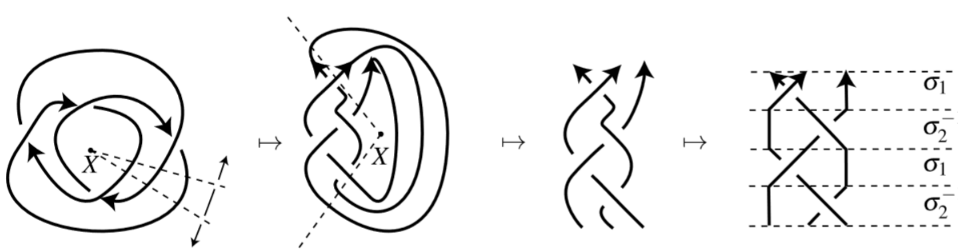
This provides an alternative strategy to defining knots and links.

Definition. *The **closure** of a braid β is given by*



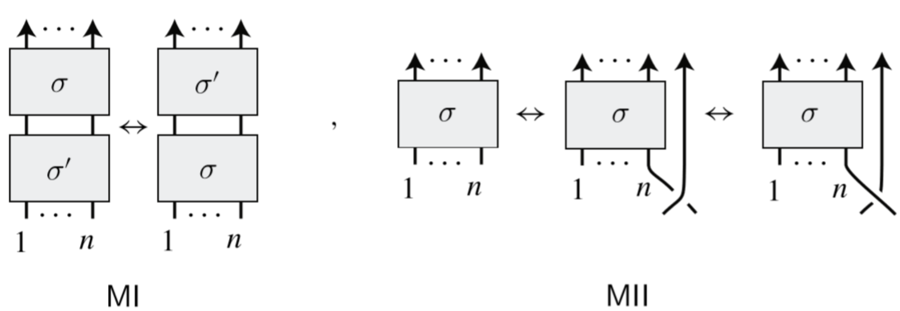
The closure of a braid is a knot if the permutation corresponding to β is an n-cycle. If not, the number of components in the link is equal to the number of disjoint cycles in this permutation.

Theorem (Alexander)**.** *Every oriented knot/link is the closure of a braid.*



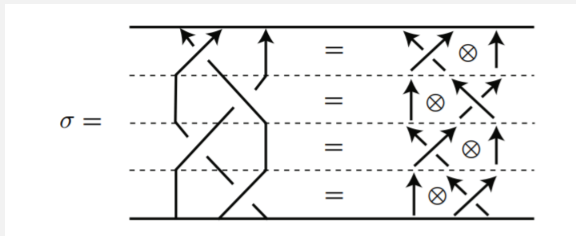
We can then think of a knot/link as an equivalence class of braids whose closure is the corresonding knot/link. To do so, we need to know which braids yield equivalent closures.

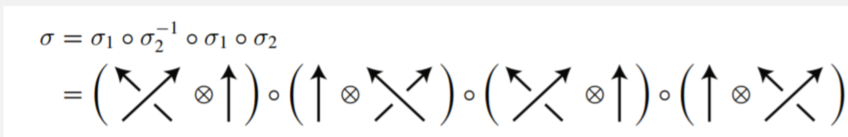
Theorem. *Two braids β, γ have equivalent closures $\widehat{\beta}, \widehat{\gamma}$ if and only if they are related by a sequence of **Markov moves**:*



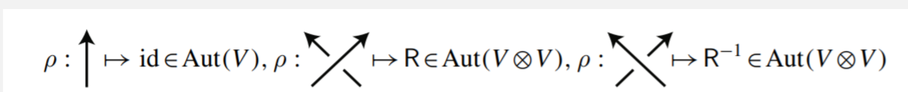
Constructing Representations of the Braid Group

Braids are special, in that they can be easily decomposed into a sequence of simple pieces (the braid word). We will associate each of these fundamental building blocks of a braid with a linear map on a vector space, and then combine them appropriately to build a linear map associated with the entire braid. In this case, we will combine vertical sections of a braid via composition and horizontal sections of a braid via the tensor product.





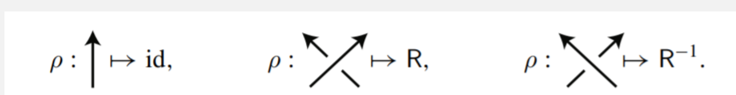
We should take our vector space to be $V^{\otimes n}$ for some V, and define the linear maps that correspond to the building blocks of a braid by



where $R \in \text{Aut}(R \otimes R)$. This yields a map

$$\rho : \mathfrak{B}_n \to \text{End}(V^{\otimes n})$$

which is obtained by taking the multiplicative and tensorial extension of



Right now, ρ is only really defined on the free group with n generators, as we don't yet know it respects the relations in $\mathfrak{B}_n$. We would like this to be a group representation of $\mathfrak{B}_n$, which means that we should have $\rho(\beta) = \rho(\beta')$ if β and β' differ by a sequence of braid moves. It turns out that the condition $\rho(\sigma_i \sigma_j) = \rho(\sigma_j \sigma_i)$ when $|i - j| > 1$ holds simply by the formulas for σ_i and σ_j.

To show that $\rho(\sigma_i \sigma_{i+1} \sigma_i) = \rho(\sigma_{i+1} \sigma_i \sigma_{i+1})$, it ends up being necessary and sufficient that

$$(\text{id}_V \otimes R) \circ (R \otimes \text{id}_V) \circ (\text{id}_V \otimes R) = (R \otimes \text{id}_V) \circ (\text{id}_V \otimes R) \circ (R \otimes \text{id}_V)$$

So if R satisfies this equation, ρ descends to the quotient for a well-defined map from the braid group $\mathfrak{B}_n$.

The above summary motivates the following definition in the context of knot theory.

Definition. *The **Yang-Baxter equation** in $End(V^{\otimes 3})$ is the equation*

$$(id_V \otimes R) \circ (R \otimes id_V) \circ (id_V \otimes R) = (R \otimes id_V) \circ (id_V \otimes R) \circ (R \otimes id_V)$$

*An invertible solution $R \in Aut(V \otimes V)$ to this equation is an **R-matrix**.*

So a solution R to the Yang-Baxter equation yields a well-defined representation $\rho : \mathfrak{B}_n \to \mathrm{End}(V^{\otimes n})$. We can use these representations to obtain invariants of links by algebraically defining a closure operation and then determining under what conditions the resulting map is invariant under Markov moves.[4]

Constructing Link Invariants from Representations of the Braid Group

Recall that there is a natural isomorphism

$$\mathrm{Hom}(V, W) \simeq V^* \otimes W$$

Then if $\rho : \mathfrak{B}_n \to \mathrm{End}(V^{\otimes n})$ is a representation, $\rho(\beta)$ can be viewed as an element of $(V^*)^{\otimes n} \otimes V^{\otimes n}$. The algebraic analogue of closure will be the *contraction map*

$$\kappa : (V^*)^{\otimes n} \otimes V^{\otimes n} \to \mathbb{F}$$
$$(e^*_{i_1} \otimes \ldots \otimes e^*_{i_n}) \otimes (e_{j_1} \otimes \ldots \otimes e_{j_n}) \mapsto e^*_{i_1}(e_{j_1}) \ldots e^*_{i_n}(e_{j_n})$$

This is the coordinate-free definition of the trace $\mathrm{Tr} : \mathrm{End}(V^{\otimes n}) \to \mathbb{F}$. We need this to be invariant under Markov moves. Markov I dictates $\widehat{\beta \circ \gamma} = \widehat{\gamma \circ \beta}$, so we need that

$$\mathrm{Tr}(\rho(\beta \circ \gamma)) = \mathrm{Tr}(\rho(\gamma \circ \beta))$$

for all $\beta, \gamma \in \mathfrak{B}_n$. This holds in general, because for linear maps A, B we have $\mathrm{Tr}(AB) = \mathrm{Tr}(BA)$. Markov II, in terms of the generators of the braid group, expresses

$$(\sigma \otimes \mathrm{id}_V) \circ (\mathrm{id}_V^{\otimes n} \otimes \sigma_n^{\pm 1}) = \sigma$$

This is achieved by the contraction, which can eliminate the final tensor factor in order to relate the representations of the two sides of this equation.

The above ideas lead us to introduce the following definition.

Definition. *The **trace operator** is the map*

$$Tr_k : End(V^{\otimes k}) \to End(V^{\otimes (k-1)})$$

given by applying $id_V^{\otimes (k-1)} \otimes \kappa \otimes id_V^{\otimes (k-1)}$ to the endomorphism

$$f \in End(V^{\otimes k}) \simeq (V^{\otimes (k-1)} \otimes V)^* \otimes (V^{\otimes (k-1)} \otimes k) \simeq (V^{\otimes (k-1)})^* \otimes V^* \otimes V \otimes (V^* \otimes (k-1))$$

[4] See Introduction to Quantum and Vassiliev Invariants [2010]

where κ is the contraction map as above. Alternatively, using coordinates Tr_k is given by

$$Tr_k(f)(e_{i_1} \otimes e_{i_2} \otimes \ldots \otimes e_{i_{k-1}}) = \sum_{j_1, j_2, \ldots, j_{k-1}, j=0}^{n-1} f_{i_1, i_2, \ldots, i_{k-1}, j}^{j_1, j_2, \ldots, j_{k-1}, j} (e_{j_1} \otimes e_{j_2} \otimes \ldots \otimes e_{j_{k-1}} \otimes e_j)$$

where $f_{i_1, i_2, \ldots, i_{k-1} j}^{j_1, j_2, \ldots, j_{k-1}, j}$ are defined by

$$f(e_{i_1} \otimes e_{i_2} \otimes \ldots \otimes e_{i_k}) = \sum_{j_1, j_2, \ldots, j_{k-1}, j=0}^{n-1} f_{i_1, i_2, \ldots, i_k}^{j_1, j_2, \ldots, j_k} (e_{j_1} \otimes e_{j_2} \otimes \ldots \otimes e_{j_k})$$

It turns out that, at this point, the algebraic conditions on the map R are too strong to yield useful invariants.[5] We fix this by introducing another map $\mu : V \to V$. WIth some work, we obtain the following theorem.

Theorem. *Let $R \in End(V \otimes V)$ be an R-matrix, and let $\mu \in End(V)$. If $(\mu \otimes \mu) \circ R = R \circ (\mu \otimes \mu)$ and $Tr_2(R^{\pm 1} \otimes \mu^{\otimes 2}) = \mu$, then $Tr(\rho_R(\sigma) \circ \mu^{\otimes n})$ is invariant under Markov moves, where ρ_R is the representation of $\mathfrak{B}_n$ constructed using the above method.*

This will provide us with a very powerful way to construct link invariants. By defining a representation of the braid group that satisfies these key properties, taking the trace of the operator associated to a braid yields an invariant of the link.

Definition. *Let V be an n-dimensional complex vector space. An **enhanced Yang-Baxter operator** is a quadruple (R, μ, a, b), where $R \in Aut(V \otimes V)$ is an R-matrix, $\mu : V \to V$ is linear, $a, b \in \mathbb{C}$ are nonzero, and*

1. $R(\mu \otimes \mu) = (\mu \otimes \mu)R$
2. $Tr_2(R^{\pm 1}(id_V \otimes \mu)) = a^{\pm 1} b\, id_V$

Definition. *Let (R, μ, a, b) be an enhanced Yang-Baxter operator. Let $K = \widehat{\beta}$ be a knot realized as the closure of a braid. Define*

$$T_{(R,\mu,a,b)}(K) = a^{\omega(\beta)} b^{-n} (Tr_1(Tr_2(\ldots Tr_n(\Phi(\beta)\mu^{\otimes n}) \ldots)))$$

where $\omega(\beta)$ is an exponential sum from the braid word.

$T_{(R,\mu,a,b)}(K)$ is an invariant of the knot K. This is the approach used to define the colored Jones polynomial.

The colored Jones polynomial

Let $V = \mathbb{C}^N$ with the standard basis $\{e_0, \ldots, e_{N-1}\}$. For a complex parameter q, notate

$$\{m\} = q^{m/2} - q^{-m/2}$$
$$\{m\}! = \{m\}\{m-1\} \ldots \{2\}\{1\}$$

[5] See [2010] for details, as we are omitting much of the algebra here.

Now, define an R-matrix

$$R(e_k \otimes e_\ell) = \sum_{i,j=0}^{N-1} R_{k,\ell}^{i,j} e_i \otimes e_j$$

where

$$R_{k\ell}^{ij} = \sum_{m=0}^{\min(N-1-i,j)} \delta_{\ell,i+m}\delta_{k,j-m} \frac{\{\ell\}!\{N-1-k\}!}{\{i\}!\{m\}!\{N-1-j\}!} q^{(i-(N-1)/2)(j-(N-1)/2)-m(i-j)/2-m(m+1)/4}$$

And define the linear map

$$\mu(e_j) = \sum_{i=0}^{N-1} \mu_j^i e_i$$

where

$$\mu_j^i = \delta_{ij} q^{(2i-N+1)/2}$$

and δ is the usual Kronecker delta.

Lemma. *The quadruple $(R, \mu, q^{(n^2-1)/4}, 1)$ is an enhanced Yang-Baxter operator.*

Definition. *The N-colored Jones polynomial is*

$$J_N(K, q) = \frac{\{1\}}{\{n\}} T_{(R,\mu,q^{(n^2-1)/4},1)}(\beta)$$

where β is chosen so that $K = \widehat{\beta}$.

$J_N(K, q)$ is an invariant of knots, and it is normalized so that $J_N(\mathcal{O}, q) = 1$.

It is possible to compute the colored Jones polynomial using the Kauffman bracket applied to the cabling of K. However, practically this becomes computationally prohibitive, as the Jones polynomial behaves unpredictably with respect to cabling.

We will proceed next time by performing a computation of the colored Jones polynomial for torus knots and also showing how this construction is related to another definition of the colored Jones polynomial.

7/5/2019 - Proving Congruences for the Kontsevich-Zagier Series

Overview

Over the next few lectures, we will develop a more streamlined argument to prove prime congruences for the types of sequences we have been examining. This techniques will be useful for our own project, as we are attempting to prove prime congruences for the F-series corresponding to the colored Jones polynomial of the torus knots $T(3, 2^t)$.

Recall the Kontsevich-Zagier 'strange' series

$$F(q) = \sum_{n \geq 0} (q)_n$$

where

$$(a)_n = (a, q)_n = \prod_{k=0}^{n-1} (1 - aq^k)$$

Write

$$F(1-q) = \sum_{n \geq 0} (1-q, 1-q)_q = \sum_{n \geq 0} \xi(n) q^n$$

We stated, although did not prove, the congruences

$$\begin{aligned} \xi(5n+3) \equiv \xi(5n+4) &\equiv 0 \bmod 5 \\ \xi(7n+6) &\equiv 0 \bmod 7 \\ \xi(11n+8) \equiv \xi(11n+9) \equiv \xi(11n+10) &\equiv 0 \bmod 11 \\ &\dots \end{aligned}$$

We would like to better understand for which primes and congruence classes the sequence ξ satisfies a congruence relation. We will discuss how to prove such congruences and why they hold for half of the primes.

Technical results

We will proceed by first presenting some lemmas which hold for general polynomials and then specialize to our case.

Definition. *The* ***Stirling numbers of the 2nd kind*** *are the numbers* $\{c_{n,j}\}$ *defined by*

$$\begin{aligned} c_{n,0} &= c_{n,j} = 0 \text{ for } j \geq n+1 \\ c_{1,1} &= 1 \\ c_{n+1,j} &= j c_{n,j} + c_{n,j-1} \text{ for } 1 \leq j \leq n+1 \end{aligned}$$

Lemma 1. *Let* $A(q), B(q) \in \mathbb{Z}[q]$. *Then*

$$\left(q \frac{d}{dq} \right)^n (A(q)B(q)) = \sum_{j=1}^{n} q^j c_{n,j} \left(\frac{d}{dq} \right)^j (A(q)B(q))$$

Proof. The proof will proceed by induction on n. When $n = 1$, the statement immediately follows. Assume the formula holds for n. Then we have

$$\begin{aligned}\left(q\frac{d}{dq}\right)^{n+1}(A(q)B(q)) &= \left(q\frac{d}{dq}\right)\sum_{j=1}^{n} q^j c_{n,j}\left(\frac{d}{dq}\right)^j (A(q)B(q)) \\ &= \left(\sum_{j=1}^{n} jq^j c_{n,j}\left(\frac{d}{dq}\right)^j (A(q)B(q))\right) + \left(\sum_{j=1}^{n} q^{j+1} c_{n,j}\left(\frac{d}{dq}\right)^{j+1} (A(q)B(q))\right)\end{aligned}$$

Since $c_{n,n+1} = 0$ and $c_{n,0} = 0$, we can increase the index on the first sum and reindex the second sum for

$$\begin{aligned}\left(q\frac{d}{dq}\right)^{n+1}(A(q)B(q)) &= \sum_{j=1}^{n+1} q^j (jc_{n,j} + c_{n,j-1})\left(\frac{d}{dq}\right)^j (A(q)B(q)) \\ &= \sum_{j=1}^{n+1} q^j c_{n+1,j}\left(\frac{d}{dq}\right)^j (A(q)B(q))\end{aligned}$$

as desired. $\square$

Lemma 2. *Let $f(q) \in \mathbb{Z}[q]$. Then*

$$\left(\frac{d}{dt}\right)^n f(qe^t)\bigg|_{t=0} = \left(q\frac{d}{dq}\right)^n f(q)$$

Proof. First apply Lemma 1 to $A(q) = f(q)$ and $B(q) = 1$ for

$$\left(q\frac{d}{dq}\right)^n f(q) = \sum_{j=1}^{n} q^j c_{n,j}\left(\frac{d}{dq}\right)^j f(q) \tag{1}$$

Next, we claim that

$$\left(\frac{d}{dt}\right)^n f(qe^t) = \sum_{j=1}^{n} q^j e^{jt} c_{n,j}\left[\left(\frac{d}{dq}\right)^j f(q)\bigg|_{q\mapsto qe^t}\right] \tag{2}$$

Note that when $n = 1$ we have

$$\left(\frac{d}{dt}\right) f(qe^t) = \left(\frac{d}{dq} f(q)\bigg|_{q\mapsto qe^t}\right) qe^t$$

by the chain rule. Induction yields

$$\begin{aligned}\left(\frac{d}{dt}\right)^{n+1} f(qe^t) &= \frac{d}{dt}\left(\sum_{j=1}^{n} q^j e^{jt} c_{n,j}\left[\left(\frac{d}{dq}\right)^j f(q)\Big|_{q\mapsto qe^t}\right]\right)\\ &= \sum_{j=1}^{n} q^j j e^{jt} c_{n,j}\left[\left(\frac{d}{dq}\right)^j f(q)\Big|_{q\mapsto qe^t}\right] + \sum_{j=1}^{n} q^j e^{jt} c_{n,j}\left[\left(\frac{d}{dq}\right)^{j+1} f(q)\Big|_{q\mapsto qe^t}\right] qe^t\\ &= \sum_{j=1}^{n+1} q^j j e^{jt} c_{n,j}\left[\left(\frac{d}{dq}\right)^j f(q) q \mapsto qe^t\right] + \sum_{j=1}^{n+1} q^j e^j c_{n,j-1}\left[\left(\frac{d}{dq}\right)^j f(q)\Big|_{q\mapsto qe^t}\right]\\ &= \sum_{j=1}^{n+1} q^j e^{jt} j(j c_{n,j} + c_{n,j-1})\left[\left(\frac{d}{dq}\right)^j f(q)\Big|_{q\mapsto qe^t}\right]\\ &= \sum_{j=1}^{n+1} q^j e^{jt} c_{n+1,j}\left[\left(\frac{d}{dq}\right)^j f(q)\Big|_{q\mapsto qe^t}\right]\end{aligned}$$

as claimed. Taking $t = 0$ in equation (2) and comparing to (1) yields the result. □

Polynomial dissections and orthogonality

There are two ideas that appear often in these papers, as well as more generally in number theory related to q-series.

Definition. *Let $h(q) \in \mathbb{Z}[q]$ and s be a positive integer. The **s-dissection** of $h(q)$ is the unique decomposition*

$$h(q) = \sum_{i=0}^{s-1} q^i A_s(i, q^s)$$

where $A_s(i, q^s)$ is a polynomial in q^s.

We will see that considering the s-dissections of polynomials obtained from partial sums of our series yields $A_s(i, q^s)$ polynomials with special divisibility properties.

Definition. *The following technique is called **orthogonality**, and it allows us to rewrite the original s-dissection equation of $h(q) \in \mathbb{Z}[q]$ to express the polynomials $A_s(i, q^s)$ in terms of $h(q)$.*

Technique. Fix an s-dissection

$$h(q) = \sum_{i=0}^{s-1} q^i A_s(i, q^s)$$

Let ζ_s be a primitive sth root of unity, and fix $0 \le i_0 \le s-1$. Since $(q^{1/s}\zeta_s^k)^s = q$, replace q with $q^{1/s}\zeta_s^k$ for the equality

$$h(q^{1/s}\zeta_s^k) = A_s(0, q) + q^{1/s}\zeta_s^k A_s(1, q) + \ldots + q^{i_0/s}\zeta_s^{i_0 k} A_s(i_0, q) + \ldots + q^{(s-1)/s}\zeta_s^{(s-1)k} A_s(s-1, q)$$

We will eliminate the i_0th coefficient by multiplying both sides by $q^{-i_0/s}\zeta_s^{-i_0 k}$ to obtain

$$q^{-i_0/s}\zeta_s^{-i_0 k} h(q^{1/s}\zeta_s^k) = q^{-i_0/s}\zeta_s^{-i_0 k} A_s(0, q) + \ldots + A_s(i_0, q) + \ldots + q^{(s-1-i_0)/s}\zeta_s^{(s-1-i_0)k} A_s(s-1, q)$$

We will need the following basic fact from number theory.

Lemma. *Suppose $t \not\equiv 0$ mod s. Then*

$$\sum_{k=0}^{s-1} \zeta_s^{-tk} = 0$$

for ζ_s a primitive root of unity.

Proof. ζ_s satisfies

$$x^s - 1 = (x-1)(x^{s-1} + x^{s-2} + \ldots + x + 1)$$

which implies

$$\sum_{k=0}^{s-1} \zeta_s^k = 0$$

for any primitive root of unity ζ_s. The group endomorphism

$$\begin{aligned} \langle \zeta_s \rangle &\to \langle \zeta_s \rangle \\ \zeta_s &\mapsto \zeta_s^{-t} \end{aligned}$$

has image generated by some $\zeta_s^{-tk_0}$, as any subgroup of $\langle \zeta_s \rangle$ is cyclic. Hence the image is generated by some primitive root of unity $\zeta_{s'}$, so we have

$$\sum_{k=0}^{s-1} \zeta_s^{-tk} = F \sum_{k=0}^{s-1} \zeta_{s'} = 0$$

where F is the order of the fibers. $\square$

Here, $t = i_0, i_0 - 1, \ldots, i_0 - s + 1$, none of which are equal to 0 modulo s. Summing both sides then yields

$$\sum_{k=0}^{s-1} q^{-i_0/s} \zeta_s^{-i_0 k} h(q^{1/s} \zeta_s^k) = s A_s(i_0, q)$$

Hence we have

$$A_s(i_0, q) = \frac{1}{s} \sum_{k=0}^{s-1} \zeta_s^{-i_0 k} q^{-i_0/s} h(q^{1/s} \zeta_s^k)$$

Note that this is an expression for the polynomial $A_s(i_0, q)$, and that the polynomials $A_s(i_0, q^s)$ are present in the dissection. $\square$

The general derivative lemma

We will first need a generalization of the Stirling numbers.

Definition. *Let the array of numbers $\{C_{N,i,j}(s)\}$ be defined by*

$$\begin{aligned} C_{0,0,0}(s) &= 1 \\ C_{N,i,0}(s) &= i^N \\ C_{N,i,j}(s) &= 0 \textit{ for } j \geq N+1 \textit{ or } j < 0 \\ C_{N+1,i,j}(s) &= (i + js) C_{N,i,j}(s) + s C_{N,i,j-1}(s) \textit{ for } 1 \leq j \leq N \end{aligned}$$

Taking $i = 0, s = 1$, and $N = n$ yields the original Stirling numbers, up to minor reindexing. The original definition came from the patterns that occur when taking multiple derivatives. The following result illustrates a similar phenomenon.

Lemma 3. *We have the following:*

1. *For all $N \geq 0$,*
$$\left(q\frac{d}{dq}\right)^N h(q) = \sum_{j=0}^{N}\sum_{i=0}^{s-1} C_{N,i,j}(s)q^{i+js}\left(\frac{d}{dq}\right)^j A_s(i,q^s)$$

2. *Let ζ_s be a primitive root of unity. For $N \geq 0$ and $i_0 \in \{0,1,\ldots,s-1\}$,*
$$\sum_{j=0}^{N} C_{N,i_0,j}(s)q^{i_0+js}\left(\frac{d}{dq}\right)^j A_s(i_0,q^s) = \frac{1}{s}\sum_{k=0}^{s-1}\zeta_s^{-i_0k}\left[\left.\left(\left(q\frac{d}{dq}\right)^N h(q)\right)\right|_{q\mapsto q\zeta_s^k}\right]$$

The first part of the lemma relates taking the derivative of a polynomial to the derivatives of the parts of the polynomial that arise in the s-dissection. The second part of the lemma relates the derivatives of these to the original polynomial.

Proof. The proof of the first part of the lemma will proceed by induction. For $N = 0$, we know $C_{0,i,0}(s) = 1$ and $C_{0,i,j} = 0$ when $j \geq N+1$, and thus the formula reduces to

$$h(q) = \sum_{i=0}^{s-1} q^i A_s(i,q^s)$$

which is true by definition. By induction we have

$$\begin{aligned}
\left(q\frac{d}{dq}\right)^{N+1} h(q) &= \left(q\frac{d}{dq}\right)\left(\sum_{j=0}^{N}\sum_{i=0}^{s-1} C_{N,i,j}(s)q^{i+js}\left(\frac{d}{dq}\right)^j A_s(i,q^s)\right)\\
&= \sum_{j=0}^{N}\sum_{i=0}^{s-1} C_{N,i,j}(s)(i+js)q^{i+js}\left(\frac{d}{dq}\right)^j A_s(i,q^s)\\
&\quad + \sum_{j=0}^{N}\sum_{i=0}^{s-1} C_{N,i,j}(s)q^{i+js}sq^s\left(\frac{d}{dq}\right)^{j+1} A_s(i,q^s)\\
&= \sum_{j=0}^{N+1}\sum_{i=0}^{s-1} C_{N,i,j}(s)(i+js)q^{i+js}\left(\frac{d}{dq}\right)^j A_s(i,q^s)\\
&\quad + \sum_{j=0}^{N+1}\sum_{i=0}^{s-1} sC_{N,i,j-1}(s)q^{i+js}\left(\frac{d}{dq}\right)^j A_s(i,q^s)\\
&= \sum_{j=0}^{N+1}\sum_{i=0}^{s-1}[(i+js)C_{N,i,j}(s) + sC_{N,i,j-1}(s)]q^{i+js}\left(\frac{d}{dq}\right)^j A_s(i,q^s)\\
&= \sum_{j=0}^{N+1}\sum_{i=0}^{s-1} C_{N,i,j}(s)q^{i+js}\left(\frac{d}{dq}\right)^j A_s(i,q^s)
\end{aligned}$$

as desired. To show part 2 of the lemma, observe that part 1 implies

$$\left(q\frac{d}{dq}\right)^N h(q) = \sum_{i=0}^{s-1} q^i \underbrace{\sum_{j=0}^{N} C_{N,i,j}(s) q^{js} \left(\frac{d}{dq}\right)^j A_s(i, q^s)}_{\text{dissections}}$$

is an s-dissection, as the right-most summands are indeed polynomials in q^s. Then by orthogonality

$$\sum_{j=0}^{N} C_{N,i_0,j}(s) q^{js} \left(\frac{d}{ds}\right)^j A_s(i_0, q^s) = \frac{1}{s}\sum_{k=0}^{s-1} \zeta_s^{-i_0 k} q^{-i_0} \left[\left(q\frac{d}{dq}\right)^N h(q)\bigg|_{q\mapsto q\zeta_s^k}\right]$$

$$\sum_{j=0}^{N} C_{N,i_0,j}(s) q^{i_0+js} \left(\frac{d}{ds}\right)^j A_s(i_0, q^s) = \frac{1}{s}\sum_{k=0}^{s-1} \zeta_s^{-i_0 k} \left[\left(q\frac{d}{dq}\right)^N h(q)\bigg|_{q\mapsto q\zeta_s^k}\right]$$

□

Stability

Next, we will begin specializing the above techniques to our problem.

Lemma. *Define the polynomials*

$$F(q, N) = \sum_{n=0}^{N} (q)_n$$

$F(q, N)$ are obtained from the partial sums of the Kontsevich-Zagier series. Let ζ_k be a kth root of unity. Then the values

$$\left(q\frac{d}{dq}\right)^\ell F(q, N)\bigg|_{q=\zeta_k}$$

are ***stable*** *for $N \geq (\ell+1)k$, namely*

$$\left(q\frac{d}{dq}\right)^\ell F(q)\bigg|_{q=\zeta_k} = \left(q\frac{d}{dq}\right)^\ell F(q, N)\bigg|_{q=\zeta_k} = \left(q\frac{d}{dq}\right)^\ell F(q, (\ell+1)k-1)\bigg|_{q=\zeta_k}$$

Stability will allow us to truncate the series $F(q)$.

Proof. For each positive integer k, $(1-q^k)^{\ell+1}$ divides $(q)_N$ when $N \geq (\ell+1)k$. This is because if $N = (\ell+1)k$, then we have

$$\begin{aligned}(q)_{(\ell+1)k} &= \prod_{k=0}^{(\ell+1)k-1} (1-q^{k+1}) \\ &= (1-q)(1-q^2)\dots(1-q^{(\ell+1)^k})\end{aligned}$$

$(1-q^k)$ divides $(1-q^k), (1-q^{2k}), \dots, (1-q^{(\ell+1)k})$. Hence $(1-q^k)^{\ell+1}$ divides $(q)_N$ for all $N \geq (\ell+1)k$.

It follows that for $0 \leq j \leq \ell$, we have

$$\left(\frac{d}{dq}\right)^j (q)_N\bigg|_{q=\zeta_k} = 0 \tag{3}$$

for $N \geq (\ell+1)k$. This is because as $(q)_N$ vanishes on ζ_k with order ℓ, so the first ℓ derivatives of $(q)_N$ will vanish.

Now take $A(q) = F(q, N)$ and $B(q) = 1$. Lemma 1 yields

$$\begin{aligned}
\left(q\frac{d}{dq}\right)^{\ell} F(q,N)\bigg|_{q=\zeta_k} &= \sum_{j=1}^{\ell} q^j c_{N,j} \left(\frac{d}{dq}\right)^{j} F(q,N)\bigg|_{q=\zeta_k} \\
&= \sum_{j=1}^{\ell} q^i c_{N,j} \sum_{n=0}^{N} \left(\frac{d}{dq}\right)^{j} (q)_n\bigg|_{q=\zeta_k} \\
&= \sum_{j=1}^{\ell} q^i c_{N,j} \sum_{n=0}^{(\ell+1)k-1} \left(\frac{d}{dq}\right)^{j} (q)_n \\
&= \left(q\frac{d}{dq}\right)^{\ell} F(q,(\ell+1)k-1)\bigg|_{q=\zeta_k}
\end{aligned}$$

by equation (3). $\square$

Proving the prime congruences

Stability will allow us to define a formal power series expansion for $F(\zeta_k e^{-t})$.

By Lemma 2, we have that

$$\left(\frac{d}{dt}\right)^{\ell} f(\zeta_k e^{-t})\bigg|_{t=0} = (-1)^{\ell} \left(q\frac{d}{dq}\right)^{\ell} f(q)\bigg|_{q=\zeta_k} \tag{4}$$

for any polynomial $f(q)$, accounting appropriately for the -1 factor. Thus we define

$$\begin{aligned}
\left(\frac{d}{dt}\right)^{\ell} F(\zeta_k e^{-t})\bigg|_{t=0} &= \left(\frac{d}{dt}\right)^{\ell} F(\zeta_k e^{-t}, N)\bigg|_{t=0} \\
&= (-1)^{\ell} \left(q\frac{d}{dq}\right)^{\ell} F(q,N)\bigg|_{q=\zeta_k}
\end{aligned}$$

These values, which correspond to the ℓth derivatives, stabilize by Lemma 4 when N is sufficiently large. Then we can define the formal series expansion

$$F(\zeta_k e^{-t}) = \sum_{\ell=0}^{\infty} \frac{1}{\ell!} (-1)^{\ell} \left(\frac{d}{dt}\right)^{\ell} F(\zeta_k e^{-t})\bigg|_{t=0} t^{\ell}$$

The proof also requires the following result.

Theorem (Zagier [2001]). *For the series $F(\zeta_k e^{-t})$, we have*

$$F(\zeta_k e^{-t}) = \sum_{n=0}^{\infty} \frac{b_n(\zeta_k)}{n!} t^n$$

where

$$b_n(\zeta_k) = \sum_{\substack{m \le 6k \\ (m,6)=1}} a(m,n,k)\zeta_k^{(m^2-1)/24}$$

The $a(m,n,k)$ can be given explicitly.

The key idea is that, when we write the coefficients b_n as functions of the roots of unity, the exponents on the roots of unity have the form $(m^2-1)/24$. These two formulations of the series $F(\zeta_k e^{-t})$ allow us to equate the coefficients for

$$b_n(\zeta_k) = (-1)^n \left(q\frac{d}{dq}\right)^n F(q,N)\bigg|_{q=\zeta_k}$$

for N sufficiently large. In particular,

$$b_0(\zeta_k) = F(q,N)\bigg|_{q=\zeta_k}$$

Definition. *The* ***pentagonal numbers*** *are given by*

$$\{(m^2-1)/24\}_{m\in\mathbb{N}} = \{0,1,2,5,7,12,\dots\}$$

where m is coprime to 6.

Definition. *Let s be a positive integer. Define*

$$S(s) = \left\{ j : 0 \le j \le s-1 \text{ such that } j \equiv \frac{m^2-1}{24} \bmod s \text{ for some } m \in \mathbb{N} \text{ coprime to } 6 \right\}$$

Consider the s-dissection

$$F(q,N) = \sum_{n=0}^{N} (q)_n = \sum_{i=0}^{s-1} q^i A_s(N,i,q^s)$$

The strategy will be to split up the sum into those for which the index is in $S(s)$ and those whose index is not.

Lemma 4 (Andrews, Sellers [2016])**.** *If $i \notin S(s)$, then $(1-q)^n$ divides $A_s(sn-1,i,q)$.*

This is the key result of the paper, and subsequent work[6] generalizes it.

Proof. We first note that the desired result is equivalent[7] to showing that for $0 \le j \le n$ and $i \notin S(s)$,

$$\left(\frac{d}{dq}\right)^j A_s(sn-1,i,q)\bigg|_{q=1} = 0 \tag{5}$$

[6] Ahlgren Kim [2015] and Ahlgren Kim Lovejoy [2015] generalize the set $S(s)$ and increase the strength of the divisibility result.

[7] The forward direction is clear. We prove the converse by induction. The case when $n=0$ is trivial. For the inductive step, suppose the first $(n+1)$ derivatives vanish on 1. Then by assumption $(1-q)^n$ divides $A_s(sn-1,i,q)$. We have

$$\left(\frac{d}{dq}\right)^{n+1}((1-q)^n p(q)) = \sum_{k=0}^{n+1}\binom{n+1}{k}\left(\left(\frac{d}{dq}\right)^{n+1-k}(1-q)^n\right)\left(\left(\frac{d}{dq}\right)^k p(q)\right)$$

for some polynomial $p(q)$, by the general Leibniz rule. Evaluating both sides on $q=1$ yields $0=p(1)$, so $(1-q)^{n+1}$ indeed divides $A_s(sn-1,i,q)$.

The proof then proceeds by induction on j. When $j=0$, substitute $q=\zeta_s$ in Lemma 3 part 2 to obtain

$$\zeta_s^i A_s(sn-1,i,1) = \frac{1}{s}\sum_{k=0}^{s-1} \zeta_s^{-ik} F(\zeta_s^{k+1}, sn-1)$$

since $C_{0,i,0}=1$. Then

$$\begin{aligned} A_s(sn-1,i,1) &= \frac{1}{s}\sum_{k=1}^{s} \zeta_s^{-i(k-1)-i} F(\zeta_s^k, sn-1) \\ &= \frac{1}{s}\sum_{k=1}^{s} \zeta_s^{-ik} b_0(\zeta_s^k) \end{aligned}$$

Fix k_0. We will now apply an orthogonality argument.[8]

$$s\zeta_s^{ik_0} A_s(sn-1,i,1) = \sum_{k=1}^{s} \zeta_s^{-ik+ik_0} b_0(\zeta_s^k)$$

Sum both sides over i for

$$\begin{aligned} \sum_{i=0}^{s-1} s\zeta_s^{ik_0} A_s(sn-1,i,1) &= \sum_{k=1}^{s}\sum_{i=0}^{s-1} \zeta_s^{-i(k-k_0)} b_0(\zeta_s^k) \\ &= sb_0(\zeta_s^k) \end{aligned}$$

Thus

$$b_0(\zeta_s^k) = \sum_{i=0}^{s-1} \zeta_s^{ik} A_s(sn-1,i,1)$$

Since $i \notin S(s)$, the exponents on the roots of unity are not equivalent to any pentagonal numbers modulo s. However, recall that we have Zagier's expression for b_0 as a linear combination of roots of unity with pentagonal number powers. Hence we invoke Lemma 2.1 of Andrews, Sellers [2016] to conclude $b_0 = 0$.

For the induction step, suppose that (5) holds for all $0 \le t \le j-1$. We want to show the claim holds for $t=j$. By Lemma 3 part 2 and the induction hypothesis, taking $q=\zeta_s$ yields

$$\begin{aligned} \sum_{k=0}^{j} C_{j,i,k}(s) q^{i+ks}\left(\frac{d}{dq}\right)^k A_s(sn-1,i,q^s) &= \frac{1}{s}\sum_{k=0}^{s-1} \zeta_s^{-ik}\left(q\frac{d}{dq}\right)^j F(q,sn-1)\Big|_{q\mapsto q\zeta_s^k} \\ C_{j,i,j}(s)\left(\frac{d}{dq}\right)^j A_s(sn-1,i,1) &= \frac{1}{s}\sum_{k=0}^{s} \zeta_s^{-ik}\left(q\frac{d}{dq}\right)^j F(q,sn-1)\Big|_{q=\zeta_s^{k+1}} \\ &= \frac{1}{s}\sum_{k=1}^{s} \zeta_s^{-i(k-1)} b_j(\zeta_s^k) \end{aligned}$$

By orthogonality, the formula for $b_j(\zeta_s^v)$, and the fact that $i \notin S(s)$, we have

$$\left(\frac{d}{dq}\right)^j A_s(sn-1,i,1) = 0$$

as $C_{t,i,t} > 0$. $\square$

[8] The rest of the proof is not quite correct and requires revision.

The moral is that for $i \notin S(s)$, there are nice divisibility properties for the polynomials in the s-dissections of our polynomials.

Definition. *Let s be a positive integer. Define*

$$T(s) = \{k : 0 \leq k \leq s-1 \text{ such that } k > \max S(s)\}$$

Example

- If $s = 11$, then

$$S(11) = \{0, 1, 2, 4, 5, 7\}$$
$$T(11) = \{8, 9, 10\}$$

These are precisely the congruence classes for which the desired equivalence holds. Next time, we will prove the following theorem.

Theorem (Andrews, Sellers [2016]). *If p is prime and $i \in T(p)$, then*

$$\xi(pn + i) \equiv 0 \text{ mod } p$$

for all $n \in \mathbb{N}$.

We also have a congruence result for prime powers, which requires some adjustments in the above work.

Our project is an attempt at a generalization of these results. The above work considers exponents of the form $(m^2 - 1)/24$, whereas a stronger result is known with exponents of the form $(m^2 - a)/b$. This will require replacing the set $T(p)$ with $T_{a,b,\xi}(p^r)$ which satisfies certain conditions.

Next time, we will examine the set $T(s)$ in the general case. We would like to classify the sets $T_{a,b,\xi}(p^r)$ for the torus knot $T(3, 4)$ and $T(3, 2^t)$.

'Strange identities' indicate that we have congruences for $\mathcal{F}(1-q)$. Do we have strange identities for the torus knots? How might we prove these?

7/8/2019 - Computing the Colored Jones Polynomial Algebraically

Recall that an enhanced Yang-Baxter operator is a quadruple (R, μ, a, b), where $R \in \mathrm{Aut}(V \otimes V)$, $\mu : V \to V$, and $a, b \in \mathbb{C}$. These should satisfy

1. $(R \otimes \mathrm{id}_V)(\mathrm{id}_V \otimes R)(R \otimes \mathrm{id}_V) = (\mathrm{id}_V \otimes R)(R \otimes \mathrm{id}_V)(\mathrm{id}_V \otimes R)$ on $V \otimes V \otimes V$
2. $R(\mu \otimes \mu) = (\mu \otimes \mu)R$ on $V \otimes V$
3. $\mathrm{Tr}_2(R^{\pm 1}(\mathrm{id} \otimes \mu)) = a^{\pm 1} b\, \mathrm{id}_V$ on V

Then we defined

$$T_{(R,\mu,a,b)}(K) = a^{-\omega(\beta)} b^{-n} \mathrm{Tr}_1(\mathrm{Tr}_2(\ldots(\mathrm{Tr}_n(\Phi(\beta) \otimes \mu^{\otimes n}))\ldots))$$

where β is an n-stranded braid so that $\widehat{\beta} = K$, $\omega(\beta)$ is the writhe of the braid, and $\Phi(\beta) \in \mathrm{End}(V^{\otimes n})$ is constructed algorithmically from the braid, with the procedure described in the previous lecture.

We will mention, omitting some of the details, that this indeed gives a well-defined invariant of knots.[9] In short, the three conditions specified in the definition of the enhanced Yang-Baxter operator provide precisely the conditions necessary to show invariance under the braid and Markov moves.

Braid relations: The definition of Φ immediately implies that it respects the far commutativity relation in the braid group. The first condition in the definition of the Yang-Baxter operator implies that ϕ respects the braid relation $\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}$.

Markov moves: $T_{(R,\mu,a,b)}$ respects the first Markov move because the operator Tr_k is invariant under cyclic permuatations of its argument, and because $\mu \otimes \mu$ commutes with R. Invariance under stabilization and destabilization follows from the third condition above.

Our interest lies mainly in the enhanced Yang-Baxter operator given by taking $V = \mathbb{C}^N$ with standard basis $\{e_0, e_1, \ldots, e_{N-1})$ and defining

$$\{m\} = q^{m/2} - q^{-m/2}$$

$$\{m\}! = \{m\}\{m-1\}\ldots\{2\}\{1\} = \prod_{j=1}^{m}(q^{j/2} - q^{-j/2})$$

The R-matrix is given by

$$R(e_k \otimes e_\ell) = \sum_{i,j=0}^{N} R_{k,\ell}^{i,j} e_i \otimes e_j$$

[9] Some of these ideas with additional explanation can be found in the notes from the previous lecture.

where

$$R^{i,j}_{k,\ell} = \sum_{m=0}^{\min(N-1-i,j)} \delta_{\ell,i+m}\delta_{k,j-m}\frac{\{\ell\}!\{N-1-k\}!}{\{i\}!\{m\}!\{N-1-j\}!}q^{(i-(N-1)/2)(j-(N-1)/2)-m(i-j)/2-m(m+1)/4}$$

And

$$\mu(e_j) = \sum_{i=0}^{N-1} \mu^i_j e_j$$

where

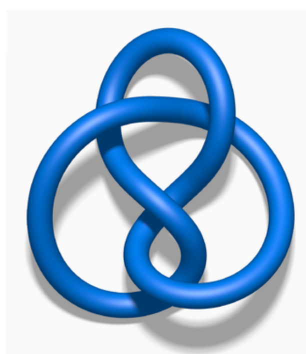
$$\mu^i_j = \delta_{i,j}q^{(2i-N+1)/2}$$

If we set $a = q^{(N^2-1)/4}$ and $b = 1$, then (R, μ, a, b) is an enhanced Yang-Baxter operator. We define the colored Jones polynomial as the normalized

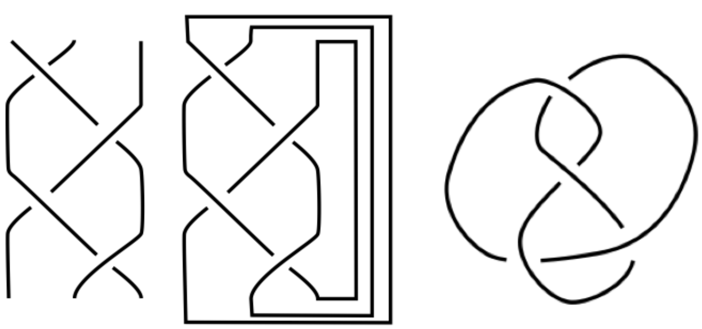
$$J_N(K, q) = \frac{\{1\}}{\{N\}}T_{(R,\mu,a,b)}(K)$$

The colored Jones polynomial of the figure eight knot

The goal today will be to produce a computation of the colored Jones polynomial for $K = 4_1$, the figure eight knot, using the above definition.



First note that the closure of the braid $\beta = \sigma_1\sigma_2^{-1}\sigma_1\sigma_2^{-1}$ is the figure eight knot.



Next, observe that the summation in the definition of $R^{i,j}_{k,\ell}$ is not in fact a sum, as only one of the summands will ever be nonzero. This is becase in order for there to be a nontrivial summand, there are relations demanded upon the indices i, j, k, ℓ by the Kronecker delta symbols. For positive crossings

1. We must have $i \leq \ell$
2. We must have $k \leq j$
3. We must have $i + j = k + \ell$

The inverse of R has matrix entries is given by

$$(R^{-1})^{i,j}_{k,\ell} = \sum_{m=0}^{\min(N-1-j,i)} \delta_{\ell,i-m}\delta_{k,j+m}\frac{\{k\}!\{N-1-\ell\}!}{\{j\}!\{m\}!\{N-1-i\}!}(-1)^m q^{-(i-(N-1)/2)(j-(N-1)/2)-m(i-j)/2+m(m+1)/4}$$

Similar reasoning yields that for negative crossings

1. We must have $i \geq \ell$
2. We must have $k \geq j$
3. We must have $i + j = k + \ell$

We have that

$$J_N(K, q) = \frac{\{1\}}{\{N\}}\mathrm{Tr}_1(\mathrm{Tr}_2(\mathrm{Tr}_3(\Phi(\beta) \circ \mu^{\otimes 3})))$$

since the writhe of the above braid is zero. It will be easier to compute

$$\mathrm{Tr}_2(\mathrm{Tr}_3(\Phi(\beta) \circ \mathrm{id}_V \otimes \mu^{\otimes 2}))$$

This coincides with $S \cdot \mathrm{id}_V$ for some scalar S, in which case[10]

$$\mathrm{Tr}_1(\mathrm{Tr}_2(\mathrm{Tr}_3(\Phi(\beta) \circ \mu^{\otimes 3}))) = S \cdot \mathrm{Tr}_1(\mu) = \frac{\{N\}}{\{1\}}S$$

as the unnormalized colored Jones polynomial of the unknot is $\{N\}$. This implies

$$J_N(K, q) = S$$

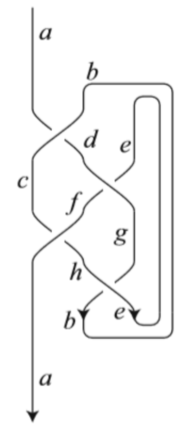
We can evaluate S in terms of R and R^{-1} as follows:

$$\sum_{b,c,d,e,f,g,h} R^{a,b}_{c,d}(R^{-1})^{d,e}_{a,h}R^{c,f}_{a,h}(R^{-1})^{h,g}_{b,e}\mu^b_b\mu^e_e$$

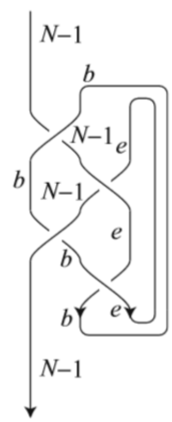
The scalar S should be independent of everything in the diagram, so we can set $a = N - 1$ to color

[10]This is nontrivial. See Kirby and Melvin [1991] for a proof by Schur's lemma.

this strand with the last representation.



By the above inequalities for positive crossings, we know that $a \leq d$ and so $d = N-1$. We also know $f = N-1$. By the above equality, we have that $b = c$. The same reasoning implies $h = b = c$. We have that $g = e$. Finally, by the above inequality for negative crossings, we have $b \geq e$.



Hence

$$\begin{aligned} T_{(R,\mu,a,b)}(K) &= \sum_{b \geq e} R^{N-1,b}_{b,N-1}(R^{-1})^{N-1,e}_{N-1,e} R^{b,N-1}_{N-1,b}(R^{-1})^{b,e}_{b,e}\mu^b_b\mu^e_e \\ &= \sum_{b \geq e}(-1)^{N-1+b}\frac{\{N-1\}!\{b\}!\{N-1-e\}!}{(\{e\}!)^2\{b-e\}!\{N-1-b\}!}q^{(-b-b^2-2be-2e^2+3N+6Nb+2Ne-3N^2)/4} \end{aligned}$$

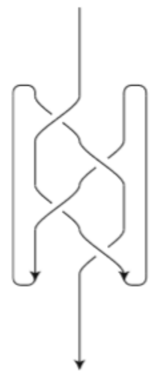
Now, in the summation for $R^{N-1,b}_{b,N-1}$ the only term present is when $m = 0$, in which case the coefficient is 1. There are no contributions from the quantum factorials, although there will be a q residual. In the summation for $(R^{-1})^{N-1,e}_{N-1,e}$ the only term present is when $m = N-1-e$, in which case it contributes

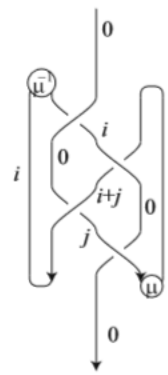
$$\frac{\{N-1\}!}{\{e\}!}q^{-((N-1)/2)(e-(N-1)/2)-(N-1-e)^2)/2+(N-1-e)(N-e)/4}$$

Similarly, $R^{b,N-1}_{N-1,b}$ has a nonzero summand when $m = 0$, in which case it contributes just a q residual. Finally, the term $(R^{-1})^{b,e}_{b,e}$ has a nonzero summand when $m = b-e$, in which case it contributes

$$\frac{\{b\}!\{N-1-e\}!}{\{e\}!\{b-e\}!\{N-1-b\}!}q^{-(b-(N-1)/2)(e-(N-1)/2)-(b-e)^2/2+(b-e)(b-e+1)/4}$$

which yields the colored Jones polynomial for the figure eight knot. One can also compute the colored Jones polynomial using the appropriate tangle:





which yields

$$J_N(K,q) = \sum_{k=0}^{N-1} \frac{\{N-1\}!}{\{N-1-k\}!} q^{k^2/4+Nk/2+k/4} \left(\sum_{i=0}^{k} (-1)^i \frac{\{k\}!}{\{i\}!\{k-i\}!} q^{-Ni-ik/2-i/2} \right)$$

Using the formula

$$\sum_{i=0}^{k} (-1)^i q^{\ell i/2} \frac{\{k\}!}{\{i\}!\{k-i\}!} = \prod_{g=1}^{k} (1 - q^{(\ell+k+1)/2-g})$$

implies

$$J_N(K,q) = \frac{1}{\{N\}} \sum_{k=0}^{N-1} \frac{\{N+k\}!}{\{N-1-k\}!}$$

We will see next time that there is an alternative technique to compute the colored Jones polynomial, using cabling and the bracket polynomial.

7/10/2019 - Congruences and Strange Identities

Last time, we spoke about the Kontsevich-Zagier strange series

$$F(q) = \sum_{n\geq 0}(q)_n$$

We are interested in the Fishburn numbers, given by

$$F(1-q) = \sum_{n\geq 0}\xi(n)q^n$$

We showed that we can truncate the series and consider an s-dissection for

$$F(q,N) = \sum_{n=0}^{N}(q)_n = \sum_{i=0}^{s-1} q^i A_s(N,i,q^s)$$

Given a positive integer s, we defined the set

$$S(s) = \left\{j : 0 \leq j \leq s-1 \text{ such that } j \equiv \frac{m^2-1}{24} \bmod s \text{ for some } m \in \mathbb{N} \text{ coprime to } 6\right\}$$

We proved that for $i \notin S(s)$, $(1-q)^n$ divides $A(sn-1,i,q)$. Today, we will further consider the set

$$T(s) = \{k : 0 \leq k \leq s-1 \text{ such that } k > \max S(s)\}$$

and show that this set corresponds to the congruence classes for which we have congruences.

Theorem (Andrews, Sellers [2016])**.** *If p is prime and $i \in T(p)$, then*

$$\xi(pn+i) \equiv 0 \textit{ mod } p$$

for all $n \in \mathbb{N}$.

Before the proof, we will require the following two theorems.

Theorem (Lucas' theorem [1878])**.** *Let m, n be nonnegative integers and p prime. Consider the base p expansions*

$$m = m_k p^k + m_{k-1}p^{k-1} + \ldots + m_1 p + m_0$$
$$n = n_k p^k + n_{k-1}p^{k-1} + \ldots + n_1 p + n_0$$

Then $\binom{m}{n} \equiv 0$ mod p if and only if $n_j > m_j$ for some j.

When we eventually strengthen the congruence result to include prime powers, it will be necessary to consider a generalized version of Lucas's theorem.

Theorem. *We can write*

$$\sum_{n=0}^{\infty}\xi(n)q^n = F(1-q,N) + O(q^{N+1})$$

We can now prove the main theorem.

Proof. We will show that, when $i \in T(p)$, the coefficient of q^{pn+i} in

$$F(1-q) = \lim_{N\to\infty} F(1-q, N)$$

is equivalent to 0 modulo p, where the limit indicates that by taking N sufficiently large we can guarantee that the coefficient of q^{pn+i} stabilizes. By Lucas' theorem, if π is an integer congruent to a pentagonal number λ modulo p, then

$$\binom{\pi}{i} \equiv 0 \bmod p$$

This is because writing

$$\begin{aligned}\pi &= \pi_k p^k + \pi_{k-1} p^{k-1} + \ldots + \pi_1 p + \pi_0 \\ \lambda &= \lambda_k p^k + \lambda_{k-1} p^{k-1} + \ldots + \lambda_1 p + \lambda_0\end{aligned}$$

implies $\pi_0 = \lambda_0 \in S(p)$, which is less than $i \in T(p)$ by definition.

By Lemma 4, we have the decomposition

$$\begin{aligned}F(q, pn-1) &= \sum_{i=0}^{p-1} q^i A_p(pn-1, i, q^p) \\ &= \sum_{\substack{i=0\\ i\in S(p)}}^{p-1} q^i A_p(pn-1, i, q^p) + \sum_{\substack{i=0\\ i\notin S(p)}}^{p-1} q^i (1-q^p)^n \alpha_p(n, i, q^p)\end{aligned}$$

for some polynomial $\alpha_p(n, i, q) \in \mathbb{Z}[q]$. So

$$\begin{aligned}F(1-q, pn-1) &\equiv \sum_{\substack{i=0\\ i\in S(p)}}^{p-1} (1-q)^i A_p(pn-1, i, (1-q)^p) + \sum_{\substack{i=0\\ i\notin S(p)}}^{p-1} (1-q)^i (1-(1-q)^p)^n \alpha_p(n, i, (1-q)^p) \\ &\equiv \sum_{\substack{i=0\\ i\in S(p)}}^{p-1} (1-q)^i A_p(pn-1, i, 1-q^p) + O(q^{pn}) \bmod p\end{aligned}$$

using that $(1-q)^p \equiv 1-q^p \bmod p$ and that

$$(1-(1-q)^p)^n = \left(-\sum_{i=1}^{p} \binom{p}{i} (-q)^i\right)^n \equiv \big(-(-q)^p\big)^n \bmod p$$

Since $A_p(pn-1, i, 1-q^p)$ is a polynomial in q^p, any term with an exponent congruent to an element of $T(p)$ modulo p arises from the expansion of $(1-q)^i$. However, the expansion of $(1-q)^i$ is

$$(1-q)^i = \sum_{k=0}^{i} \binom{i}{k} (-q)^k$$

So if k is congruent to an element of $T(p)$ modulo p, $\binom{i}{k} = 0$ by Lucas' theorem, as $i \in S(p)$. Therefore any variable with exponent congruent to an element of $T(p)$ modulo p has coefficient zero modulo p.

This argument holds for the stabilized terms in $F(1-q, pn-1)$, so for all $j < pn$ such that j is congruent to an element of $T(p)$ modulo p, the coefficient of q^j is equivalent to 0 modulo p. Letting n go to infinity concludes the proof. $\square$

The case of prime powers is very similar. One can either proceed by choosing coefficients more carefully or employing a stronger divisibility result and an alternative decomposition.

Identifying congruence classes and their asymptotics

Recall that we claimed there exist congruences for half of the primes. Can we (explicitly) construct $T(p)$?

Definition. *Let p be an odd prime, and let $a \in \mathbb{Z}$. The **Legendre symbol** is given by*

$$\left(\frac{a}{p}\right) = \begin{cases} 1 & a \equiv x^2 \textit{ mod } p \textit{ for some nonzero } x \in \mathbb{Z} \\ -1 & a \not\equiv x^2 \textit{ mod } p \textit{ for all } x \in \mathbb{Z} \\ 0 & \textit{if } p \textit{ divides } a \end{cases}$$

*In the first case, a is a **quadratic residue** modulo p. In the second case, a is a **quadratic nonresidue**.*

We will require some basic properties of the Legendre symbol.

Proposition. *We have the following properties.*

1. $\left(\frac{a}{p}\right) = \left(\frac{b}{p}\right)$ *if* $a \equiv b$ *mod* p
2. $\left(\frac{ab}{p}\right) = \left(\frac{a}{p}\right)\left(\frac{b}{p}\right)$
3. $\left(\frac{-1}{p}\right) = (-1)^{(p-1)/2}$
4. *Let p and q be distinct odd primes. Then*

$$\left(\frac{p}{q}\right)\left(\frac{q}{p}\right) = (-1)^{\frac{p-1}{2}\frac{q-1}{2}}$$

   *This is the **law of quadratic reciprocity**.*

5. *Fix a nonsquare integer a. The set*

$$\{p \textit{ prime} : \left(\frac{a}{p}\right) = 1\}$$

   has natural density 1/2.

Definition. *A set S of primes has **natural density** $\delta(S)$ if the limit*

$$\lim_{x\to\infty} \frac{|\{p \le x : p \in S\}|}{|\{p \le x : p \text{ prime}\}|}$$

exists and is equal to $\delta(S)$.

Theorem. *Let $p \neq 23$ be an odd prime. Then $T(p)$ is nonempty if and only if $p = 23k + r$ for $k \ge 0$ and $0 < r < 23$, such that $\left(\frac{r}{23}\right) = -1$.*

Proof. First note that $T(p)$ is empty if and only if $p - 1 \in S(p)$, equivalently

$$p - 1 \equiv (m^2 - 1)/24 \bmod p$$

$$m^2 \equiv -23 \bmod p$$

for some m coprime to 6. This holds if and only if $\left(\frac{-23}{p}\right) = 1$.

Suppose $p = 23k + r$ such that $\left(\frac{r}{23}\right) = -1$. Then

$$\begin{aligned}
\left(\frac{-23}{p}\right) &= \left(\frac{-1}{p}\right)\left(\frac{23}{p}\right) \\
&= (-1)^{(p-1)/2}(-1)^{11(p-1)/2}\left(\frac{p}{23}\right)^{-1} \\
&= (-1)^{11(p-1)^2/2}\left(\frac{p}{23}\right) \\
&= \left(\frac{r}{23}\right) = -1
\end{aligned}$$

by the properties of the Legendre symbol, since $p \equiv r \bmod 23$, and since $p - 1$ is even. Thus $T(p)$ is nonempty.

Now suppose $T(p)$ is nonempty. Write $p = 23k + r$ with $k \ge 0$ and $0 < r < 23$. Then we have

$$-1 = \left(\frac{-23}{p}\right) = \left(\frac{r}{23}\right)$$

as above. $\square$

Corollary. *There are congruences for the coefficients of the Fishburn numbers for half of the primes.*

What about the case $p = 23$? What about prime powers?

The idea will be to instead consider the set

$$S^*(p) = \{j : 0 \le j \le p-1 \text{ such that } j \equiv \frac{m^2 - 1}{24} \text{ for some } m \in \mathbb{N} \text{ coprime to 6 and } 24j \not\equiv -1 \bmod p\}$$

and similarly define $T^*(p)$. Then we have the following theorem.

Theorem (Garvan [2015]). *Let $p \geq 5$ be a prime. If $j \in T^*(p)$, then*

$$\xi(pn + j) \equiv 0 \ mod\ p$$

for all $n \in \mathbb{N}$.

Example

- We have $22 \in S(23)$, but $22 \notin S^*(23)$ and $T^*(23) = \{18, 19, 20, 21, 21, 22\}$. This yields additional congruences.

We can use similar techniques along with *Kumer's theorem*,[11] to prove the following theorem.[12]

Theorem (Straub [2015]). *Let p be a prime. If $j \in \{1, 2, \ldots, p - 1 - \max S^*\}$, then*

$$\xi(p^r n - j) \equiv 0 \ mod\ p$$

for all $n \in N$.

Strange identities[13]

We will now attempt to understand how to obtain these congruences for our series as well as tbe source of these results.

Theorem (Zagier [2001]). *We have*

$$F(q) = \frac{1}{2} \sum_{n=1}^{\infty} n \chi_{12}(n) q^{(n^2-1)/24}$$

for

$$\chi_{12}(n) = \begin{cases} 1 & n \equiv 1, 11, \ mod\ 12 \\ -1 & n \equiv 5, 7 \ mod\ 12 \\ 0 & otherwise \end{cases}$$

Remark. *The statement of equality in the theorem is meant in the sense that the radial limit yields $F(\zeta)$. More formally, taking $q = \zeta e^{-t}$ in the right-hand expression and letting $t \to 0^+$ yields $F(\zeta)$.*

This is a *strange identity.* It says that the expression $F(q)$ is sufficiently well-behaved on roots of unity to be approximated by a complex function on the open disc.

An important moral is that *behind every strange identity lies an actual q-series identity.*

[11] A generalization of Lucas's theorem

[12] See pages 1687-1688 in Straub's paper. He uses nothing more than the divisibility result in Andrews, Sellers [2016].

Behind a Strange Identity Lies a q-Series Identity

We have the identity

$$\sum_{n=0}^{\infty}[(q)_n - (q)_\infty] = \frac{1}{2}\sum_{n=1}^{\infty} n\chi_{12}(n)q^{(n^2-1)/24} + \left(\frac{1}{2} - \sum_{n=1}^{\infty}\frac{q^n}{1-q^n}(q)_\infty\right)$$

Taking $q = \zeta e^{-t}$ and $t \to 0^+$ implies $(q)_\infty \to 0$, and Zagier's result follows.

Questions: How can we prove a strange identity for $T(3,4)$? What about the general $T(3,2^t)$? How is such an identity related to congruences?

To answer the first question, we will follow Hikami and Kirillov. We first define

$$H(x) = H(x,q) = \sum_{n=1}^{\infty}\chi_{24}(n)q^{(n^2-25)/48}x^{(n-5)/2}$$

where

$$\chi_{24}(n) = \begin{cases} 1 & n \equiv 5, 19 \bmod 24 \\ -1 & n \equiv 11, 13 \bmod 24 \\ 0 & \text{otherwise} \end{cases}$$

We have the following key result.

Proposition. *We can write*

$$H(x) = \sum_{n=0}^{\infty}(x)_{n+1}x^{2n}\left(\sum_{k=0}^{\lfloor (n-1)/2\rfloor} x^{2k-1}q^{2k(k+1)}\begin{bmatrix} n \\ 2k+1\end{bmatrix}_q + \sum_{k=0}^{\lfloor n/2\rfloor} x^{2k}q^{2k+1}\begin{bmatrix} n+1 \\ 2k+1\end{bmatrix}_q\right)$$

Before the next result we will need some facts.

Theorem (Classical q-binomial theorem)**.** *We have*

$$\sum_{n=0}^{\infty}\frac{(a)_n}{(q)_n}z^n = \frac{(az)_\infty}{(z)_\infty}$$

Theorem (Watson's quintuple product identity)**.** *We have*

$$\sum_{k\in\mathbb{Z}} q^{k(3k-1)/2}x^{3k}(1-xq^k) = (q, x, qx^{-1}, q)_\infty(qx^2, qx^{-2}, q^2)_\infty$$

where

$$(a_1, \dots, a_m)_n = (a_1, \dots, a_m; q)_n = (a, q)_n \dots (a_m, q)_n$$

Theorem (Slater's identity)**.** *We have*

$$(q)_\infty\sum_{n=0}^{\infty}\frac{q^{2n(n+1)}}{(q)_{2n+1}} = (q^3, q^5, q^8, q^8)_\infty(q^2, q^{14}, q^{16})_\infty$$

Lemma. *We have the following three identities.*

$$H(x) = (qx)_\infty \sum_{n=0}^{\infty} \frac{q^{2n(n+1)}}{(x^2q)_{2n+1}} x^{6n} \\ + (1-x)\sum_{n=0}^{\infty}[(qx)_n - (qx)_\infty]x^{2n}\left(\sum_{k=0}^{\infty} x^{2k-1}q^{2k(k+1)}\left(\begin{bmatrix} n \\ 2k+1\end{bmatrix}_q + x\begin{bmatrix} n+1 \\ 2k+1\end{bmatrix}_q\right)\right)$$

and

$$\frac{1}{2}\sum_{n=0}^{\infty} n\chi_{24} q^{(n^2-25)/48} - \frac{5}{2}(q^3,q^5,q^8,q^8)_\infty(q^2,q^{14},q^{16})_\infty \\ = (q^3,q^5,q^8,q^8)_\infty(q^2,q^{14},q^{16})_\infty\left(-\sum_{k=1}^{\infty}\frac{q^k}{1-q^k}\right) \\ + (q)_\infty\sum_{n=0}^{\infty}\frac{q^{2n(n+1)}}{(q)_{2n+1}}\left(6n + 2\sum_{k=1}^{2n+1}\frac{q^k}{1-q^k}\right) - \sum_{n=0}^{\infty}((q)_n - (q)_\infty)\big(T_n(q) + T_{n+1}(q)\big)$$

where

$$T_n(q) = \sum_{k=0}^{\lfloor (n-1)/2 \rfloor} q^{2k(k+1)}\begin{bmatrix} n \\ 2k+1\end{bmatrix}_q$$

and finally

$$\mathcal{F}(q) = -\frac{1}{2}\sum_{n=0}^{\infty} n\chi_{24}(n) q^{(n^2-25)/48}$$

where $\mathcal{F}$ is the series obtained from the colored Jones polynomial of the knot $T(3,4)$, and equality is meant in the sense of a strange identity.

Proof. First, note that the right hand side of the first equation is

$$\underbrace{(qx)_\infty\sum_{n=0}^{\infty}\frac{q^{2n(n+1)}}{(x^2q)_{2n+1}}x^{6n}}_{(1)} + RHS - \underbrace{(x)_\infty\sum_{n=0}^{\infty}x^{2n}\sum_{k=0}^{\infty}x^{2k-1}q^{2k(k+1)}\left(\begin{bmatrix} n \\ 2k+1\end{bmatrix}_q + x\begin{bmatrix} n+1 \\ 2k+1\end{bmatrix}_q\right)}_{(2)}$$

using that

$$(1-x)(qx)_n = (x)_{n+1}$$
$$(1-x)(qx)_\infty = (x)_\infty$$

We will show that (2) equals (1). We have

$$(x)_\infty\sum_{n=0}^{\infty}x^{2n}\sum_{k=0}^{\infty}x^{2k-1}q^{2k(k+1)}\left(\frac{(q)_n}{(q)_{2k+1}(q)_{n-2k+1}} + x\frac{(q)_{n+1}}{(q)_{2k+1}(q)_{n-2k}}\right) \\ = (x)_\infty\sum_{k\ge 0}\frac{x^{2k-1}q^{2k(k+1)}}{(q)_{2k+1}}\underbrace{\left(\sum_{n\ge 0}x^{2n+4k+2}\frac{(q)_{n+2k+1}}{(q)_n} + \sum_{n\ge 0}x^{2n+4k+1}\frac{(q)_{n+2k+1}}{(q)_n}\right)}_{(*)}$$

where we are replacing n with $n+2k+1$ to obtain the first sum, and n with $n+2k$ to obtain the second sum. This is equal to

$$(x)_\infty(1+x)\sum_{k\geq 0} x^{6k}q^{2k(k+1)}\frac{(x^2q^{2k+2})_\infty}{(x^2)_\infty}$$

since (*) is

$$(1+x)x^{4k+1}\sum_{n\geq 0}\frac{(q)_{n+2k+1}}{(q)_n}x^{2n} = (1+x)x^{4k+1}(q)_{2k+1}\sum_{n\geq 0}\frac{(q^{2k+2})_n}{(q)_n}x^{2n}$$

Apply the binomial theorem with $a=q^{2k+2}$ and $z=x^2$ for

$$(xq)_\infty\sum_{k\geq 0}\frac{x^{6k}q^{2k(k+1)}}{(x^2q)_{2k+1}}$$

using that $(x)_\infty=(1-x)(xq)_\infty$, which completes the proof of the first part of the lemma.

To show the second part of the lemma, differentiate both sides of the previous equality with respect to x, and then let $x\to 1$. For the right hand side, it will be necessary to use the product rule

$$\frac{d}{dx}\prod_{k=1}^{n} f_k(x) = \left(\sum_{k=1}^{n}\frac{\frac{d}{dx}f_k(x)}{f_k(x)}\right)\prod_{k=1}^{n} f_k(x)$$

The third part follows by taking $q=\zeta e^{-t}$ in the second part and letting $t\to 0^+$. $\square$

Robert conjectures a strange identity in general.

Conjecture. *Let $\mathcal{F}_t(q)$ be the Kontsevich-Zagier series associated to the general family of torus knots $T(3,2^t)$. Then we have the strange identity*

$$\mathcal{F}_t(q)=\frac{1}{2}\sum_{n=0}^{\infty} n\chi_{3\cdot 2^{t+1}}(n)q^{(n^2-(3-2^{t+1})^2)/(3\cdot 2^{t+2})}$$

where

$$\chi_{3\cdot 2^{t+1}}(n)=\begin{cases}1 & n\equiv 3-2^{t+1}, 2^{t+3}-3 \ mod\ 3\cdot 2^{t+1}\\ -1 & n\equiv 2^{t+2}-3, 2^{t+1}+3 \ mod\ 3\cdot 2^{t+1}\\ 0 & otherwise\end{cases}$$

We will now address why strange identities are important. Consider the generalized version of the strange identity

$$P^{(v)}_{a,b,\chi}(q)=\sum_{n\geq 0} n^v\chi(n)q^{(n^2-a)/b}$$

where $v\in\{0,1\}$, $a\geq 0, b>0$ are integers, and $\chi:\mathbb{Z}_{\geq 0}\to\mathbb{C}$ is a function that satisfies

1. $\chi(n)\neq 0$ only if $(n^2-a)/b\in\mathbb{Z}$
2. For each root of unity ζ, the function $n\mapsto \zeta^{(n^2-a)/b}\chi(n)$ is periodic and has mean value 0.

Now let

$$F(q) = \sum_{n\geq 0}(q)_n f_n(q)$$

where $f_n(q) \in \mathbb{Z}[q]$. For positive integers s and N, we can similarly truncate

$$F(q, N) = \sum_{n=0}^{N}(q)_n f_n(q) = \sum_{i=0}^{s-1} q^i A_{F,s}(n, i, q^s)$$

Define

$$S_{a,b,\chi}(s) = \left\{0 \leq j \leq s-1 : j \equiv \frac{n^2 - a}{b} \bmod s \text{ for some } n \in \mathbb{N} \text{ such that } \chi(n) \neq 0\right\}$$

$$\lambda(N, s) = \lfloor \frac{N+1}{s} \rfloor$$

Theorem (Ahlgren, Kim, Lovejoy [2018])**.** *Suppose we have*

$$F(q) = P^{(v)}_{a,b,\chi}(q)$$

If $i \notin S_{a,b,\chi}(s)$, then $(q)_{\lambda(N,s)}$ divides $A_{F,s}(N, i, q)$.

This divisibility result implies the prime power congruences. Note that when $a = 1, b = 24, \chi = \chi_{12}, v = 1$, we have Zagier's strange identity. When $a = 25, b = 48, \chi = \chi_{24}, v = 1$, we have the strange identity for $T(3, 4)$. When $a = 3 \cdot 2^{t+1}, b = 3 \cdot 2^{t+2}, \chi = \chi_{3\cdot 2^{t+1}}, v = 1$, it is conjectured we have a general identity for $T(3, 2^t)$.

We must

1. For $T(3, 4)$, confirm that conditions 1 and 2 are satisfied.[a]
2. Compute some examples to verify the theorem ($s = 5, N = 8$).
3. For $T(3, 2^t)$, check that conditions 1 and 2 are satisfied.
4. Determine the prime and prime power congruences in general for $T(3, 4)$.

[a] For example, see Lemma 5.1 in the Ahlgren, Kim, Lovejoy paper.

7/12/2019 - Framing and the Skein Algebra

The structure of the project is broadly to attempt to prove number theoretic dualities via knot theoretic computations. Today we will begin to discuss the necessary *skein-theory.*

Skein Algebras

Our first goal will be to understand the computation of the colored Jones polynomial of the trefoil via the Kauffman bracket, due to Masbuam [2003]. We will attempt to emulate this technique for other knots. We will employ the *Kauffman bracket skein algebra.* This method is distinct from other computations of the colored Jones polynomial, and it has connections to quantum algebra,[14] 3-manifold invariants, and mathematical physics.[15]

Definition. *A **skein algebra** over a ring R, associated to an orientable 3-manifold M, is the R-module with elements that are formal linear combinations of links in M up to isotopy.*

For our purposes, we will take $R = \mathbb{Z}[A, A^{-1}]$ and either $M = S^3$ or $M = S^1 \times D^2$ (the solid torus). These skein algebras are very large.

Remark. *We will mention the algebra structure on these skein objects, but in general we will only consider them as R-modules. In particular, maps on skein algebras will not usually respect their multiplication.*

That being said, it is often very natural to examine the skein algebra of a thickened surface, namely a 3-manifold of the form $S \times [0, 1]$. In this case, we can define a multiplication by stacking two copies of the ambient manifold to obtain a product analogous to the disjoint union of the two links. We will see this concretely in the case of the solid torus.

Elements of skein algebras are in fact linear combinations of *framed* links.

Definition. *A **framing** of a link L is an embedded, orientable surface which deformation retracts onto L.*

These framings are distinguished by their number of twists. The surface must be orientable, so it will have an integral number of full twists. In a diagrammatic representation of a link, we can label

[14]The study of the representation theory of quantum groups

[15]See the paper *The colored Jones polynomial of doubles of knots*, Tanaka [2008], which adapts Masbaum's method in a similar way that we will. Also see Lickorish, chapters 12-14.

each component with the integral number of full twists on each framing. Given a link diagram, there is a usual framing obtained by viewing the surface to be parallel to the plane of the diagram. This is the ***blackboard framing*** of a link.

Note that there are right- and left-handed twists, which can be distinguished by the direction in which your hand twists when traveling along the surface. Right-handed twists are clockwise, and left-handed twists are counterclockwise.

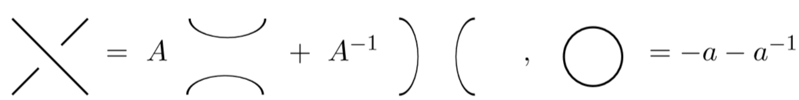
The number of twists of a framing is well-defined. We can 'cut' the surface at a point and then measure the number of resulting twists when the surface is unraveled.

Remark. *The Reidemeister I move increases or decreases the framing number by 1.*

The Reidemeister I move results in a *full* twist, rather than a half twist, as the surface is orientable and 'pulling the crossing tight' is an isotopy.

Remark. *The Reidemesiter II and Reidemeister III moves do not affect the framing of a link.*

Definition. *Let A be a formal variable and M an orientable 3-manifold. The* ***Kauffman bracket skein algebra*** *of M, denoted $\mathcal{K}(M)$, is the $\mathbb{Z}[A, A^{-1}]$-module generated by framed links in M up to isotopy with relations*



where $a = A^2$. In other words, $\mathcal{K}(M)$ is the general skein algebra of M quotient the submodule generated by the above relations.

From now on, the term 'skein algebra' will refer to the Kauffman bracket skein algebra.

Every element of $\mathbb{Z}[A, A^{-1}]$ is an element of the skein algebra, via the scalar product with the empty link. The second relation indicates that, given a framed link, one can produce two new links by resolving a local crossing. The $\mathcal{K}(M)$ element represented by that framed link can be expressed as a sum of the resolutions.

This is the framework in which the Masbuam computation will take place. We will consider a heavily decorated knot diagram, and the colored Jones polynomial of this knot will be the associated element of the skein algebra.

The above relations imply the following result.

Proposition. *Let $L^{\pm 1}$ be the link obtained from L by increasing the framing by 1. Then*

$$L^{\pm 1} = -A^{\pm 3} L$$

in $\mathcal{K}(M)$. This is the ***framing relation****.*

Proof. The proof is not difficult. Introducing a twist on L can be represented diagramatically via the corresponding Reidemeister I move. Applying the relations above and simplifying yields the desired equality, using the fact that multiplication in $\mathcal{K}(M)$ is given by disjoint union.[16] $\square$

The skein algebra of S^3 is isomorphic to $\mathbb{Z}[A, A^{-1}]$ and thus is particularly simple. This is not always the case for other 3-manifolds, as repeatedly applying the bracket relations does not necessarily yield homotopically trivial unlinks.

Computing the colored Jones polynomial corresponds to examining more complex framings and decorations of a diagram and then applying a modified isomorphism to obtain the corresponding polynomial. We would like to find such a diagram for $T(3,4)$ to simplify computation of the colored Jones polynomial.

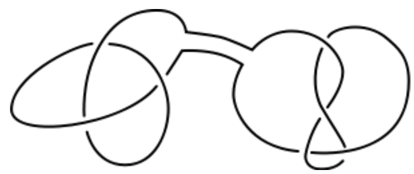
[16] See the notes from 7/19/2019

7/15/2019 - Satellite Knots, Cabling, and the Volume Conjecture

Much of the topological side of the project will be computations in the Skein algebra.

Operations on knots

Definition. *The **connected sum** of K and J, denoted $K\#J$, is the operation defined on two knots by*



Note that the connected sum is only well-defined when the knots are oriented, as this forces them to be connected in one particular way (otherwise, there are two distinct ways to connect the knots).

Remark. *The unknot is a unit for connected sum. Thus connected sum makes knots into a commutative monoid.*

Definition. *A knot is **composite** if it is the connected sum of two nontrivial knots.*

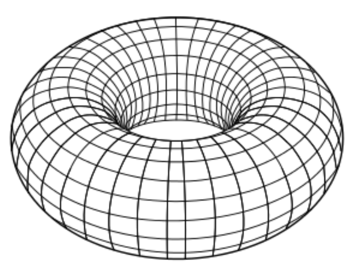
Definition. *A knot K is **prime** if, for any decomposition $K = K_1\#K_2$, either K_1 or K_2 is equivalent to the unknot.*

Theorem (Prime decomposition theorem)**.** *Every knot K can be decomposed as a connected sum $K = K_1\#\dots\#K_n$ of nontrivial, prime knots. Furthermore, this decomposition is unique up to reordering.*

One way to prove the theorem is with Seifert surfaces.[17]

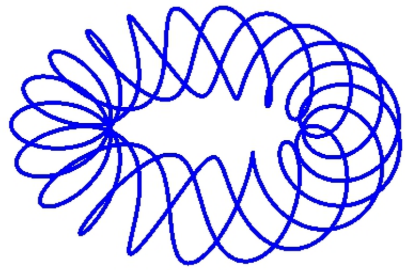
Torus knots

Let T be the standard 2-torus in S^3, obtained as a surface of revolution. T is unknotted, in the sense that we can view it as the boundary of a regular neighborhood $N = S^1 \times D^2$ of the unknot.



[17] Livingston is a good reference, which details surgery on Seifert surfaces.

Definition. *The (p, q) torus knot, where p, q are relatively prime, is obtained from the simple closed curve in T that winds p times around longitudinally and q times around meridionally.*



One can also view the torus as the quotient $\mathbb{R}^2/\mathbb{Z}^2$, in which case the torus knot is given as the image of the line in $\mathbb{R}^2$ with slope p/q after mapping the torus into S^3.

The exterior of $T_{p,q} = S^3 \setminus T_{p,q}$ consists of three pieces:

1. The solid torus inside of T
2. The solid torus outside of T
3. The annular region in T given by removing the simple closed curve of slope p/q. The annulus winds p times around T longitudinally and q times around T meridionally.

Definition. *The knot group is the invariant $G_k = \pi_1(S^3 \setminus K)$.*

In this case, the knot group admits a two-generator, one-relation presentation. We can appropriately consider neighborhoods of these pieces to apply Van Kampen's theorem. Each of the solid torii give one generator, and gluing along the annulus gives the relation. Thus

$$G_{T_{p,q}} = \langle x, y : x^p y^{-q} \rangle$$

as the annulus has core S^1 and is simultaneously equal to x^p and y^q.

Proposition. *The Alexander polynomial of $T_{p,q}$ is given by*

$$\Delta_{T_{p,q}}(t) = \frac{(1-t)(1-t^{pq})}{(1-t^p)(1-t^q)}$$

up to multiplication by $\pm t^k$.

This formula is easy to derive using *Fox calculus* from the presentation of $T_{p,q}$.

Proposition. *The Jones polynomial of $T_{p,q}$ is given by*

$$V_{T_{p,q}}(t) = \frac{t^{(p-1)(q-1)/2}(1 - t^{p+1} - t^{q+1} + t^{p+q})}{(1-t^2)}$$

This is a nontrivial computation to which we will return.

We can also describe $T_{p,q}$ by consider the cylinder $S^1 \times I$. Let there be p lines on the cylinder, evenly spaced. Then glue the ends of the cylinder along a twist of angle $2\pi q/p$.

Satellite knots

These different definitions of the torus knot are intended to emphasize certain properties they have. Torus knots are special cases of cabled knots, and cabled knots are special cases of satellite knots.

Definition. *The* ***satellite construction*** *is a procedure involving two knots, the* ***companion*** *C in S^3 and the* ***pattern*** *P in T^2.*

We will see that we need a framing on C. Usually, this is accomplished by considering C as a diagram with the blackboard framing. The satellite knot is obtained by removing from S^3 a regular neighborhood of C, and gluing the solid torus containing P in its place.

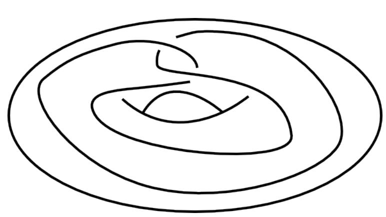
The framing is important, as it provides a trivialization of the normal bundle. This allows us to glue the solid torus into the regular neighborhood in an unambiguous way.[18]

So if we take C to be the unknot and P to the (p,q) curve, then the result is $T_{p,q}$.

Definition. *If P is the (p,q) curve, then this construction yields the (p,q) cabling of a knot K, denoted by $C(P)$.*

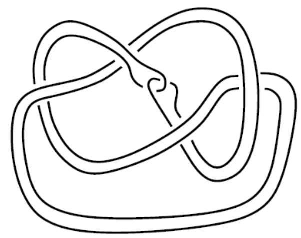
Generally, cabling makes knots much more intricate. It also preserves primeness, in the sense that if P is a pattern in the solid torus with wrapping number[19] greater than 1, any associated satellite knot is prime if and only if P is prime.[20] Cabling makes knots overtly more complicated, but the following procedure pushes complexity deeper into the knot.

Definition. *Let C a knot, and let P be the Whitehead double knot:*



[18]Intuitively, the framing tells us how much to 'twist' the solid torus when gluing it.
[19]The wrapping number of a knot is the minimum number of times it intersects a disc whose boundary is a meridian of the torus.
[20]Livingston [1981].

*The **Whitehead double**[21] of C is given by the resulting satellite knot.*



One indicator of satellite complexity is the number of times the pattern winds around the solid torus. For example, the (p, q) cabling pattern winds around the torus p times, and the Whitehead double knot does not wind around the torus at all.

Remark. *Connected sum is a special case of the satellite construction, given by embedding the second knot into the torus and letting one of the strands go around the longitude.*

Definition. *A knot is **simple** if it is not a satellite with nontrivial companion.*

For example, torus knots are simple, even though they are satellite knots (as they have trivial companion). We can now state a classification result in classical knot theory.

Theorem (Thurston)**.** *Every simple knot is either a torus knot or is hyperbolic.*[22]

Cabling will be important in our project, as it turns out the colored Jones polynomial admits a description based on computing the Kauffman bracket of a particular cabling of the original knot by *Jones-Wenzl idempotents.*

The volume conjecture

In 1995, Rinat Kashaev defined a link invariant using 'quantum dilogarithms.' It can be described using the following R-matrices.

Let

$$(x)_n = \prod_{i=1}^{n} (1 - x)$$

for $n \geq 0$. Define

$$\begin{aligned} \theta : \mathbb{Z} &\to \{0, 1\} \\ n &\mapsto \begin{cases} 1 & N > n \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

[21] Tanaka [2008] computes the colored Jones polynomial of the Whitehead double of a knot by adapiting Masbaum's argument.

[22] A knot is *hyperbolic* if its complement admits a hyperbolic structure with finite volume.

For integers x, let $\operatorname{res}(x)$ be its residue modulo N. Let $\zeta = e^{2\pi i/N}$. Let

$$(R_k)^{cd}_{ab} = \frac{N\zeta^{1+c-b+(a-d)(c-b)}\theta(\operatorname{res}(b-a-1)+\operatorname{res}(c-d))\cdot\theta(\operatorname{res}(a-c)+\operatorname{res}(d-b))}{(\zeta)_{\operatorname{res}(b-a-1)}(\zeta^{-1})_{\operatorname{res}(a-c)}(\zeta)_{\operatorname{res}(c-d)}(\zeta^{-1})_{\operatorname{res}(d-b)}}$$
$$(\mu_k)^i_j = -\zeta^{1/2}\delta_{i,j+1}$$

Lemma. $(R_k, \mu_k, -\zeta^{1/2}, 1)$ *is an enhanced Yang-Baxter operator, with associated knot invariant* $\langle K\rangle_N$*, called* ***Kashaev's invariant****.*

Kashaev computed the asymptotic limit of $\langle K\rangle_N$ as N goes to infinity. He showed that it increases exponentially and observed that the growth rate is proportional to the hyperbolic volume

$$\operatorname{Vol}(K) = \operatorname{Vol}(S^3 \setminus K)$$

when K is hyperbolic.

Conjecture. *If K is hyperbolic, then*

$$\lim_{N\to\infty} \frac{1}{N}\log(\langle K\rangle_N) = \frac{1}{2\pi}\, Vol(S^3 \setminus K)$$

In 2001, Murakami and Murakami proposed a generalization that extends to all knots and involves the colored Jones polynomial via the following observation: take $q = \zeta = e^{2\pi i/N}$. Then the colored Jones polynomial of K coincides with $\langle K\rangle_N$.

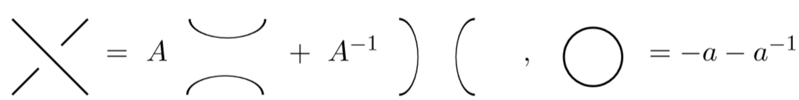
Tanaka also proved in 2008 that the volume conjecture implies that the colored Jones polynomial detects the unknot.

7/18/2019 - Skein Algebras and the Colored Jones Polynomial

Today we will introduce the skein-theoretic techniques Masbuam develops to compute colored Jones polynomials. First let

$$\begin{aligned} q &= a^2 = A^4 \\ \{n\} &= a^n - a^{-n} \\ \{n\}! &= \{n\}\{n-1\}\dots\{1\} \\ [n] &= \frac{\{n\}}{\{1\}} \\ [n]! &= [n][n-1]\dots[1] \\ \begin{bmatrix} n \\ i \end{bmatrix} &= \frac{[n]!}{[i]![n-i]!} \end{aligned}$$

Recall that the Kauffman relations



in the skein algebra $\mathcal{K}(M)$ imply the *framing relation*, which says

$$L^{\pm 1} = -A^{\pm 3} L$$

It is important to note that, as we consider framed links in manifolds besides S^3, the second Kauffman relation says that a *homotopically trivial* unknot is equal to $-a - a^{-1}$. The fact that there are unknotted links in other manifolds that are not homotopically trivial is part of what makes skein algebras more complex in the general case. However, the first homotopy and homology groups of a manifold do not completely determine the skein algebra.

Proposition. *We have $\mathcal{K}(S^3) \simeq \mathbb{Z}[A, A^{-1}]$, where the isomorphism is given by evaluating the Kauffman bracket.*[23]

The skein algebra of the solid torus

Consider the skein algebra of the solid torus $S^1 \times D^2$. We have

$$\mathcal{B} = \mathcal{K}(S^1 \times D^2) = \mathbb{Z}[A, A^{-1}][z]$$

where z is given by a single component circling the torus once. This is because we can always resolve crossings in a diagram until we obtain unknotted components, which are either homotopically trivial or wrap around the torus once.[24]

In this case, by viewing $S^1 \times D^2 \simeq S^1 \times I \times I$, we can define multiplication in the skein algebra by gluing two copies of the thickened annulus on top of each other. It is evident that this operation is commutative.

[23] This is nontrivial. See Kauffman [1986] and his state model for the Jones polynomial.
[24] For if there is a loop that wraps around the torus more than once, it contains a crossing yet to be resolved.

Definition. *Let $\mathcal{B}^{even}$ be the subalgebra generated by z^2.*

There is a distinguished basis of $\mathcal{B}$ defined recursively by

$$\begin{aligned} e_0 &= 1 \\ e_1 &= z \\ &\dots \\ e_i &= ze_{i-1} - e_{i-2} \end{aligned}$$

This basis will turn out to be the *Jones-Wenzl idempotents.*

Definition. *The* ***twist map*** *$t : \mathcal{B} \to \mathcal{B}$ is the module homomorphism induced by a right-handed Dehn twist on the solid torus and then extended linearly to all of $\mathcal{B}$.*

The twist map acts nontrivially on $\mathcal{B}$, and it has z as an eigenvector by the framing relation. In fact, we have a more general statement about eigenvectors of t.

Proposition. *The basis $\{e_0, e_1, \dots\}$ defined above for $\mathcal{B}$ consists of eigenvectors of t.*

So we can write

$$t(e_i) = \mu_i e_i$$

where

$$\mu_i = (-1)^i A^{i^2+2i}$$

Proof. The proof will proceed by induction. The base case is trivial, as $t(1) = 1$. We must understand how a Dehn twist acts on the product ze_{i-1}. The rest . □

Definition. *Define the bilinear map $\langle \cdot \rangle : \mathcal{B} \times \mathcal{B} \to \mathbb{Z}[A, A^{-1}]$ by cabling the 0-framed Hopf link with the two arguments, viewing the result as a link in S^3, and taking the Kauffman bracket. It is thus defined on pure links in the solid torus and extended linearly.*

Lemma. *We have*

$$\langle e_i, 1 \rangle = (-1)^i [i+1]$$

Proof. When $i = 0$, the claim is immediate. For the inductive step, observe

$$\begin{aligned} \langle e_i, 1 \rangle &= \langle ze_{i-1} - e_{i-2}, 1 \rangle \\ &= \langle ze_{i-1}, 1 \rangle - \langle e_{i-2}, 1 \rangle \\ &= (-a - a^{-1})(-1)^{i-1}[i] - (-1)^{i-2}[i-1] \\ &= (-1)^i \frac{1}{\{1\}} \big((a + a^{-1})(a^i - a^{-i}) - (a^{i-1} - a^{-(i-1)}) \big) \\ &= (-1)^i [i+1] \end{aligned}$$

since the addition of z yields a homotopically trivial unlink component in S^3. □

Lemma. *For $f(z) \in \mathcal{B} \simeq (\mathbb{Z}[A, A^{-1}])[z]$, we have*

$$\langle f(z), e_i \rangle = f(\lambda_i)\langle 1, e_i \rangle$$

where $\lambda_i = -A^{2(i+1)} - A^{-2(i+1)}$.

Proof. We first claim

$$\langle z^j, e_i \rangle = (-A^{2(i+1)} - A^{-2(i+1)})^j \langle 1, e_i \rangle$$

When $j = 0$ the claim is immediate. For the inductive step, note that $\langle z^j, e_i \rangle$ is the Kauffman bracket of j disjoint circles each linked once with e_i. [Incomplete]

Then we have

$$\begin{aligned} \langle f(z), e_i \rangle &= \langle \sum_{j=0}^{\deg f} f_j z^j, e_i \rangle = \sum_{j=0}^{\deg f} f_j \langle z^j, e_i \rangle \\ &= \sum_{j=0}^{\deg f} f_j \lambda_i^j \langle 1, e_i \rangle = f(\lambda_i)\langle 1, e_i \rangle \end{aligned}$$

□

Now define

$$R_n = \prod_{\ell=0}^{n-1} (z - \lambda_{2\ell})$$

By construction

$$\langle R_n, e_{2i} \rangle = 0$$

for all $i < n$. This is simply because

$$\langle R_n, e_{2i} \rangle = \left(\prod_{\ell=0}^{n-1} (\lambda_{2i} - \lambda_{2\ell}) \right) \langle 1, e_i \rangle$$

by the above lemma.

Lemma. *We have*

$$\langle R_n, z^{2k} \rangle = 0$$

for all $k < n$.

Proof. Consider the linear subspace

$$\mathcal{B}_{2n}^{\text{even}} \subset \mathcal{B}^{\text{even}} \simeq (\mathbb{Z}[A, A^{-1}])[z^2]$$

that consists of polynomials in z of degree less than $2n$. Certainly $z^{2k} \in \mathcal{B}_{2n}^{\text{even}}$. Furthermore, the elements $e_0, e_2, \ldots, e_{2i}$ for $i < n$ are linearly independent in $\mathcal{B}_{2n}^{\text{even}}$. The dimension of $\mathcal{B}_{2n}^{\text{even}}$ is n, so these elements are in fact a basis. Thus

$$\begin{aligned}\langle R_n, z^{2k}\rangle &= \langle R_n, \sum_{i=0}^{n-1} a_i e_{2i}\rangle \\ &= \sum_{i=0}^{n-1} a_i \langle R_n, e_{2i}\rangle \\ &= 0\end{aligned}$$

□

We are now equipped to tackle Masbaum's method.

The colored Jones polynomial via skein algebras

The idea will be to find a particular nice form for an element $\omega \in \mathcal{B}$ that satisfies

$$\langle \omega, x\rangle = \langle t(x), 1\rangle$$

for all $x \in \mathcal{B}^{\text{even}}$. Namely, the geometric operation of twisting on the underlying 3-manifold can be replicated algebraically by the addition of ω around x.

Theorem (Habiro [2000]). *ω exists, and it can be written*

$$\omega = \sum_{n=0}^{\infty} c_n R_n$$

where

$$c_n = (-1)^n \frac{a^{n(n+3)/2}}{\{n\}!}$$

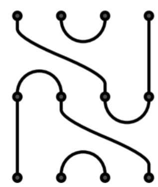
A large portion of the Masbum paper is devoted to determining an expression for ω^p, which corresponds to p full twists.

The element ω, in this particularly nice form, will allow us to use the cancellation properties of the R_n to simplify bracket computations.

The Temperley-Lieb algebra

Consider the square with n marked points on its top face and n marked poins on its bottom face. An (n, n) tangle diagram is formed by joining these endpoints and decorating any intersections with the appropriate over/under crossings.

Definition. *The* ***nth Temperley-Lieb algebra*** TL_n *is the* $\mathbb{Q}(A)$*-algebra*[25] *generated by* (n, n)*-tangle diagrams modulo the Reidemeister relations. The addition is formal, and the multiplication is given by concatanation of two diagrams.*



Note that some papers[26] use the ground ring $\mathbb{Z}[A, A^{-1}]$, while others[27] use $\mathbb{Z}(A, A^{-1})$.[28]

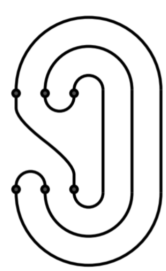
At this point, all diagrams will be assumed to have the blackboard framing, so an integer k decorating a link component will indicate k parallel strands.

We can reduce an (n, n)-tangle diagram to a combination of elements of the form

U_1 $\qquad$ U_{n-1}

So the elements $U_1, \ldots, U_{n-1}$ generate TL_n.

Definition. *Given an element* $\tau \in TL_n$*, the* ***closure*** *of* τ *is given by connecting the corresponding strands of the diagram for* τ *and viewing the result as a link in* S^3*.*

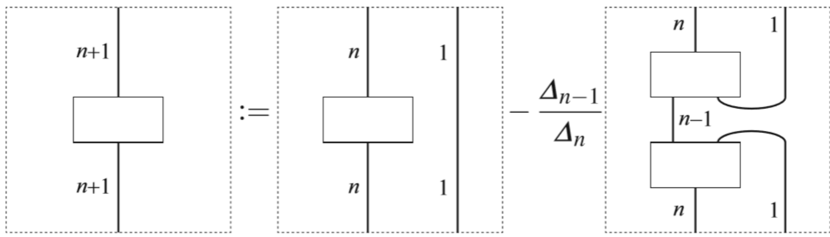


[25]The field $\mathbb{Q}(A)$ is taken from Masbaum and Vogel.
[26]Namely Masbaum, Tanaka, Przytyck
[27]Elhamadadi and Hajij
[28]Lickorish uses $\mathbb{C}$.

Definition. *The **Jones-Wenzl** idempotents are the distinguished elements of* TL_n *denoted by* $f^{(i)}$*, defined below.*

Definition. *Let* $\Delta_n \in \mathbb{Z}(A, A^{-1})$ *be the Kauffman bracket of the closure of* $f^{(i)}$*.*

Then we define $f^{(i)}$ recursively by



Proposition. *We have the following:*

1. $f^{(n)}U_i = U_i f^{(n)} (= 0)^{29}$
2. $(f^{(i)})^2 = f^{(i)}$
3. $\Delta_n = (-1)^n \frac{A^{2(n+1)} - A^{-2(n+1)}}{A^2 - A^{-2}}$

Definition. *The **trace map*** $g : TL_n \to \mathcal{B}$ *is given by closing the tangle and viewing the result as a link in* $S^1 \times D^2$*.*

Note that g does not respect the multiplication on TL_n. Furthermore, there may be minor problems with coefficient compatability in the ground ring, as TL_n is an $\mathbb{Q}(A)$-module and $\mathcal{B}$ is an $\mathbb{Z}[A, A^{-1}]$-module.

Remark. *We have*

$$g(f^{(i)}) = e_i$$

as they satisfy the same recursive relations.

Definition. *The unnormalized* N*-colored Jones polynomial of a framed link* L *is the Kauffman bracket of* L *cabled by* e_{N-1}*.*

$$J'_N(L) = (-1)^{N-1}\langle L(e_{N-1})\rangle$$

Then we normalize as follows for the N*-colored Jones polynomial.*

$$J_N(L) = \frac{J'_N(L)}{J'_N(\mathcal{O})}$$

where $\mathcal{O}$ *is the unknot.*

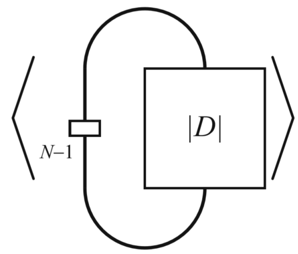
This cabling is prohibitively expensive to compute with the Kauffman relations. The approach will be to use a 'graphical calculus' to expand, replacing twists with ω appropriately and using the vanishing properties of the R_n terms to vastly simplify the result.

[29] In Lickorish [1992] we have $f^{(n)}U_i = U_i f^{(n)} = 0$ when $i \leq n$ and $f^{(n)}U_i = f^{(n)}U_i$ when $n + 2 \leq i$.

7/22/2019 - Colored Jones Polynomial in Skein Algebras, the Volume Conjecture

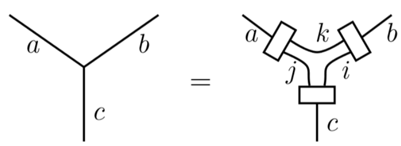
We will pick up where we left off last time in the computation of the colored Jones polynomial of the trefoil. We've seen that working in skein algebras is often difficult. However, Masbaum and Vogel developed a shorthand method for calculating $J_N(L)$.

Recall that $J_N(K)$ is given, up to normalization, by the bracket of L cabled with the $(N-1)$th Jones-Wenzl idempotent.



Masbaum and Vogel represent these idempotents using *trivalent graphs*. As before, all diagrams will have the blackboard framing, which is not necessarily the zero framing. An integer n decorating a diagram denotes n parallel strands.

Definition. *A triple* $(a, b, c) \in \mathbb{N}^3$ *is* ***admissible*** *if* $a + b + c \in 2\mathbb{N}$ *and* $|a - b| \le c \le a + b$. *Given an admissible triple, a* ***trivalent vertex*** *is a vertex*



where

$$\begin{cases} i = \frac{b+c-a}{2} \\ j = \frac{a+b-c}{2} \\ k = \frac{a+b-c}{2} \end{cases}$$

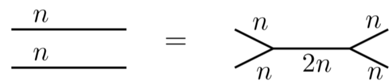
(i, j, k) *are the* ***internal colors*** *of the vertex.*

Note that (a, b, c) is admissible if and only if there exist a triple of internal colors. The process of converting diagrams into trivalent graphs and then performing operations on these graphs is known as *graphical calculus.*

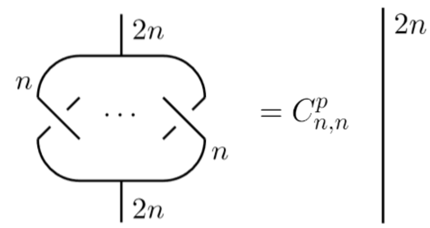
From now on, it will be assumed that a Jones-Wenzl idempotent is attached to every group of strands.[30] The following two theorems are nontrivial (see Masbaum [2003]) and give important equalities in the skein algebra.

[30] It does not matter how many of these elements are placed on a family of strands, as they are idempotent in TL_n, in which multiplication is given by concatenation.

Theorem. *We have*[31]



Theorem. *We have*



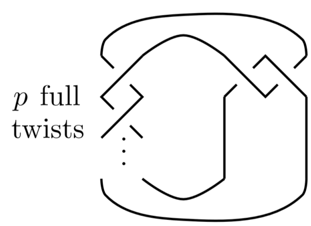
where there are p twists in the center region of the left diagram and $C^p_{n,n}$ is some coefficient.

In particular, when $p = 1$ we have

$$C^1_{n,n} = A^{-n(n+1)}\{n\}!$$

The colored Jones polynomial of the trefoil

Masbaum computes $J_N(K_p)$ for the *twist knots*



We will just consider the case when $p = 1$. Let $K = K_1 = T(3, 2)$. The writhe of this diagram is $\omega = -2$. To make the blackboard framing into the zero framing, add two positive Reidemeister I moves.[32]

We would like to evaluate the unnormalized

$$J'_N(K) = (-1)^{N-1}\langle K(e_{N-1})\rangle$$

where $K(e_{N-1})$ is the knot K cabled with e_{N-1}.

Lemma. *We can write*

$$e_{N-1} = \sum_{n=0}^{N-1} (-1)^{N-n-1} \begin{bmatrix} N+n \\ N-1-n \end{bmatrix} R_n$$

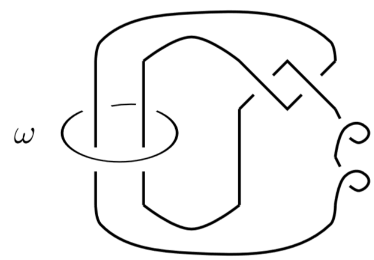
[31] It seems this needs justification, as Masbuam has a slightly different result.
[32] Note that ommitting this step would yield the correct Jones polynomials up to the framing relation.

Proof. The proof proceeds by induction. [Incomplete] □

This is a $\mathbb{Z}[A, A^{-1}]$-linear combination of elements

$$R_n = \prod_{i=0}^{n-1} (z - \lambda_{2i})$$

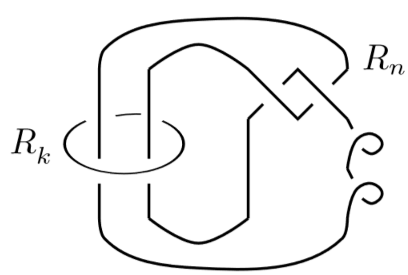
In the skein algebra, we have that the trefoil is equal to



So evaluating $\langle K(e_{N-1}) \rangle$ amounts to evaluating $\langle \omega, \widetilde{K}(e_{N-1}) \rangle$, where $\widetilde{K}$ is the knot in $S^1 \times D^2$ obtained by removing the twist. Furthermore, we can break up

$$\begin{aligned} \langle K(e_{N-1}) \rangle &= \langle \omega, \widetilde{K}(e_{N-1}) \rangle \\ &= \left\langle \sum_{n=0}^{\infty} c_n R_n, \widetilde{K}\left(\sum_{n=0}^{N-1} (-1)^{N-n-1} \begin{bmatrix} N+n \\ N-1-n \end{bmatrix} R_n \right) \right\rangle \\ &= \sum_{n=0}^{N-1} (-1)^{N-n-1} \begin{bmatrix} N+n \\ N-1-n \end{bmatrix} \sum_{k=0}^{\infty} c_k \langle R_k, \widetilde{K}(R_n) \rangle \end{aligned}$$

by linearity into summands of the form



Consider the expression $\langle R_k, \widetilde{K}(R_n) \rangle$. If we fill in the component R_k with a solid disk, $\widetilde{K}(R_n)$ intersects this disk evenly many times.[33]

We can simplify $\widetilde{K}(R_n)$ in the expression $\langle R_k, \widetilde{K}(R_n) \rangle$ first. Every resolution of crossings for $\widetilde{K}(R_n)$ is a polynomial in even powers of z. This is because we can first resolve crossings on the right side of R_n. Then the resolved diagram still intersects the disk spanning R_k evenly many times, as we have not changed the diagram in this region. Finally, the modulo 2 intersection number of

[33] As even though $\widetilde{K}(R_n)$ is cabled, two copies of each strand pass through the disk.

the resolved state and the spanning disk is well-defined, so any isotopy preserves the fact that the resolution intersects an even number of times.

Hence $\widetilde{K}(R_n) \in \mathcal{B}^{\text{even}}$. Next, recall $\langle R_i, z^{2j} \rangle = 0$ when $j < i$.

R_n has z-degree n by definition. Furthermore, $\widetilde{K}(R_n)$ has z-degree at most $2n$, since it is cabled twice through the disk spanning R_k. Thus when $k > n$, we can write R_n as a linear combination of terms z^{2i} where $i < k$. Therefore $\langle R_k, \widetilde{K}(R_n) \rangle = 0$.

It is also true that when $n > k$ we have $\langle R_k, \widetilde{K}(R_n) \rangle = 0$ as well. This can be demonstrated in at least two ways:

1. It possible to isotopy the two-component link so that R_k and R_n switch roles in the diagram. Then the same argument implies $\langle R_n, R_k \rangle = 0$ when $n > k$.
2. Consider a Seifert surface for $\widetilde{K}(R_n)$ and observe that R_k pierces this surface twice. Let this surface be a cross-section of some torus. Then the same argument implies $\langle R_k, \widetilde{K}(R_n) \rangle = 0$ when $n > k$.

Then the sum becomes finite for

$$\sum_{n=0}^{N-1} (-1)^{N-n-1} \begin{bmatrix} N+n \\ N-1-n \end{bmatrix} c_n \langle R_n, \widetilde{K}(R_n) \rangle$$

Note that we have

$$\langle R_n, \widetilde{K}(R_n) \rangle = \langle R_n, \widetilde{K}(R_n) \rangle + \underbrace{\langle R_n, \widetilde{K}(e_n - R_n) \rangle}_{0} = \langle R_n, \widetilde{K}(e_n) \rangle$$

since $e_n - R_n$ has z-degree less than n by definition, and thus $\widetilde{K}(e_n - R_n)$ can be written as a linear combination of terms z^{2i} where $i < n$.

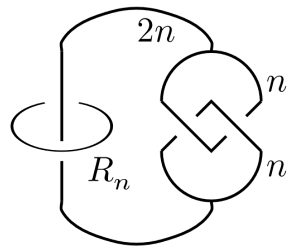
Therefore the sum is

$$\sum_{n=0}^{N-1} (-1)^{N-n-1} \begin{bmatrix} N+n \\ N-1-n \end{bmatrix} c_n \langle R_n, \widetilde{K}(e_n) \rangle$$

To resolve the framing, observe that we can recast the above link in the torus, where the kinked regions are now separated by themselves. Then we have

$$\langle R_n, \widetilde{K}(e_n) \rangle = \langle R_n, t^2(\widetilde{K}'(e_n)) \rangle = \mu_n^2 \langle R_n, \widetilde{K}'(e_n) \rangle$$

where K' is the unkinked trefoil in this form. Now each adjusted summand $\langle R_n, \widetilde{K}'(e_n)\rangle$ is equal to



where we are combining parallel strands using the first theorem above. The second theorem implies this is equal to $C^1_{n,n}\langle R_n, e_{2n}\rangle$. Putting everything together yields

$$\langle R_n, \widetilde{K}(R_n)\rangle = \mu_n^2 C^1_{n,n}\langle R_n, e_{2n}\rangle$$

Recall $\langle f(z), e_i\rangle = f(\lambda_i)\langle 1, e_i\rangle$. Then[34]

$$\langle R_n, e_{2n}\rangle = \left(\prod_{i=0}^{n-1}(\lambda_{2n} - \lambda_{2i})\right)\langle 1, e_{2n}\rangle = (-1)^n \frac{\{2n+1\}!}{\{1\}}$$

Thus the unnormalized colored Jones polynomial is given by

$$\begin{aligned}
J'_N(K) &= (-1)^{N-1}\sum_{n=0}^{N-1}(-1)^{N-n-1}\begin{bmatrix} N+n \\ N-1-n\end{bmatrix} c_n\langle R_n, \widetilde{K}(R_n)\rangle \\
&= (-1)^{N-1}\sum_{n=0}^{N-1}(-1)^{N-n-1}\begin{bmatrix} N+n \\ N-1-n\end{bmatrix} c_n\mu_n^2 C^1_{n,n}\langle R_n, e_{2n}\rangle \\
&= (-1)^{N-1}\sum_{n=0}^{N-1}(-1)^{N-n-1}\begin{bmatrix} N+n \\ N-1-n\end{bmatrix} c_n A^{(n^2+2n)^2} A^{-n(n+1)}\{n\}!\frac{\{2n+1\}!}{\{1\}}
\end{aligned}$$

[34] This equality requires justification.

The volume conjecture

Definition. *A knot K in S^3 is **hyperbolic** if the complement $M = S^3 \setminus K$ admits a complete metric of constant curvature -1 with finite volume.*

Examples

- The trefoil is not hyperbolic.
- The figure-eight knot is hyperbolic.

We also had the following theorem.

Theorem (Thurston)**.** *Every knot in S^3 is a torus knot, hyperbolic knot, or satellite knot.*

Conjecture. *The hyperbolic volume of a knot is related to Kashaev's dilogarithm invariant by*

$$\lim_{N\to\infty} \frac{1}{N} \log |\langle K \rangle_N| = \frac{1}{2\pi} Vol(S^3 \setminus K)$$

After the conjecture was proposed, Kashaev and Tirkkonen proved that for torus knots K

$$\lim_{N\to\infty} \frac{1}{N} \log |\langle K \rangle_N| = 0$$

Theorem (Murakami, Murakami)**.** *Kashaev's invariant $\langle K \rangle_N$ coincides with the colored Jones polynomial $J_N(K, q)$ by specializing with a root of unity $q = e^{2\pi i/N}$.*

They generalized Kashaev's conjecture to all knots using *simplicial volume.*

Simplicial volume and the generalized volume conjecture

Definition. *A surface F in a 3-manifold M is **incompressible** if the inclusion $i : F \hookrightarrow M$ induces an injective map $\pi_1 i : \pi_1 F \hookrightarrow \pi_1 M$.*

Intuitively, $F \subset M$ is incompressible if there are no nontrivial curves in F which bound a disk in M.

Definition. *Two surfaces F', F'' are **parallel** in M if they cobound a thickened surface $F \times I$ in M, namely $F' = F \times \{0\}$ and $F'' = F \times \{1\}$. If M is a 3-manifold with boundary, then F is **boundary parallel** if it is parallel to a connected component of ∂M.*

Two surfaces are parallel if and only if they are isotopic in M.

The next result is the Jaco-Shalen-Johannson decomposition, which holds more generally for Haken 3-manifolds when either $\partial M = \varnothing$ or when ∂M is a union of torii.

Theorem (JSJ decomposition). *Let K be a knot in S^3. Then there exists a maximal set of incompressible torii in $M = S^3 \setminus K$ such that*

1. *No torus is boundary parallel*
2. *No two torii are parallel*

Definition. *A 3-manifold M is **Seifert-fibered** if it can be realized as a S^1-bundle over a surface with finitely many singular fibers.*

Theorem. *Let $\mathcal{T} = \{T_i : T_i \subset M\}$ be a maximal set of torii. Denote the result of cutting M along all of the torii in $\mathcal{F}$ by $M|\mathcal{T}$. The connected components of $M|\mathcal{F}$ are either hyperbolic or Seifert-fibered.*[35]

Definition. *The **simplicial volume** of $S^3 \backslash K$ is the sum of the hyperbolic volumes of the hyperbolic pieces of $M|\mathcal{T}$.*

The torus knots have complements that are Seifert-fibered, so they have no hyperbolic volume, as expected. This definition also allows us to extend the conjecture to satellite knots, whose complements consists of pieces that are both hyperbolic and Seifert-fibered.

Remark. *Simplicial volume coincides with Gromov norm up to some overall factor.*

Then we can now state the generalized volume conjecture.

Conjecture. *Let K be any knot. Then*

$$\lim_{N\to\infty} \frac{1}{N} \log \left| J_N(K, q = e^{2\pi i/N}) \right| = \frac{1}{2\pi} Vol(S^3 \setminus K)$$

The conjecture has been verified for very low crossing knots:

1. 3_1 was verified by Kashaev, Tirkkonen
2. 4_1 was verified by Ekholm
3. 5_1 was verified by Kashaev, Tirkkonen
4. 5_2 was verified by Kashaev, Yokota
5. $6_1, 6_2, 6_3$ were verified by Ohtsuki, Yokota
6. Whitehead doubles of torus knots were verified by Zheng

Verification of the volume conjecture for the figure-eight knot

Let K be the figure-eight knot. Recall that

$$J_N(K, q) = \frac{1}{\{N\}} \sum_{j=0}^{N-1} \frac{\{N+j\}!}{\{N-1-j\}!}$$

[35] This is a highly nontrivial result, relying upon the geometrization theorem and particular properties of knot complements.

where

$$\{m\} = q^{m/2} - q^{-m/2}$$

We can cancel terms and simplify for

$$\sum_{j=0}^{N-1} \{N+j\}\{N+j-1\}\dots\{N+1\}\{N-1\}\dots\{N-j\}$$

$$= \sum_{j=0}^{N-1} \prod_{k=1}^{j} \{N+k\}\{N-k\}$$

$$= \sum_{j=0}^{N-1} \prod_{k=1}^{j} (q^{(N+k)/2} - q^{-(N+k)/2})(q^{(N-k)/2} - q^{-(N-k)/2}$$

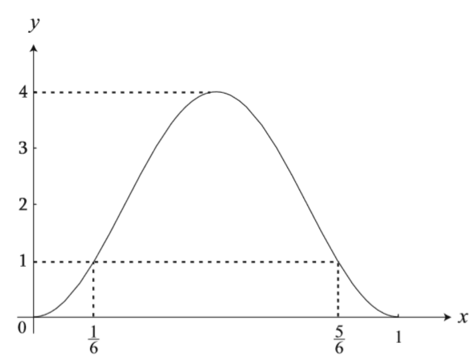
Taking $q = e^{2\pi i/N}$ and using the fact that cos is even yields

$$\sum_{j=0}^{N-1} \prod_{k=1}^{j} 2i \sin\big(\frac{2\pi(N+k)}{N}\big) \cdot 2i \sin\big(\frac{2\pi(N-k)}{N}\big) = \sum_{j=0}^{N-1} \prod_{k=1}^{j} -(-1)4\sin^2\big(\frac{k\pi}{N}\big)$$

since sin is periodic and odd. Define

$$g_N(j) = \prod_{k=1}^{j} 4\sin^2\big(\frac{k\pi}{N}\big)$$

Consider the graph of $y = 4\sin^2(\pi x)$.



As a function of j, $g_N(j)$ is is

1. decreasing when $0 < j < N/6$
2. increasing when $N/6 < j < 5N/6$
3. decreasing when $5N/6 < j < N$

Thus $g_N(j)$ obtains its maximal value when $j = \lfloor 5N/6 \rfloor$. Furthermore, since $g_N(j) > 0$ we have

$$g_N(\lfloor 5N/6 \rfloor) < \sum_{j=0}^{N-1} g_N(j) < N g_N(\lfloor 5N/6 \rfloor)$$

Taking log and dividing by N yields

$$\frac{1}{N}\log g_N(\lfloor 5N/6\rfloor) < \frac{1}{N}\log\left(\sum_{j=0}^{N-1} g_N(j)\right) < \frac{1}{N}\log(N g_N(\lfloor 5N/6\rfloor)) = \frac{1}{N}\log N + \frac{1}{N}\log g_N(\lfloor 5N/6\rfloor)$$

Note that as N becomes large, the left and right sides converge, and hence are equal in the limit to the center expression by the squeeze theorem. Taking the limit as N goes to infinity yields

$$\begin{aligned}
\lim_{N\to\infty}\frac{1}{N}\log|J_N(K, q = 2\pi i/N)| &= \lim_{N\to\infty}\frac{1}{N}\left(\sum_{j=0}^{N-1} g_N(j)\right) \\
&= \lim_{N\to\infty}\frac{1}{N}\log g_N(\lfloor 5N/6\rfloor) \\
&= \lim_{N\to\infty}\frac{1}{N}\sum_{k=1}^{\lfloor 5N/6\rfloor} 2\log\left(2\sin\frac{k\pi}{N}\right) \\
&= \frac{2}{\pi}\int_0^{5\pi/6}\log(2\sin x)\,dx \\
&= -\frac{2}{\pi}\Lambda(5\pi/6)
\end{aligned}$$

where Λ is the *Lobachevsky function* given by

$$\Lambda(t) = -\int_0^t \log(2\sin x)\,dx$$

$\Lambda(t)$ gives the volume of an ideal terahedron[36] in terms of its dihedral angles α, β, γ.

$$\mathrm{Vol}(T) = \Lambda(\alpha) + \Lambda(\beta) + \Lambda(\gamma)$$

Lemma. *We have the following.*

1. $\Lambda(\zeta + \pi) = \Lambda(\zeta)$
2. $\Lambda(-\zeta) = -\Lambda(\zeta)$
3. $\Lambda(2\zeta) = 2\Lambda(\zeta) + 2\Lambda(\zeta + \pi/2)$

We can use these identities for

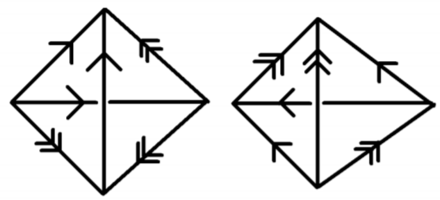
$$\begin{aligned}
\Lambda(5\pi/6) = \Lambda(\pi - \pi/6) = \Lambda(-\pi/6) &= -\Lambda(\pi/6) \\
= -\frac{1}{2}\Lambda(\pi/3) + \Lambda(2\pi/3) &= -\frac{3}{2}\Lambda(\pi/3)
\end{aligned}$$

Substituting this in the above equation yields

$$-\frac{2}{\pi}\Lambda(5\pi/6) = \frac{3}{\pi}\Lambda(\pi/3) = \frac{1}{2\pi}6\Lambda(\pi/3) = \frac{1}{2\pi}\mathrm{Vol}(S^3 \setminus K)$$

[36] For example, see Milnor [1982].

where the last equality utilizes the fact that $S^3 \setminus K$ can be obtained by gluing two ideal regular tetrahedra together.



Midterm Presentation Slides

The midterm presentation slides are available on the next page.

Quantum Invariants and Modularity

Fields Institute, Summer 2019

Colin Bijaoui, Beckham Myers, Aaron Tronsgard, Shaoyang Zhou
Professors Hans Boden, Robert Osburn, Will Rushworth

July 25, 2019

Knots and Invariants

A mathematical knot can be imagined as a knot in a piece of string, where the ends of the string have been fused together.

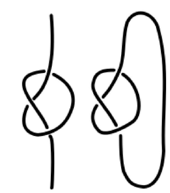


Figure: Knot in string (left) with ends fused together (right).

Two knots are *equivalent* if there is a manipulation of one into the other without cutting the string or passing the string through itself.

Knots and Invariants

A central question in knot theory is the *classification problem*.

Knot invariants are mathematical objects associated to a knot such that two equivalent knots are associated to the same object.

The colored Jones polynomial of a knot K is a particular family of (Laurent) polynomials $\{J_N(K)\}_{N\in\mathbb{N}}$ associated to every knot.

Given a knot K, we can construct two infinite power series that agree with the Nth colored Jones polynomial at Nth roots of unity. The F-series is of *noncyclotomic* form, and the U-series is of cyclotomic form.

The Colored Jones Polynomial and Number Theory

Theorem (Ahlgren-Kim, Straub, 2015)

Let

$$F(q) := \sum_{n\geq 0}(1-q)(1-q^2)\dots(1-q^n)$$

and consider

$$F(1-q) = \sum_{n\geq 0}^{\infty}\xi(n)q^n$$

Let p be a prime and $r\in\mathbb{N}$. If $i\in T^*(p^r)$ then

$$\xi(np^r+i)\equiv 0 \bmod p^r$$

for all $n\in\mathbb{N}$.

The Colored Jones Polynomial and Number Theory

Theorem (Bryson, Ono, Pitman, Rhoades, 2012)

Let F be defined as in the previous slide, and let

$$U(x,q) := \sum_{n\geq 0} (-xq;q)_n(-x^{-1}q)_n q^{n+1}$$

where

$$(a;q)_n = \prod_{i=0}^{n-1}(1-aq^i)$$

Then

$$F(\zeta_N^{-1}) = U(-1,\zeta_N)$$

for all roots of unity ζ_N.

Project Outline

Given a knot K, we can consider the schematic

$$\begin{tikzcd}
 & \underbrace{J_N(K^*,q)}_{\text{cyclotomic}} \arrow[r, rightsquigarrow, "x=-1", "q=\zeta_N"'] & U(x,q) \arrow[r, rightsquigarrow] & \text{`mock modular form'} \arrow[d, dashed, "q\mapsto q^{-1}"] \\
K \arrow[ur] \arrow[dr] & & & \\
 & \underbrace{J_N(K,q)}_{\text{non-cyclotomic}} \arrow[r, rightsquigarrow, "q=\zeta_N"] & F(q) \arrow[r, rightsquigarrow] & \text{`quantum modular form'}
\end{tikzcd}$$

Project Goal

Current progress on the problem has included

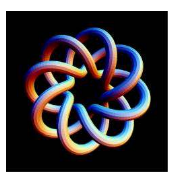
- The case of the *trefoil knot*
- The case of the infinite family of *twist knots*

Project Goal

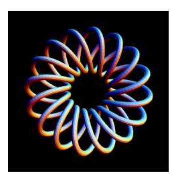
Explicitly work out the above schematic and the relevant theorems for the family of torus knots $T(3,2^t)$, where $t \geq 2$.

$T(3,2)$

$T(3,4)$



$T(3,8)$



$T(3,16)$

Generalized $\mathcal{F}_t$-series - Evidence for Congruences

We have discovered numerical evidence for prime power congruence relations in the coefficients of the series $\mathcal{F}_t(1-q)$, where $\mathcal{F}_t$ is obtained from the knot $T(3,2^t)$. Computations with Mathematica suggest the following hold:

$$\xi_2(5n+4) \equiv 0 \bmod 5$$
$$\xi_2(7n+4) \equiv \xi_2(7n+5) \equiv \xi_2(7n+6) \equiv 0 \bmod 7$$
$$\xi_2(11n+8) \equiv \xi_2(11n+9) \equiv \xi_2(11n+10) \equiv 0 \bmod 11$$
$$\cdots$$

for the knot $T(3,4)$ and

$$\xi_3(5n+4) \equiv 0 \bmod 5$$
$$\xi_3(7n+5) \equiv \xi_3(7n+6) \equiv 0 \bmod 7$$
$$\cdots$$

for the knot $T(3,8)$.

Proof Outline

We are looking to prove that

- The series $\mathcal{F}_t(q)$ satisfies the *strange identity*

$$\mathcal{F}_t(q) \underbrace{=}_{\text{strange!}} \frac{1}{2}\sum_{n\geq 0} n\chi_{3\cdot 2^{t+1}}(n) q^{(n^2-(3-2^{t+1})^2)/(3\cdot 2^{t+2})}$$

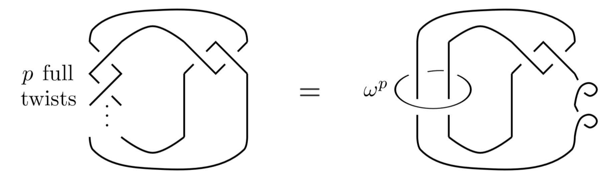
- The strange identity implies the desired prime power congruences.

To do this, we are working to generalize methods that were used in the case of the trefoil.

$\mathcal{U}_t$-series - Computation of the Colored Jones Polynomial

We also must compute the $\mathcal{U}_t$-series for the knots $T(3, 2^t)$.

- The idea will be to compute a certain expansion of the colored Jones polynomial.
- We will use elements of the *skein algebra* to resolve twist regions in the knot, replacing them with an element ω which is computationally simpler.



Conclusion

If we are successful, we will have established a number-theoretic duality between the generalized $\mathcal{F}_t$ and $\mathcal{U}_t$ series via a knot theoretic computation.

- The last step, which may be beyond the scope of this summer project is to establish some interesting properties of modularity on $\mathcal{F}_t$ and $\mathcal{U}_t$.
- If we follow the pattern of previous examples, this provides a duality between what are called mock modular forms and quantum modular forms

Conclusion

Knot theory provides a deeper, more general perspective with which to understand duality and congruence results that appear in number theory.

7/30/2019 - Progress Update and Jones-Wenzl Idempotents

We gave the midterm presentation to Robert, Hans, and Will. Afterwards, Beckham presented a computation of the colored Jones polynomial for the $T(3,4)$ knot using graphical calculus.

Prime power congruences

Aaron will present a proof of the prime power congruences for the. We can write

$$\mathcal{F}(q) = \sum_{i=0}^{s-1} q^i A_s(N, i, q^s)$$

The Ahlgren, Kim, Lovejoy divisibility result implies for $i \notin S(s)$

$$(q)_{\lambda(N,s)} | A_s(N, i, q)$$

We we only need the weaker result

$$(1-q^n)^r | A_s(N, i, q)$$

We truncate the series for

$$\mathcal{F}(1-q) = \sum_{i \in S(p^r)} (1-q)^i A_{p^r}(N, i, (1-q)^{p^r}) + \sum_{i \notin S(p^r)} (1-q)^i A_{p^r}(N, i, (1-q)^{p^r})$$

The is equal to

$$(1-(1-q)^{p^r})^n \sum_{i \notin S(p^r)} f_i(q)$$

for all n such that $\lambda(N, p^r) \geq np^r$. Howevever, we need that

$$(1-(1-q)^{p^r})^n = O(q^{np^r-(r-1)(p^r-1)}) \text{ mod } p^r$$

This is because $(1-q)^{p^r} = 1 + qg(q) - q^{p^r}$, and hence

$$(1-(1-q)^{p^r})^n = (qg(q) - q^{p^r})^n = \sum_{i=0}^{n} \binom{n}{i} (qg(q))^i (-q^{p^r})^{n-i} \equiv \sum_{i=0}^{r-1} \binom{n}{i} (qg(q))^i (-q^{p^r})^{n-i} \text{ mod } p^r$$

Thus

$$\mathcal{F}(1-q, N) = \sum_{i \in S(p^r)} (1-q)^i A_{p^r}(N, i, (1-q)^{p^r}) + O(q^{np^r-(r-1)(p^r-1)})$$

which is a linear combination of terms of the form $(1-q)^{i+kp^r}$, for $i \in S(p^r), k \in \mathbb{N}$.

Theorem. *For $m \in T(p^r)$, we have*

$$\binom{i+kp^r}{m+\alpha p^r} \equiv 0 \text{ mod } p^r$$

Lemma. *Let $n \in S(p)$ and $r \geq 2$.*

1. *If $n \not\equiv -24^{-1}$ mod p, then $n + \ell p \in S(p^r)$ for $0 \leq \ell \leq p^{r-1}$.*

2. If $n \equiv -24^{-1}$ mod p, then $n + p^r - p \notin S(p^r)$.

To show the first part of the lemma, it is necessary to invoke Hensel's lemma:

Lemma (Hensel's Lemma)**.** *If $f(r) \equiv 0$ mod p and $f'(r) \not\equiv 0$ mod p, there exists $s \equiv r$ mod p such that $f(s) \equiv 0$ mod p^n for all $n \geq 1$.*

We can write $n \equiv (3m^2 - m)/2 \bmod p$ by definition of the pentagonal numbers. If we let $f(x) = (3x^2 - x)/2 - p\ell$, we have $f(m) \equiv 0 \bmod p$. $f'(n) \not\equiv 0$ because $n \equiv -24^{-1} \bmod p$.

To prove the second part of the lemma, suppose $n + p^r - p \in S(p^r)$. Then

$$n + p^r - p \equiv \frac{m^2 - 1}{24} \bmod p$$

for some m coprime to 6. [The rest of the argument is missing.]

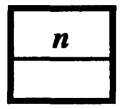
Let n_0 be the largest element in $S(p)$ not congruent to $-24^{-1} \bmod p$. Then $n_0 + p^r - p \in S(p^r)$ is the largest element in $S(p^r)$. We are looking at $\binom{i+kp^r}{m+\alpha p^r}$ for $m > n_0 + p^r - p \geq i$. Thus we can write $m = m_0 + p^r - p$ for $m_0 > n_0$.

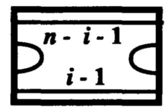
$$m = m_0 + (p-1)p + (p-1)p^2 + \ldots + (p-1)p^{r-1}$$

All the digits in the base-p expansion of m are maximal, where m_0 is strictly larger than the first digit in the expansion of i. We require the case when $r = 1$. $\binom{i+kp}{m\alpha p} \equiv 0 \bmod p$ and Kummer's theorem imply that since $\binom{n}{m}$ is the number of carries in the base p addition of $n - m$ and m, there will be always be carry in every digit.

Jones-Wenzl idempotents

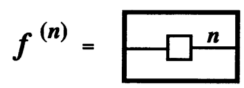
We will work in the (Kauffman) skein module of the n-stranded disk and the annulus $A = S^1 \times I \times I$, with the usual diagrammatic crossing relations. Recall that the n-stranded skein module of the disk is in fact the Temperley-Lieb algebra TL_n, with multiplication given by the obvious stacking operation. It is generated by the identity 1 and the elements $U_1, \ldots, U_{n-1}$.



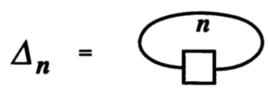


We know $\mathcal{K}(A) \simeq \mathbb{Z}[A, A^{-1}][z]$. The goal today will be to understand the Jones-Wenzl idempotents and their properties.

Definition. *The **Jones-Wenzl** idempotents are the elements*



We also define Δ_n as the trace of $f^{(n)}$.



Lemma. *If A^4 is not a kth root of unity for all $1 \leq k \leq n$, then there exists a unique element $f^{(n)} \in TL_n$ such that*

1. $f^{(n)} U_k = 0 = U_k f^{(n)}$ *for* $1 \leq k \leq n-1$
2. $1 - f^{(n)}$ *lies in the algebra generated by* $U_1, \ldots, U_{n-1}$
3. $f^{(n)} f^{(n)} = f^{(n)}$
4. $\Delta_n = (-1)^n \frac{A^{2n+2} - A^{-(2n+2)}}{A^2 - A^{-2}}$

Proof. We first prove uniqueness using properties 1 and 2. Let $g^{(n)}$ be another such element that satisfies the above properties. Then $1 - g^{(n)}$ lies in the algebra generated by $U_1, \ldots, U_{n-1}$, in which case

$$f^{(n)}(1 - g^{(n)}) = 0 = g^{(n)}(1 - f^{(n)})$$

So

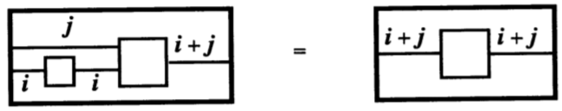
$$1 - f^{(n)} = (1 - f^{(n)})(1 - g^{(n)}) = 1 - g^{(n)}$$
$$f^{(n)} = g^{(n)}$$

Next, observe that property 3, idempotency, follows from properties 1 and 2. This is because

$$0 = f^{(n)}(1 - f^{(n)}) = f^{(n)} - f^{(n)} f^{(n)}$$

Hence we must construct an element that satisfies properties 1 and 2.

Note that we have an absorption result



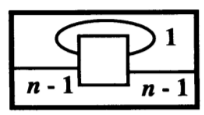
$$\widehat{f^{(i)}}f^{(i+j)} = f^{(i+j)}$$

where $f^{(i)}$ is stabilized appropriately. This is because

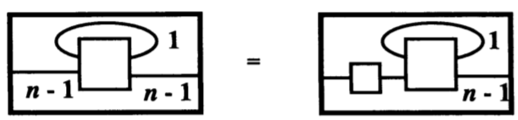
$$(f^{(i)} - 1)f^{(i+j)} = 0$$

as $f^{(i)} - 1$ lies in the subalgebra generated by $U_1, \ldots, U_{i-1}$ and is hence annihilated by $f^{(i+j)}$.

Now consider the element $x \in TL_{n-1}$ given by



Then $f^{(n-1)}x = x$ by the absorption result.



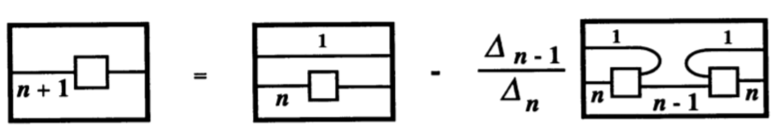
However, since x is a linear combination of 1 and products of $U_1, \ldots, U_{n-2}$

$$f^{(n-1)}x = \lambda f^{(n-1)}$$

for some scalar λ. Therefore $x = \lambda f^{(n-1)}$. We can solve for λ by taking the trace of both sides of this equation. The closure of x is Δ_n, and the closure of $\lambda f^{(n-1)}$ is $\lambda\Delta_{n-1}$. Thus

$$\lambda = \frac{\Delta_n}{\Delta_{n-1}}$$

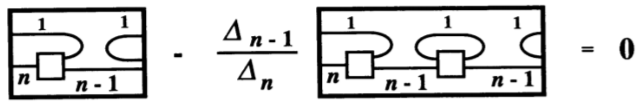
We can now proceed to define $f^{(n)}$. Set $f^{(0)}$ to be the empty diagram and $f^{(1)}$ to be the unknotted one-stranded diagram. Recursively define



$$f^{(n+1)} = f^{(n)} - \underbrace{\frac{\Delta_{n-1}}{\Delta_n}}_{\lambda^{-1}} f^{(n-1)}$$

Note that the leading term is always $n + 1$ parallel strands with coefficient 1. We will verify that this definition indeed satisfies the desired properties.

1. We know $f^{(n)}U_k = 0 = U_k f^{(n)}$ immediately for all $1 \leq k \leq n-1$. Composition with U_n yields



The right rectangle has center component x, which we can replace with $\lambda f^{(n-1)}$ and consequently absorb.

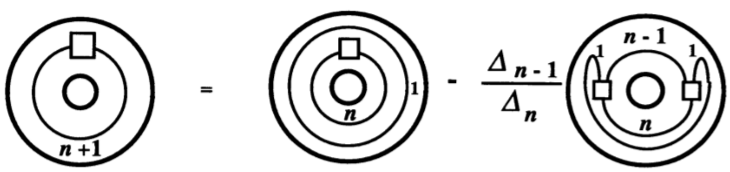
2. $1 - f^{(n)}$ is in the subalgebra generated by $U_1, \ldots, U_{n-1}$, as the identity component in the recursion expansion necessarily has coefficient 1 by construction.

3. Idempotence follows from the previous two properties, as remarked above.

4. We must show

$$\Delta_n = (-1)^n \frac{A^{2n+2} - A^{-(2n+2)}}{A^2 - A^{-2}}$$

We have the map $TL_n \to \mathcal{K}(A)$ given by closing an element around the annulus. Then $f^{(n)}$ is sent to polynomial in z, denoted by $S_n(z)$. Note $S_0(z) = 1$ and $S_1(z) = z$. The recursive formula implies



$$S_{n+1}(z) = zS_n(z) - S_{n-1}(z)$$

as the last diagram is in fact the closure of x using idempotence, which is equal to the closure of $\lambda f^{(n-1)}$. These are the *Chebyshev polynomials of the 2nd kind.*

We extend the map $TL_n \to \mathcal{K}(A) \to \mathcal{K}(S^3)$ by setting $z = -A^2 - A^{-2}$. Thus if Δ_n is the closure of $f^{(n)}$, Δ_n satisfies the recursion

$$\Delta_{n+1} = (-A^2 - A^{-2})\Delta_n - \Delta_{n-1}$$

and hence by induction

$$\begin{aligned}\Delta_{n+1} &= (-A^2 - A^{-2})(-1)^n \frac{A^{2n+2} - A^{-(2n+2)}}{A^2 - A^{-2}} - (-1)^{n-1}\frac{A^{2n} - A^{-2n}}{A^2 - A^{-2}} \\ &= (-1)^{n+1}\frac{A^{2(n+1)} - A^{-(2(n+1)+2)}}{A^2 - A^{-2}}\end{aligned}$$

$\square$

8/7/2019 - Bailey's Machinery

Today we will discuss a collection of ideas encapsulated by the term *Bailey machinery*. In particular, we will be interested in its applications to classical q-series identities in general, and our problem in particular.

Bailey's lemma

Lemma (Weak Bailey's Lemma)**.** *Let* $\{\alpha_n\}$ *and* $\{\beta_n\}$ *be two sequences of rational functions in* q*. Suppose*

$$\beta_n = \sum_{r=0}^{n} \frac{\alpha_r}{(q)_{nr}(aq)_{n+r}}$$

Then

$$\sum_{n}^{\infty} a^n q^{n^2} \beta_n = \frac{1}{(aq)_\infty} \sum_{n=0}^{\infty} a^n q^{n^2} \alpha_n$$

Proof. We have

$$\begin{aligned}
\sum_{n=0}^{\infty} a^n q^{n^2} \beta_n &= \sum_{n=0}^{\infty} a^n q^{n^2} \sum_{r=0}^{n} \frac{\alpha_r}{(q)_{n-r}(aq)_{n+r}} \\
&= \sum_{r=0}^{\infty} \sum_{n=r}^{\infty} \frac{a^n q^{n^2} \alpha_r}{(q)_{n-r}(aq)_{n+r}} \\
&= \sum_{r=0}^{\infty} \alpha_r \sum_{n=0}^{\infty} \frac{q^{(n+r)^2} a^{n+r}}{(q)_n (aq)_{n+2r}}
\end{aligned}$$

by taking $k = n - r, n = k + r$ and then replacing k with n. We then employ Cauchy's identity[37] by taking $x = aq^{1+2r}$ and using that

$$\frac{(aq^{1+2r})_\infty}{(aq^{1+2r})_n} = \frac{(aq)_\infty}{(aq)_{n+2r}}$$

□

This can be in fact generalized, using *Bailey's transform* and the *q-Pfaff-Saalshitz identity* to obtain the following result.

Theorem (Bailey's Lemma)**.** *Let* $\{\alpha_n\}$ *and* $\{\beta_n\}$ *be two sequences of rational functions in* q*. Suppose*

$$\beta_n = \sum_{r=0}^{n} \frac{\alpha_r}{(q)_{n-r}(aq)_{n+r}}$$

[37] This states

$$\sum_{n=0}^{\infty} \frac{q^{n^2-n} x^n}{(q)_n (x)_n} = \frac{1}{(x)_\infty}$$

Then

$$\beta'_n = \sum_{r=0}^{\infty} \frac{\alpha'_r}{(q)_{n-r}(aq)_{n+r}}$$

where

$$\alpha'_r = \frac{(\rho_1)_r(\rho_2)_r\left(\frac{aq}{\rho_1\rho_2}\right)^r \alpha_r}{\left(\frac{aq}{\rho_1}\right)_r\left(\frac{aq}{\rho_2}\right)_r}$$

and

$$\beta'_n = \sum_{i=0}^{n} \frac{(\rho_1)_i(\rho_2)_i\left(\frac{aq}{\rho_1\rho_2}\right)_{n-i}\left(\frac{aq}{\rho_1\rho_2}\right)^i}{(q)_{n-i}\left(\frac{aq}{\rho_1}\right)_n\left(\frac{aq}{\rho_2}\right)_n}\beta_i$$

Note that the weak Bailey's lemma implies Bailey's lemma by letting n, ρ_1, ρ_2 go to infinity and using

$$\lim_{\rho_i\to\infty}(\rho_i)_r\left(\frac{1}{\rho_i}\right)^r = (-1)^r q^{r(r-1)/2}$$

$$\lim_{\rho_i\to\infty}\left(\frac{aq}{\rho_i}\right) = 1$$

for $i = 1, 2$. So the above statement becomes

$$\sum_{i=0}^{n} \frac{q^{i^2}a^i\beta_i}{(q)_{n-i}} = \sum_{r=0}^{n} \frac{q^{r^2}a^r\alpha_r}{(q)_{n-r}(aq)_{n+r}}$$

Letting n go to infinity yields the result since

$$\lim_{n\to\infty}\frac{1}{(q)_{n-i}} = \frac{1}{(q)_\infty}$$

$$\lim_{n\to\infty}\frac{1}{(aq)_{n+r}} = \frac{1}{(aq)_\infty}$$

Bailey pairs

It is now natural to make the following definition.

Definition. *A* ***Bailey pair relative*** *to a is a pair of sequences* (α_n, β_n) *that satisfy*

$$\beta_n = \sum_{r=0}^{n} \frac{\alpha_r}{(q)_{n-r}(aq)_{n+r}}$$

The key idea is that, given a Bailey pair (α_n, β_n) we can construct a new Bailey pair (α'_n, β'_n) via Bailey's lemma. Iterating this yields a *Bailey chain*

$$(\alpha_n, \beta_n) \to (\alpha'_n, \beta'_n) \to (\alpha''_n, \beta''_n) \to \ldots$$

Note that given $\{\alpha_n\}$, we can find $\{(\beta_n)\}$ using the definition. Given $\{\beta_n\}$, we can find $\{\alpha_n\}$ via the inversion formula

$$\alpha_n = (1 - aq^{2n}) \sum_{i=0}^{n} \frac{(aq)_{n+i-1}(-1)^{n-i} q^{\binom{n-i}{2}}}{(q)_{n-i}} \beta_i$$

Application to a classical identity

We have the following example, which is one of the easiest possible choices for β_n.

Lemma (Unit Bailey pair)**.** *Consider the pair (α_n, β_n) defined by*

$$\beta_n = \begin{cases} 1 & n = 0 \\ 0 & n > 0 \end{cases}$$

$$\alpha_n = \begin{cases} 1 & n = 0 \\ (-1)^n q^{n(n-1)/2}(1 + q^n) & \end{cases}$$

Then (α_n, β_n) is a Bailey pair relative to $a = 1$.

Proof. We have

$$\begin{aligned} \sum_{r=0}^{n} \frac{\alpha_r}{(q)_{n-r}(aq)_{n+r}} &= \frac{1}{(q)_n^2} + \sum_{r=0}^{n} \frac{(-1)^r q^{r(r-1)/2}(1+q^r)}{(q)_{n-r}(q)_{n+r}} \\ &= \frac{1}{(q)_{2n}} \sum_{r=-n}^{n} (-1)^r q^{r(r-1)/2} \begin{bmatrix} 2n \\ n-r \end{bmatrix} \end{aligned}$$

since

$$\begin{bmatrix} 2n \\ n-r \end{bmatrix} = \frac{(q)_{2n}}{(q)_{n-r}(q)_{n+r}}$$

implies that the right hand side above is

$$\underbrace{\frac{1}{(q)_n^2}}_{r=0} + \frac{1}{(q)_{2n}} \sum_{r=1}^{n} (-1)^r q^{r(r-1)/2} \begin{bmatrix} 2n \\ n-r \end{bmatrix} + \frac{1}{(q)_{2n}} \underbrace{\sum_{r=0}^{n} (-1)^r q^{r(r+1)/2} \begin{bmatrix} 2n \\ n-r \end{bmatrix}}_{r \mapsto -r} = \frac{1}{(q)_{2n}} \sum_{r=0}^{2n} (-1)^{r-n} q^{\binom{r-n}{2}} \begin{bmatrix} 2n \\ r \end{bmatrix}$$

by taking $s = r + n$ and then replacing s with r. We invoke the q-binomial theorem[38] to then write

$$\frac{(q^{-n})_{2n}(-1)^n q^{\binom{n+1}{2}}}{(q)_{2n}}$$

by taking $z = q^{-n}$ and replacing n with $2n$. This is 1 precisely when $n = 0$ and 0 otherwise, which completes the proof. $\square$

[38] Using the specialized version

$$(z)_n = \sum_{r=0}^{n} \begin{bmatrix} n \\ r \end{bmatrix} (-1)^r q^{r(r-1)/2} z^r$$

We can then construct the next pair in the Bailey chain, which will give us a famous identity.

Corollary. *Let (α_n, β_n) be as in the previous lemma. Then taking*

$$\alpha'_n = \begin{cases} 1 & n = 0 \\ (-1)^n q^{n(3n-1)/2}(1+q^n) & n > 0 \end{cases}$$
$$\beta'_n = \frac{1}{(q)_n}$$

yields a Bailey pair (α'_n, β'_n).

Proof. Let ρ_1, ρ_2 go to infinity in Bailey's lemma and simplify to obtain

$$\alpha'_n = a^n q^{n^2} \alpha_n$$
$$\beta'_n = \sum_{k=0}^{n} \frac{a^k q^{k^2}}{(q)_{n-k}} \beta_k$$

where this is true for any Bailey pair relative to any base. Substituing (α_n, β_n) from above yields the desired result. □

Theorem (1st Rogers-Ramanujan identity)**.** *We have*

$$\sum_{n=0}^{\infty} \frac{q^{n^2}}{(q)_n} = \frac{1}{(q;q^5)_\infty (q^4;q^5)_\infty}$$

This is the beginning of a very long story that is connected to ideas in algebraic K-theory, modular forms, conformal field theory, vertex operator algebra, and knots.

Proof. Substitute (α'_n, β'_n) in the weak Bailey's lemma to obtain

$$\begin{aligned} \sum_{n=0}^{\infty} \frac{q^{n^2}}{(q)_n} &= \frac{1}{(q)_\infty}\left[\sum_{n=1}^{\infty}(-1)^n q^{n(3n-1)/2}(1+q^n) + 1\right] \\ &= \frac{1}{(q)_\infty}\left[\underbrace{\sum_{n=1}^{\infty}(-1)^n q^{n(3n-1)/2}}_{n\mapsto -n} + \sum_{n=1}^{\infty}(-q)^n q^{n(3n+1)/2} + 1\right] \\ &= \frac{1}{(q)_\infty}\sum_{n\in\mathbb{Z}}(-1)^n q^{n(3n+1)/2} \\ &= \frac{1}{(q;q^5)_\infty (q^4;q^5)_\infty} \end{aligned}$$

where have used the *Jacobi triple product*[39] and taking $z = -q^{1/2}$ and replacing q with $q^{5/2}$. □

[39]This states
$$\sum_{n\in\mathbb{Z}} z^n q^{n^2} = (q^2;q^2)_\infty(-zq : q^2)_\infty(-q/z;q^2)_\infty$$

Our strange identity

Recall that in order to prove the strange identity for $T(3,4)$, we needed

1. The quintuple product identity
$$\sum_{k\in\mathbb{Z}} q^{k(3k-1)/2}x^{3k}(1-xq^k) = (q,x,qx^{-1};q)_\infty(qx^2,qx^{-2};q^2)_\infty$$

2. Slater's identity
$$(q)_\infty \sum_{n=0}^{\infty} \frac{q^{2n(n+1)}}{(q)_{2n+1}} = (q^3,q^5,q^8;q^8)_\infty(q^2,q^{14};q^{16})_\infty$$

It turns out that Slater's identity comes from Bailey pairs.

Remark (Exercise)**.** *Use the quintuple product identity and the Bailey pair (α_n, β_n) with respect to $a = q$ given by*

$$\begin{aligned}
\alpha_{3n-1} &= q^{3n^2-2n}\\
\alpha_{3n} &= q^{3n^2+2n}\\
\alpha_{3n+1} &= -q^{3n^2+4n+1} - q^{3n^2+2n}
\end{aligned}$$
$$\beta_n = \frac{q^{n^2+n}}{(q^2)_{2n}}$$

Also recall we have
$$H(x) = \sum_{n=0}^{\infty} (x)_{n+1}x^{2n}T(x,n,q)$$
where
$$T(x,n,q) = \sum_{k=0}^{\lfloor (n-1)/2\rfloor} x^{2k-1}q^{2k(k+1)}\begin{bmatrix} n \\ 2k+1\end{bmatrix} + \sum_{k=0}^{\lfloor n/2\rfloor} x^{2k}q^{2k(k+1)}\begin{bmatrix} n+1 \\ 2k+1\end{bmatrix}$$

1. We showed
$$H(x) = (qx)_\infty \sum_{n=0}^{\infty} \frac{q^{2n(n+1)}}{(x^2q)_{2n+1}}x^{6n} + (1-x)\sum_{n=0}^{\infty}((qx)_n - (qx)_\infty)x^{2n}T(x,n,q)$$
via the difference equation.

2. We then took the derivative of both sides with respect to x, letting x go to 1. On the H-side, this yields part of the summation part of the strange identity
$$\frac{1}{2}\sum_{n=0}^{\infty} n\chi_{24}(n)q^{(n^2-25)/48} - \underbrace{\frac{5}{2}\sum_{n=0}^{\infty}\chi_{24}(n)q^{(n^2-25)/48}}$$
Taking $q \mapsto q^8$ and $x = q^3$ in the underbrace and applying the quintuple product identity yields $(q^3,q^5,q^8;q^8)_\infty(q^2,q^{14};q^{16})_\infty$.

3. If we take $x = 1$ in the left term in the above H-equality, we have Slater's identity, behind which is a Bailey pair.

We have the same exact story for the family of torus knots $T(2, 2t+1)$, which Hikami developed explicitly. Namely, he defined

$$H_t^{(0)}(x) = \sum_{n=0}^{\infty} \chi_{8t+4}^{(0)}(n) q^{(n^2-(2t-1)^2)/(8(2t+1))} x^{(n-(2t-1))/2}$$

and proves

$$H_t^{(0)}(x) = \sum_{k_1,\ldots,k_t=0}^{\infty} (x)_{k_t+1} x^{k_t} \prod_{i=1}^{t-1} q^{k_i^2+k_i} x^{2k_i} \begin{bmatrix} k_{i+1} \\ k_i \end{bmatrix}$$

He uses the q-binomial coefficient to rewrite

$$H_t^{(0)}(x) = (qx)_\infty \sum_{k_1,\ldots,k_{t-1}=0}^{\infty} \frac{q^{k_1^2+\ldots+k_{t-1}^2+k_1+\ldots+k_{t-1}}}{(xq)_{k_{t-1}}} x^{2\sum_{i=1}^{t-1} k_i + k_{t-1}} \prod_{i=1}^{t-2} \begin{bmatrix} k_{i+1} \\ k_i \end{bmatrix}$$
$$+ (1-x) \sum_{k_1,\ldots,k_t=0}^{\infty} ((qx)_{k_t} - (qx)_\infty) x^{k_t} \prod_{i=1}^{t-1} q^{k_i^2+k_i} x^{2k_i} \begin{bmatrix} k_{i+1} \\ k_i \end{bmatrix}$$

It turns out that the left term arises from a Bailey pair, morally as a generalization of the Rogers-Ramanujan identity. If we take $x = 1$, then this is $(q, q^{2t}, q^{2t+1}; q^{2t+1})_\infty$ by using the Bailey chain.[40] Then $H_t^{(0)}(1)$ is the same product (by taking the derivative with $a = 0$ and $\mu = t$).

Now, we are examining the general family $T(3, 2^t)$. We have the function

$$H_t(x) = \sum_{n=0}^{\infty} \chi_{3 \cdot 2^{t+1}}(n) x^* q^{**}$$

for suitable powers * and **. Then using the quintuple product identity yields

$$H_t(1) = (q^{2^t-1}, q^{2^t+1}, q^{2^{t+1}}; q^{2^{t+1}})_\infty (q^2, q^{2^{t+2}-2}; q_\infty^{2^{t+2}}$$

The idea will be to work backwards, given the seed Bailey pair, using the Bailey machinery to desired multisum version.

In the case $t = 3$, what double sum is equal to $(q^7, q^9, q^{14}; q^{16})_\infty (q^2, q^{30}; q^{32})_\infty$.

[40]See the proof of proposiition 11 in Hikami's paper for details.

8/12/2019 - Project Update and Khovanov Homology

For the knot theory side of the project, we presented[41] two approaches to the computation of $T(3,4)$ and made some remarks about generalization to the entire family $T(3,2^t)$. In summary, we would like to understand

1. Tanaka's 'bridge' lemma 3.3
2. The incorrect power of q in our current computation of the colored Jones polynomial of the modified trefoil using ω
3. The inconsistency between our computation and that of Tanaka for the diagram
4. The mistakes present in our current computation of the colored Jones polynomial of $T(3,4)$
5. If it is possible to use the pretzel knot diagram for $T(3,4)$ to eliminate the multiple sums and maximize symmetry
6. More generally, how the colored Jones polynomial of a knot with an additional strand attached at two points can be reduced to the bracket of the knot itself

We now currently believe that the framing should be left unresolved until a final surgery presentation of the knot is obtained. Three applications of ω means that the writh of the knot will be -2. A corresponding μ_n should then be included in each modified trefoil computation.

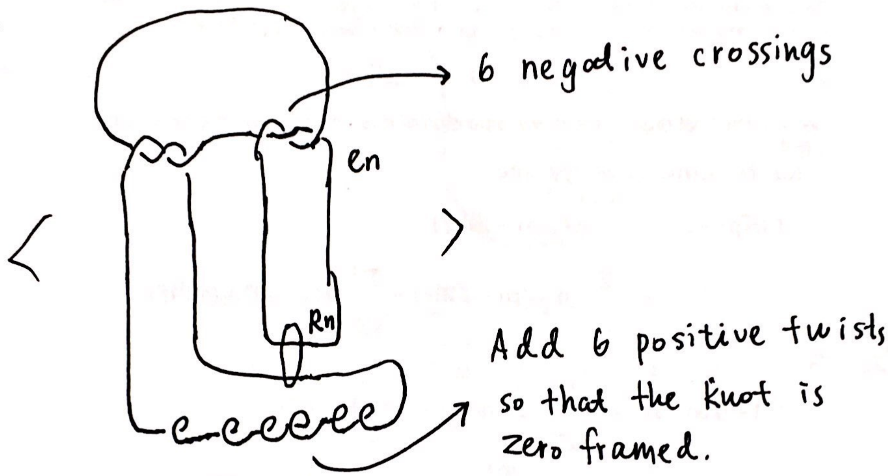
Aaron also presented an update on his progress proving the strange identity.

[41] See the below write-up by Shaoyang.

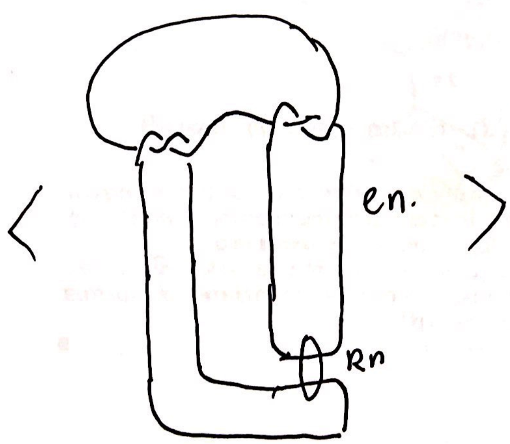
$$J_N(K) = \sum_{n=0}^{N-1} (-1)^{N-n-1} \begin{bmatrix} N+n \\ N-n-1 \end{bmatrix} \sum_{k=0}^{\infty} c_k \langle R_k, \hat{K}(R_n) \rangle$$

$$= \sum_{n=0}^{N-1} (-1)^{N-n-1} \begin{bmatrix} N+n \\ N-n-1 \end{bmatrix} \langle R_n, \tilde{K}(e_n) \rangle$$

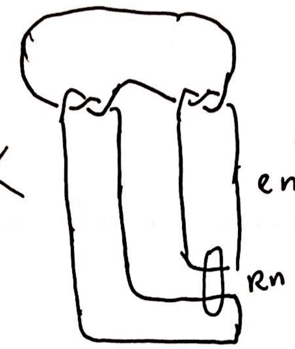
where $\langle R_n, \hat{K}(e_n) \rangle$ has the configuration



So $\langle R_n, \hat{K}(e_n) \rangle = \mu_n^6 \cdot \langle R_n, \tilde{K}'(e_n) \rangle$, where $\langle R_n, \tilde{K}'(e_n) \rangle$ has the configuration

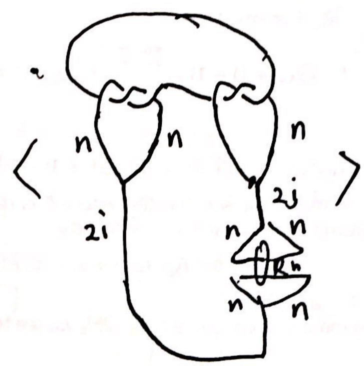


Using the fusion equation twice,

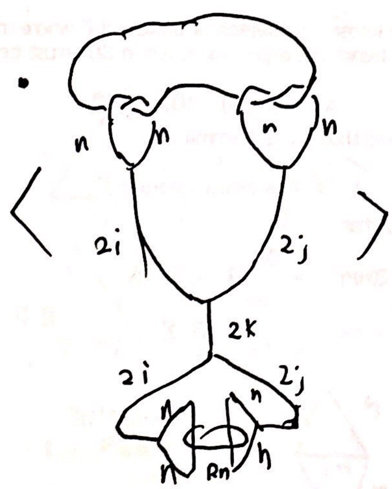
$\Big\langle$ 

$\Big\rangle =$

$$\sum_{i,j=0}^{n} \frac{\langle e_{2i}\rangle}{\theta(n,n,2i)} \cdot \frac{\langle e_{2j}\rangle}{\theta(n,n,2j)}$$



$$= \sum_{i,j=0}^{n} \frac{\langle e_{2i}\rangle}{\theta(n,n,2i)} \cdot \frac{\langle e_{2j}\rangle}{\theta(n,n,2j)} \cdot \sum_{k}^{i+j} \frac{\langle e_{2k}\rangle}{\theta(2i,2j,2k)}$$

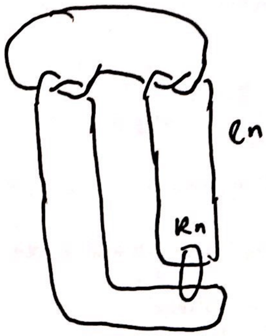


(Fusion on $2i, 2j$).

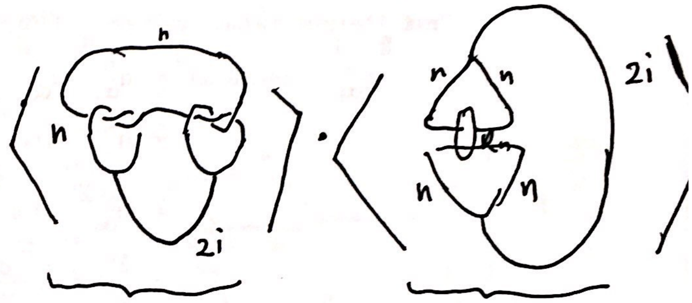
When $k=0$, one has $i=j$ and $\displaystyle\sum_{k}^{i+j} \frac{\langle e_{2k}\rangle}{\theta(2i,2j,2k)}$

$$= \frac{\langle e_0\rangle}{\theta(2i,2i,0)} = \frac{1}{\langle e_{2i}\rangle}.$$

Therefore,

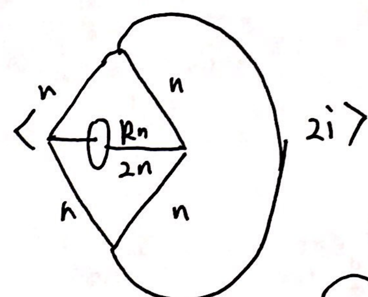


$$= \sum_{i=0}^{n} \frac{\langle e_{2i}\rangle}{\theta(n,n,2i)^2}\cdot\frac{1}{\langle e_{2i}\rangle}$$



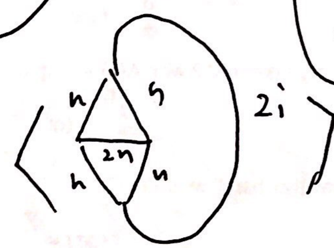
Part ① Part ②

Now Part ② =

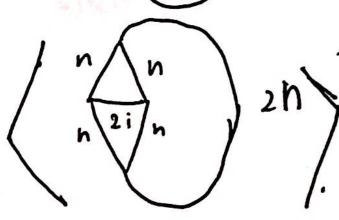


verified via tetrahedron coefficient formula

$$= \frac{\langle R_n, e_{2n}\rangle}{\langle e_{2n}\rangle}\cdot$$



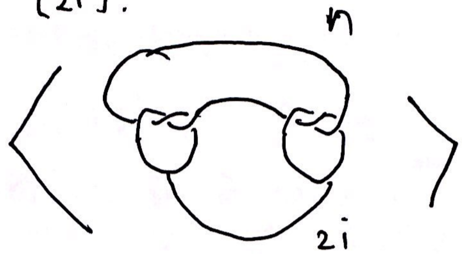
$$= \frac{\langle R_n, e_{2n}\rangle}{\langle e_{2n}\rangle}$$

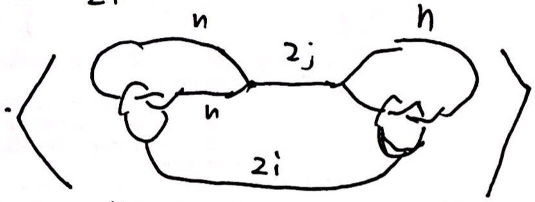


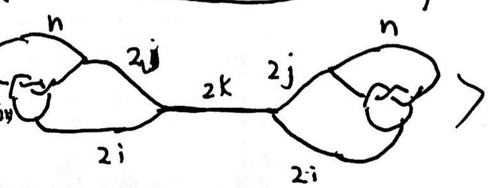
$$= \frac{\langle R_n, e_{2n}\rangle}{\cancel{\langle e_{2n}\rangle}} \cdot \frac{([i]!)^2}{[2i]!} \cdot \cancel{\langle e_{2n}\rangle} \quad \text{(Masbaum Lemma 3.1)}$$

$$= \langle R_n, e_{2n}\rangle \cdot \frac{([i]!)^2}{[2i]!}$$

$$= (-1)^n \frac{\{2n+1\}!}{\{1\}} \cdot \frac{([i]!)^2}{[2i]!}$$

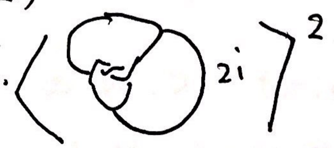
Now Part ① = 

$$= \sum_{j=0}^{n} \frac{\langle e_{2j}\rangle}{\theta(n,n,2j)} \cdot$$ 

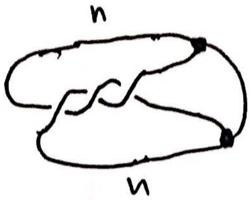
$$= \sum_{j=0}^{n} \frac{\langle e_{2j}\rangle}{\theta(n,n,2j)} \cdot \sum_k \frac{\langle e_{2k}\rangle}{\theta(2i,2j,2k)}$$ 

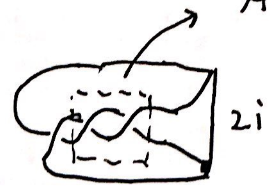
Set $k=0$, then $i=j$, $\sum_k \frac{\langle e_{2k}\rangle}{\theta(2i,2j,2k)} = \frac{\langle e_0\rangle}{\theta(2i,2i,0)} = \frac{1}{\langle e_{2i}\rangle}$

and $\sum_{j=0}^{n} \frac{\langle e_{2j}\rangle}{\theta(n,n,2j)} = \frac{\langle e_{2i}\rangle}{\theta(n,n,2i)}$

Therefore, Part ① $= \frac{1}{\theta(n,n,2i)} \cdot$ 

Now $\Big\langle$  $\Big\rangle$

$= \Big\langle$  $\Big\rangle$

$= \langle k_1(e_n) \rangle$, where $k_1 =$ 

$$\text{So } \langle k_1(e_n)\rangle = \langle \omega^-, \tilde{k}_1(e_n)\rangle$$

$$= \Big\langle \sum C_k^- R_k, \tilde{k}_1\Big(\sum_{j=0}^{n} (-1)^{n-j} \begin{bmatrix} n+1+j \\ n-j \end{bmatrix} R_j\Big)\Big\rangle$$

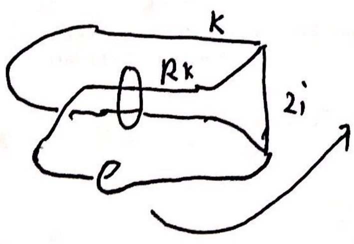
$$= \sum_{j=0}^{n} (-1)^{n-j} \begin{bmatrix} n+1+j \\ n-1 \end{bmatrix} \sum C_k^- \langle R_k, \tilde{k}_1(R_j)\rangle$$

$$= \sum_{j=0}^{n} (-1)^{n-j} \begin{bmatrix} n+1+j \\ n-1 \end{bmatrix} C_j^- \langle R_j, \tilde{k}_1(R_j)\rangle$$

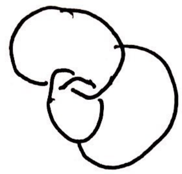
$$= \sum_{k=0}^{n} (-1)^{n-k} \begin{bmatrix} n+1+k \\ n-1 \end{bmatrix} C_k^- \langle R_k, \tilde{k}_1(R_k)\rangle$$

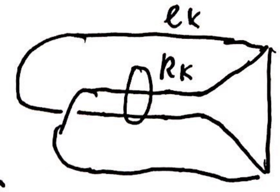
$$= \sum_{k=0}^{n} (-1)^{n-k} \begin{bmatrix} n+1+k \\ n-1 \end{bmatrix} C_k^- \langle R_k, \tilde{k}_1(e_k)\rangle$$

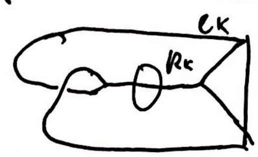
where $\langle R_k, \tilde{k}_1(e_k)\rangle$ has the configuration

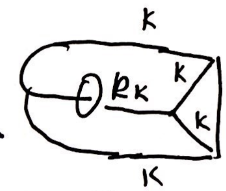


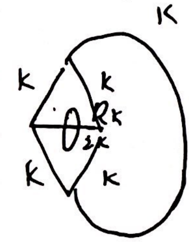
add a positive twist so that the diagram is zero framed.

Therefore, $\Big\langle$  $_{2i} \Big\rangle$

$= \sum_{k=0}^{n} (-1)^{n+k} \begin{bmatrix} n+1+k \\ n-1 \end{bmatrix} c_k^{-} \mu_k \Big\langle$  $_{2i} \Big\rangle$

Fusion
$= \sum_{k=0}^{n} (-1)^{n+k} \begin{bmatrix} n+1+k \\ n-1 \end{bmatrix} c_k^{-} \mu_k \Big\langle$  $_{2i} \Big\rangle$

Half-twist
$= \sum_{k=0}^{n} (-1)^{n+k} \begin{bmatrix} n+1+k \\ n-1 \end{bmatrix} c_k^{-} \mu_k \cdot A^{-k^2} \Big\langle$  $_{2i} \Big\rangle$

$= \sum_{k=0}^{n} (-1)^{n+k} \begin{bmatrix} n+1+k \\ n-1 \end{bmatrix} c_k^{-} \mu_k A^{-k^2} \Big\langle$  $_{2i} \Big\rangle$

Part ②
$= \sum_{k=0}^{n} (-1)^{n+k} \begin{bmatrix} n+1+k \\ n-1 \end{bmatrix} c_k^{-} \mu_k A^{-k^2} \cdot (-1)^k \frac{\{2k+1\}!}{\{1\}} \cdot \frac{([i]!)^2}{[2i]!}$

<u>Remark</u>. When $i=0$, $\Big\langle$  $\Big\rangle = \langle$ trefoil cabled w/. $e_n \rangle$

our formular for $\Big\langle$  $\Big\rangle$ matches the CJP

up to q^{n^2-1}.

In total, $J_N(K) =$

$$\sum_{n=0}^{N-1} (-1)^{N-n-1} \begin{bmatrix} N+n \\ N-n-1 \end{bmatrix} \mu_n^{6} \cdot \sum_{i=0}^{n} \frac{\langle e_{2i} \rangle}{\theta(n,n,2i)^3} \Big\langle$$

$$_{2i} \Big\rangle^{2} \cdot (-1)^n \frac{\{2n+1\}!}{\{1\}} \frac{([i]!)^2}{[2i]!}$$

where $\Big\langle$

$$_{2i} \Big\rangle = \sum_{k=0}^{n} (-1)^{n-k} \begin{bmatrix} n+1+k \\ n-1 \end{bmatrix} a_k^{-} \mu_k A^{-k^2} (-1)^k \frac{\{2k+1\}!}{\{1\}} \frac{([i]!)^2}{[2i]!}$$

Khovanov Homology

We will have a short mini-course on *Khovanov homology*.[42] The goals will be to

1. Define Khovanov homology
2. Find examples of knots/links with the same Khovanov homology
3. Discuss the Rasmussen invariant and slice genus

Defining Khovanov homology

The following is a review of the material in Dror Bar-Natan's paper (see there for more details). The idea is to find a homology theory with Euler characteristic equal to the Jones polynomial. Given a diagram D with n crossings, we define

$$\widehat{J}(D) = (-1)^{n_-} q^{n_+ - 2n_-} \langle D \rangle$$

where n_+ is the number of positive crossings, n_- is the number of negative crossings, $n = n_+ + n_-$, and $q = t^{1/2}$. The bracket here is defined by the relation

$$\langle \times \rangle = \langle \asymp \rangle - q \langle)(\rangle$$

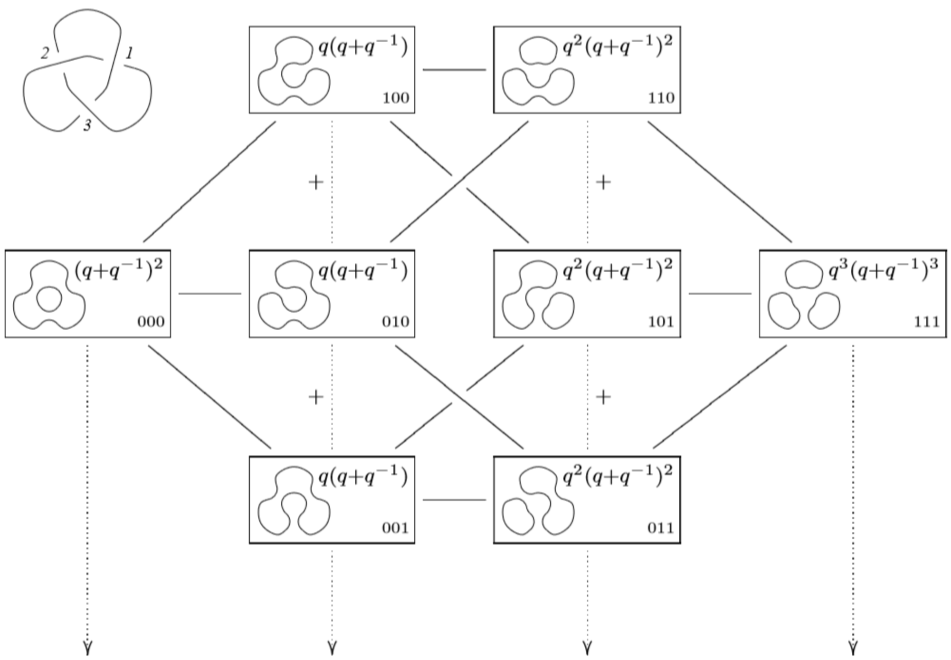
Remark. *Despite the unusual normalization, this definition indeed yields the regular the Jones polynomial.*

We can consider the 2^n states obtained by all possible smoothings of each crossing. Denote by $r(S)$ the number of 1-smoothings in the total smoothing S, and denote by $k(S)$ the number of cycles in the total smoothing S. Then we have

$$\widehat{J}(D) = \sum_S (-1)^{r(S)+n_-} q^{r(S)+n_+-2n_-} (q + q^{-1})^{k(S)}$$

[42] Some references are

1. *A categorification of the Jones polynomial* by Khovanov [2001]
2. *Five lectures on Khovanov homology* by Paul Turner
3. *On Khovanov's categorification of the Jones polynomial* by Bar-Natan
4. *Knots with identical Khovanov homology* by Liam Watson
5. *Khovanov homology and slice genus* by Jacob Rasmussen



(1)

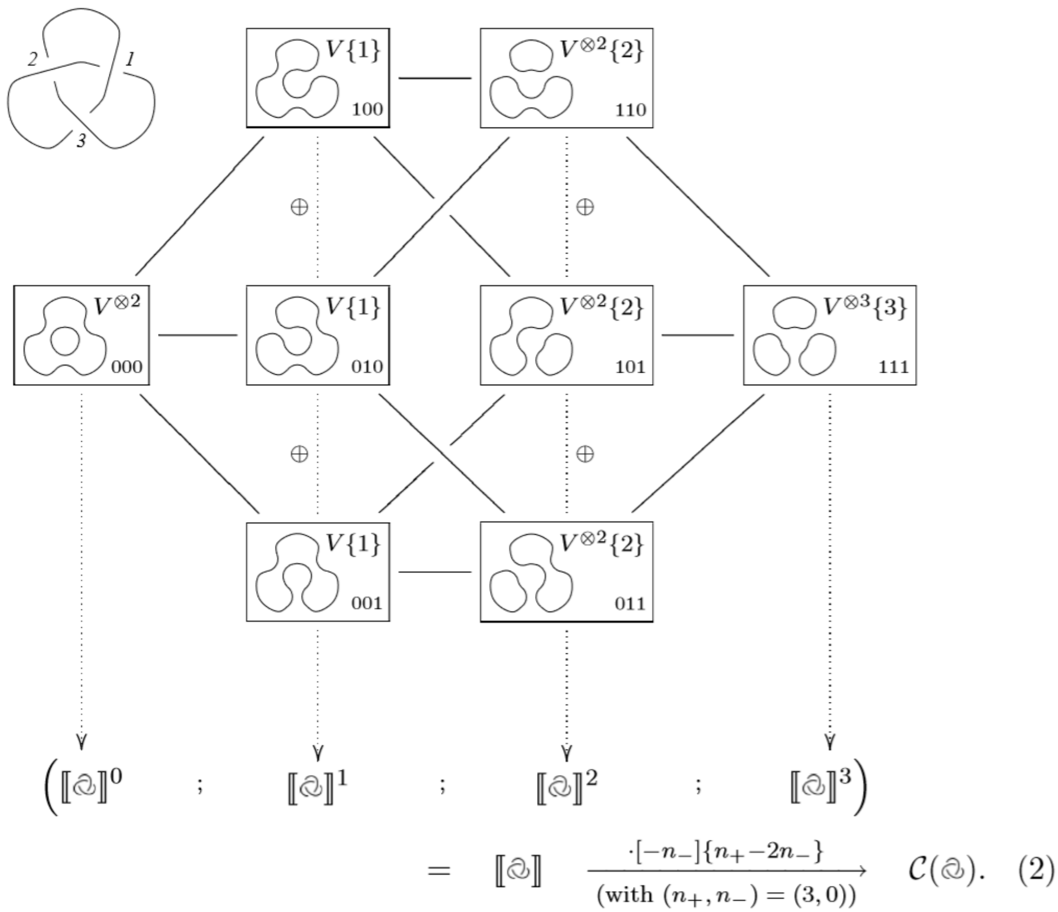
$$(q+q^{-1})^2 \quad - \quad 3q(q+q^{-1}) \quad + \quad 3q^2(q+q^{-1})^2 \quad - \quad q^3(q+q^{-1})^3$$

$$= \; q^{-2}+1+q^2-q^6 \xrightarrow[\text{(with } (n_+,n_-)=(3,0))]{\cdot(-1)^{n_-}q^{n_+-2n_-}} q+q^3+q^5-q^9 \xrightarrow{\cdot(q+q^{-1})^{-1}} J(\circledcirc) = q^2+q^6-q^8.$$

We follow through with an analagous categorification of the process to obtain Khovanov homology. Take a two-dimensional vector space

$$V = \mathbb{Q}\oplus\mathbb{Q} = \mathbb{Q}\underbrace{(1,0)}_{1}\oplus\mathbb{Q}\underbrace{(0,1)}_{x}$$

We can consider the tensor product $V^{\otimes k}$ and will write $11x$ for the element $1\otimes 1\otimes x$. We then replace the corners of the above cube with the appropriate vector spaces so that the dimensions of the chain complex spaces work out.



We can assign degree 1 to the basis element 1 and degree -1 to the basis element x. For example, the element $11x$ has degree 1. Then we can include the degree shifts in the summation by adjusting the dimensions of the spaces and grading on the chain complexes.

Lemma. *For a finite chain complex*

$$\mathcal{C} : 0 \to W_m \to W_{m+1} \to \ldots \to W_n \to 0$$

we have

$$\chi(\mathcal{C}) = \chi(H_*(\mathcal{C}))$$

Thus the graded Euler characteristic of $\mathcal{C}$ is indeed the Jones polynomial of K by construction.

We will next construct a differential of degree 0, which just means that the differential respects the grading on the chain groups. Thus we can view this construction as either a chain complex of graded vector spaces or a graded chain complex.

To define the differential, at each state S we will construct a map from the associated vector space to the states obtained by adjusting one of the 0-smoothings to a 1-smoothing. In the case where this results in two cycles merged into one, we use the join map

$$\begin{aligned} m : V \otimes V &\to V \\ 1x &\mapsto x \\ x1 &\mapsto x \\ xx &\mapsto 0 \end{aligned}$$

When a cycle is split into two, we use the split map

$$\begin{aligned} \Delta : V &\to V \otimes V \\ 1 &\mapsto 1x + x1 \\ x &\mapsto xx + 11 \end{aligned}$$

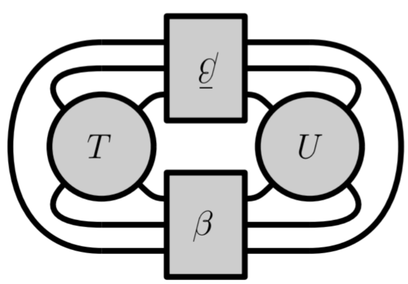
With appropriate shift on the chain complex groups, these maps a degree-preserving. We then add appropriate signs to the sums of these maps so that these indeed yield a differential. See Bar-Natan's paper for details.

Khovanov homology as an invariant

Theorem (Kronheier and Mrowka)**.** *As an invariant, Khovanov homology detects the unknot.*

Despite its strength, Khovanov homology is not a complete invariant. It is possible to come up with infinitely many examples with the same Khovanov homology. We will describe a reciple to generate these.

Given a braid $\beta \in \mathcal{B}_3$ and two tangles T, U, we can consider the knot



denoted by $K_\beta(T, U)$. Applying some number of twists to the above regions before the tangles yields a distinct knot with the same Khovanov homology.[43]

[43] See Watson's paper for a five page proof.

We can define *Lee homology* by adjusting the definitions of the maps to

$$\begin{aligned} m' : V \otimes V &\to V \\ 1x &\mapsto x \\ x1 &\mapsto x \\ xx &\mapsto 0 + 1 \end{aligned}$$

and

$$\begin{aligned} \Delta : V &\to V \otimes V \\ 1 &\mapsto 1x + x1 \\ x &\mapsto xx + 11 \end{aligned}$$

The degree now changes by 4, including the degree shift present between the different heights on the resolution cube. This yields a new homology $\mathrm{Lee}(K)$, which is trivial for knots and for links is given by $\mathrm{rank}(\mathrm{Lee}(K)) = 2^{\mu}$, where μ is the number of components of L.

Note that since these maps no longer preserve degree, we are simply taking the direct sum of everything at a particular height in the resolution cube.

Rasmussen observed that the ith degree gives a filtration

$$\mathcal{C} = \underbrace{F^1\mathcal{C}}_{\{v:i(v)\geq 1\}} \supset F^2\mathcal{C} \supset F^3\mathcal{C} \supset \ldots \supset F^9\mathcal{C} \supset 0$$

in the case of the trefoil. The differentials preserve this filtration, which induces a filtration on homology

$$F^m H_* \supset F^{m+1} H_* \supset \ldots \supset 0$$

If we let $v \in F^k H_*$ be a homology class, we define the degree of v to be the maximum degree of possible representatives x for

$$q([v]) = \max\{q(x) : [x] = v\}$$

Take

$$s_{\max} = \max\{q(v) : v \neq 0, v \in H_*\}$$
$$s_{\min} = \min\{q(v) : v \neq 0, v \in H_*\}$$

Lemma. *$s_{\max}$ and $s_{\min}$ differ by 2, and hence we define* ***Rasmussen's s-invariant*** *to be*

$$s(K) = \frac{S_{\max} + S_{\min}}{2}$$

With spectral sequences, we can understand the s-invariant as saying something about cobordisms between knots. Given a cobordism between two links, we know that the surface is generated by three elementary pieces: the 'birth' cap, 'death' cap, and 'pair of pants'. Thus we can break any cobordism into these components and sequentially examine how they act on the Khovanov homology. For example, killing a cycle in the cobordism yields the induced map

$$\begin{aligned}\mathcal{C}(L \sqcup \mathcal{O}) &\to \mathcal{C}(L) \\ 1 &\mapsto 0 \\ x &\mapsto 1 \in \mathbb{Q}\end{aligned}$$

We can similarly define the morphisms associated to the birth element and pair of pants, using the m and Δ maps. This makes Khovanov homology into a functor from the category of embedded 1-dimensional manifolds in S^3 with cobordisms as morphisms. A similar construction applies to Lee homology.

Definition. *The **slice genus** of K is the minimum genus of all surfaces in D^4 that bound K.*

We can consider such a surface that bounds K to be a cobordism between K and the unknot. This induces a morphism from the Lee homology of K to the Lee homology of the unknot. This is always an isomorphism, but this is nontrivial to show.

If we take nonzero $x \in \mathrm{Lee}(K)$, then the image of x under the induced morphism on homology is $s(K)$. This implies $s(K) < 2g$, where g is the slice genus of K. A similar argument applied to the mirror of K yields $|s(K)| < 2g$. Hence the s-invariant is a lower bound on the slice genus of K.

8/15/2019 - Project Update

Cyclotomic expansion via skein-theoretic computations

I'll first describe how to obtain the cyclotomic expansion of the colored Jones polynomial from a skein-theoretic derivation such as Masbaum's. A computation that involves ω yields an expression for the colored Jones polynomial of the form

$$J_N(K) = \sum_{n=0}^{N-1} \begin{bmatrix} N+n \\ N-1-n \end{bmatrix} R(n)$$

where $R(n)$ is some expression. Examining the binomial coefficient yields

$$\begin{bmatrix} N+n \\ N-1-n \end{bmatrix} = \frac{[N+n][N+n-1]\dots[1]}{[N-1-n][N-1-n-1]\dots[1][2n+1][2n]\dots[1]}$$

$$= \frac{[N+n][N+n-1]\dots[N-n]}{[2n+1][2n]\dots[1]}$$

$$= \frac{\{N+n\}\{N+n-1\}\dots\{N-n\}}{\{2n+1\}\{2n\}\dots\{1\}} = \frac{\{N+n\}\{N+n-1\}\dots\{N-n\}}{\{2n+1\}!}$$

Now observe

$$\begin{aligned}
(q^{1-N})_n(q^{1+N})_n &= \left[(1-q^{1-N})(1-q^{2-N})\dots(1-q^{n-N})\right]\cdot\left[(1-q^{1+N})(1-q^{2+N})\dots(1-q^{n+N})\right] \\
&= \left[\frac{q^{(N-1)/2}-q^{-(N-1)/2}}{q^{(N-1)/2}}\cdot\frac{q^{(N-2)/2}-q^{-(N-2)/2}}{q^{(N-2)/2}}\cdot\dots\cdot\frac{q^{(N-n)/2}-q^{-(N-n)/2}}{q^{(N-n)/2}}\right] \\
&\quad\cdot(-1)^n\left[\frac{q^{(N+1)/2}-q^{-(N+1)/2}}{q^{-(N+1)/2}}\cdot\frac{q^{(N+2)/2}-q^{-(N+2)/2}}{q^{(-(N+2)/2}}\dots\frac{q^{(N+n)/2}-q^{-(N+n)/2}}{q^{-(N+n)/2}}\right] \\
&= (-1)^n\frac{1}{q^{-1-2-\dots-n}}\left[\{N-1\}\{N-2\}\dots\{N-n\}\right]\cdot\left[\{N+1\}\{N+2\}\dots\{N+n\}\right] \\
&= (-1)^n q^{n(n+1)/2}\frac{\{N+n\}\{N+n-1\}\dots\{N-n\}}{\{N\}}
\end{aligned}$$

Hence

$$\begin{bmatrix} N+n \\ N-1-n \end{bmatrix} = (-1)^n\frac{\{N\}}{\{2n+1\}!}q^{-n(n+1)/2}(q^{1-N})_n(q^{1+N})_n$$

and thus

$$J_N(K) = \sum_{n=0}^{\infty}\left[(-1)^n\frac{\{N\}}{\{2n+1\}!}q^{-n(n+1)/2}R(n)\right](q^{1-N})_n(q^{1+N})_n$$

Cyclotomic expansion for pretzel knots

Presented below is a computation which, combined with the above work, should provide the cyclotomic expansion of the colored Jones polynomial for all pretzel knots (although the computation is for $T(3,4)$, the argument easily generalizes).

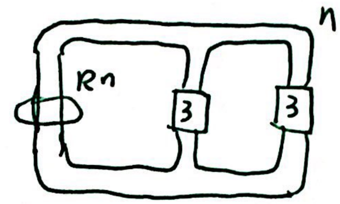
$$\langle K(e_{N-1})\rangle = \langle w^{-1}, \tilde{K}(e_{N-1})\rangle$$

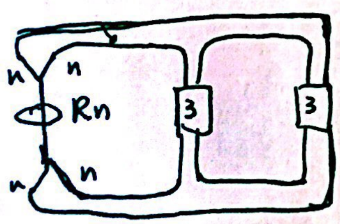
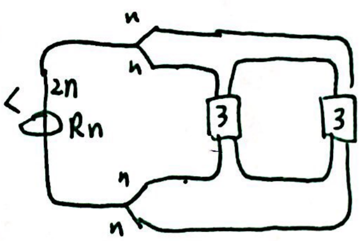
$$= \langle \sum c_k^- R_k, \tilde{k}\Big(\sum_{n=0}^{N-1} (-1)^{N-1-n} \begin{bmatrix} N+n \\ N-1-n \end{bmatrix} R_n\Big\rangle$$

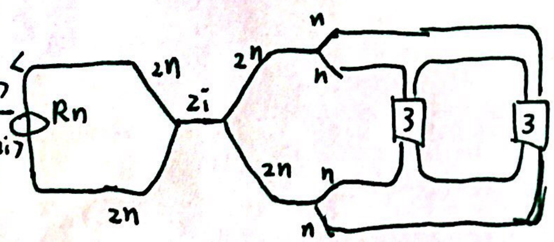
$$= \sum_{n=0}^{N-1} (-1)^{N-1-n} \begin{bmatrix} N+n \\ N-1-n \end{bmatrix} \langle \sum_k c_k^- R_k \tilde{k}(R_n)\rangle$$

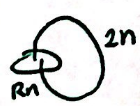
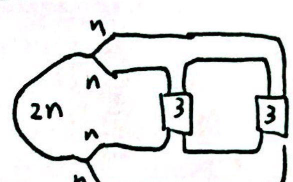
$$= \sum_{n=0}^{N-1} (-1)^{N-1-n} \begin{bmatrix} N+n \\ N-1-n \end{bmatrix} c_n^- \langle R_n, \tilde{k}(R_n)\rangle$$

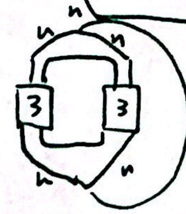
$$= \sum_{n=0}^{N-1} (-1)^{N-1-n} \begin{bmatrix} N+n \\ N-1-n \end{bmatrix} c_n^- \langle R_n, \tilde{k}(e_n)\rangle$$

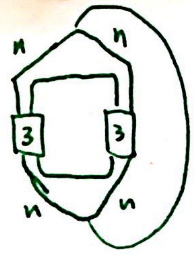
where $\langle R_n, \tilde{k}(e_n)\rangle = \Big\langle$  $\Big\rangle$

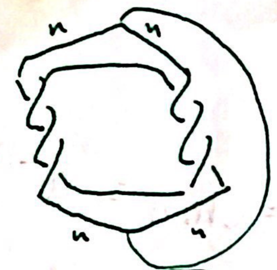
$= \Big\langle$  $\Big\rangle = \Big\langle$  $\Big\rangle$

$= \sum_i \frac{\langle e_{2i}\rangle}{\langle n,n,2i\rangle} \Big\langle$  $\Big\rangle$

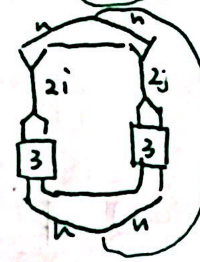
$\overset{\text{Bridge}}{\underset{i=0}{=}} \frac{1}{\langle e_{2n}\rangle} \cdot \Big\langle$  $\Big\rangle \Big\langle$  $\Big\rangle$

$= \frac{1}{\langle e_{2n}\rangle} \cdot \langle R_n, e_{2n}\rangle \cdot \Big\langle$  $2n \Big\rangle$

For $\langle$ 

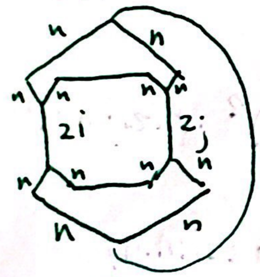
$2n \ \rangle = \langle$ 

$\rangle$

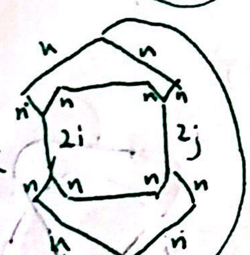
$\overset{\text{Fusion}}{=} \sum_{i,j=0}^{n} \frac{\langle e_{2i}\rangle\langle e_{2j}\rangle}{\theta(n,n,2i)\,\theta(n,n,2j)}$ 

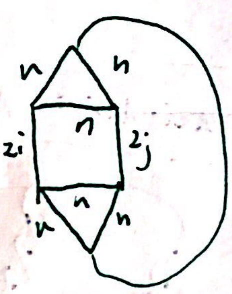
$2n$

$\overset{\text{Half Twist}}{=} \sum_{i,j=0}^{n} \frac{\langle e_{2i}\rangle\langle e_{2j}\rangle}{\theta(n,n,2i)\,\theta(n,n,2j)}\, \gamma(2i;n,n)^3 \cdot \gamma(2j;n,n)^3$

$\langle$ 

$2n \ \rangle.$

Now $\langle$ 

$2n \ \rangle = \langle$ $2i$ 

$2n \ \rangle$

$$\overset{\text{Masbaum Vogel}}{=} \frac{\left\langle \begin{matrix} 2n & n & n \\ n & 2j & 2i \end{matrix} \right\rangle \left\langle \begin{matrix} 2i & n & n \\ n & 2j & 2n \end{matrix} \right\rangle}{\theta(2n,2i,2j)} \overset{\text{Computation}}{=} \frac{\left\langle \begin{matrix} 2n & n & n \\ n & 2j & 2i \end{matrix} \right\rangle^2}{\theta(2n,2i,2j)}.$$

For $\left\langle \begin{matrix} 2n & n & n \\ n & 2j & 2i \end{matrix} \right\rangle = \left\langle \begin{matrix} A & B & E \\ D & C & F \end{matrix} \right\rangle$, $\Sigma = 5n+2j+2i$

one has $a_1 = A+B+E/2 = 2n$

$a_2 = B+D+F/2 = n+i$

$a_3 = C+D+E/2 = n+j$

$a_4 = A+C+F/2 = n+j+i$

$b_1 = \Sigma - A - D/2 = n+j+i$

$b_2 = \Sigma - E - F/2 = 2n+j$

$b_3 = \Sigma - B - C/2 = 2n+i$

Note that $n+j+i \le n+n+i = 2n+i$

$n+j+i \le n+j+n = 2n+j$.

So $\min\{b_1, b_2, b_3\} = n+j+i$.

(i) When $n+j+i \le 2n \Leftrightarrow j+i \le n$,

$\max\{a_1, a_2, a_3, a_4\} = 2n$

(ii) When $n+j+i \ge 2n \Leftrightarrow j+i \ge n$

$\max\{a_1, a_2, a_3, a_4\} = n+j+i$

Therefore, if (i) $j+i \le n$, one has $2n \le \varsigma \le n+j+i$

$\Rightarrow n \le j+i$. Contradition.

(ii) $j+i \ge n$, one has $n+j+i \le \varsigma \le n+j+i$

$\Rightarrow \varsigma = n+j+i$

So $\begin{pmatrix} a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & \end{pmatrix} = \dfrac{(-1)^{n+j+i}[n+j+i+1]!}{[0]!\,[n-i]!\,[n-j]!\,[i+j-n]!\,[j]!\,[i]!\,[0]!}$

$$= \frac{(-1)^{n+j+i}[n+j+i+1]!}{[n-\cancel{i}]![n-\cancel{j}]![i+\cancel{j}-n]![\cancel{j}]![\cancel{i}]!}$$

$$\text{And } \prod_{i=1}^{3}\prod_{j=1}^{4}[b_i-a_j]! = [i+j-\cancel{n}]![\cancel{j}]![\cancel{i}]![\cancel{0}]!$$

$$[j]![n+j-i]![n]![n-\cancel{i}]!$$

$$[i]![n]![n+i-j]![n-\cancel{j}]!$$

$$\text{Therefore, } \begin{pmatrix} a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & \end{pmatrix} \frac{\prod_{i=1}^{3}\prod_{j=1}^{4}[b_i-a_j]!}{[A]![B]![C]![D]![E]![F]!}$$

$$= \frac{(-1)^{n+j+i}[n+j+i+1]![n+j-i]![n+i-j]![\cancel{n}]![\cancel{n}]![i]![j]!}{[2n]![\cancel{n}]![\cancel{n}]![2j]![2i]![n]!}$$

$$= \frac{(-1)^{n+j+i}[n+j-i]![n+i-j]![n+j+i+1]![i]![j]!}{[2n]![2j]![2i]![n]!}$$

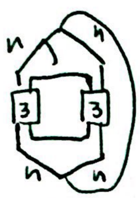
$$\left\langle \begin{matrix} 2n & n & n \\ n & 2j & 2i \end{matrix} \right\rangle^2 = \frac{([n+j-i]!)^2([n+i-j]!)^2([n+j+i+1]!)^2([i]!)^2([j]!)^2}{([2n]!)^2([2j]!)^2([2i]!)^2([n]!)^2}$$

provided. $i+j \geq n$.

Since $\theta(2n, 2i, 2j) = (-1)^{n+i+j} \dfrac{[n+i+j+1]!\,[n+i-j]!\,[n+j-i]!\,[i+j-n]!}{[2n]!\,[2i]!\,[2j]!\,[n]!}$

one has
$$\frac{\left\langle \begin{matrix} 2n & n & n \\ n & 2j & 2i \end{matrix} \right\rangle^2}{\theta(2n, 2i, 2j)} = \theta(2n, 2i, 2j) \cdot \left(\frac{[i]!\,[j]!}{[n]!\,[i+j-n]!} \right)^2$$

provided $i+j \geq n$.

Therefore, $\Big\langle$  $2n \Big\rangle$
$$= \sum_{\substack{i,j=0 \\ i+j \geq n}}^{n} \frac{\langle e_{2i} \rangle \langle e_{2j} \rangle}{\theta(n, n, 2i)\, \theta(n, n, 2j)} \cdot \gamma(2i; n, n)^3 \gamma(2j; n, n)^3 \cdot \theta(2n, 2i, 2j) \cdot \left(\frac{[i]!\,[j]!}{[n]!\,[i+j-n]!} \right)^2$$

$$= \sum_{i=0}^{n} \sum_{j=n-i}^{n} \frac{\langle e_{2i} \rangle \langle e_{2j} \rangle}{\theta(n, n, 2i)\, \theta(n, n, 2j)}\, \theta(2n, 2i, 2j)\, \gamma(2i; n, n)^3 \gamma(2j; n, n)^3 \left(\frac{[i]!\,[j]!}{[n]!\,[i+j-n]!} \right)^2$$

$$:= G(n)$$

$$\langle K(e_{N-1}) \rangle = \sum_{n=0}^{N-1} (-1)^{N-1-n} \begin{bmatrix} N+n \\ N-1-n \end{bmatrix} C_n^{-}\, \mu_n^{6} \cdot \frac{\langle R_n, e_{2n} \rangle}{\langle e_{2n} \rangle} \cdot G(n) .$$

8/21/2019 - Project Update

After implementing the above work in Mathematica several times and adjusting for writhe appropriately following a suggestion from Hans, it was determined that the computation is correct for knots like the trefoil and the unknot, but is incorrect for $T(3,4)$. The formula produces a polynomial with too many terms and the incorrect coefficients.

Explaining the error in the computation

Recall that, after decorating a knot diagram K with ω to remove a full twist, we wish to compute the bracket of the Hopf link $\langle \omega, \widetilde{K}(e_{N-1})\rangle$, where $\widetilde{K}(e_{N-1})$ is the modified, untwisted knot cabled with the Jones-Wenzl idempotent e_{N-1}. We then consider the linear combinations

$$\omega = \sum_{k=0}^{\infty} c_k R_k$$
$$e_{N-1} = \sum_{n=0}^{N-1} (-1)^{N-1-n} \begin{bmatrix} N+n \\ N-1-n \end{bmatrix} R_n$$

and expand

$$\langle \omega, \widetilde{K}(e_{N-1})\rangle = \sum_{k=0}^{\infty} \sum_{n=0}^{N-1} (-1)^{N-1-n} \begin{bmatrix} N+n \\ N-1-n \end{bmatrix} c_k \langle R_k, \widetilde{K}(R_n)\rangle$$

The key idea is to argue that $\langle R_k, \widetilde{K}(R_n)\rangle = 0$ unless $k = n$. *It is these high cancellation properties that vastly simplify the computation.* This is the purpose of introducing the element ω and working hard to express it in the basis $\{R_n\}$.

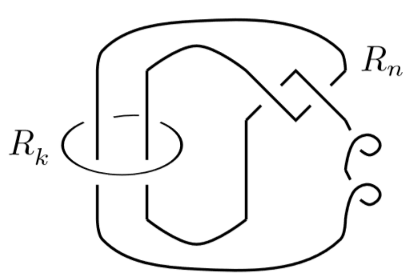
Let's review why indeed $\langle R_k, \widetilde{K}(R_n)\rangle = 0$ unless $k = n$. Recall that we have the following crucial properties[44] of the $\{R_n\}$ basis, which we defined

$$R_n = \prod_{i=0}^{n-1} (z - \lambda_{2i})$$

Lemma. $\langle R_n, e^{2i}\rangle = 0$ *for all* $i < n$.

Lemma. $\langle R_n, z^{2i}\rangle = 0$ *for all* $i < n$.

In short, the second lemma states that circling via R_n in a Hopf link configuration annihilates the portion of the even subalgebra $\mathcal{B}^{even} \subset (\mathbb{Z}[A, A^{-1}])[z] = \mathcal{B}$ of z-degree less than $2n$.



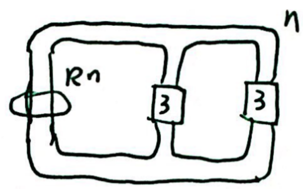
[44]See page 54 of these notes for proofs of these.

Consider the above picture, and first suppose $n < k$. There are two strands of R_n (which are really cables of the Jones-Wenzl idempotents) that pierce the spanning disc of R_k. Now apply all necessary Kauffman relations so that the R_n component of the link has no crossings.

First, since two strands of the original R_n link pierce the disc spanning R_k, all of the resulting states will have an even intersection number with this disc.[45] Second, there cannot be *more* strands that pass through R_k. Thus the element $\widetilde{K}(R_n)$ lies in the even subalgebra of $\mathcal{B} \simeq (\mathbb{Z}[A, A])[z]$ with z-degree less than $2n$. So by the above lemma we have $\langle R_k, \widetilde{K}(R_n)\rangle = 0$.

Now suppose $k > n$. ***Since*** $\widetilde{K}(R_n)$ ***is the unknot*** there is an isotopy of this link taking the diagram to one of the form $\langle R_n, J(R_k)\rangle$ for some knot J (note that R_k and R_n have switched roles. And since in the original diagram R_k pierces a Seifert surface for $\widetilde{K}(R_n)$ twice, after isotopy we can arrange it so that $J(R_k)$ pierces the spanning disc for R_n twice. Then the same argument implies $\langle R_k, \widetilde{K}(R_n)\rangle = \langle R_n, J(R_k)\rangle = 0$. Thus we have the desired cancellation properties.

In short, the problem with our computation is that the link



has component $\widetilde{K}(e_n)$ which is not actually the unknot. In fact, anytime the number of twists on one of the twist regions is greater than or equal to 3, the link component is the connected sum of a nontrivial knot and another knot. Hence such a decomposition means that we no longer have the high cancellation properties of ω. The resulting formula for a *knotted* link component is then

$$\sum_{n=0}^{N-1} \sum_{k=n}^{\infty} (-1)^{N-1-n}(-1)^{N-1-n} \begin{bmatrix} N+n \\ N-1-n \end{bmatrix} c_k \langle R_k, \widetilde{K}(R_n)\rangle$$

Moving forward

We now have a much clearer picture of the element ω. Namely, it can be used to easily simplify the computation of the colored Jones polynomials for knots with unknotting number 1 (as applying ω to the appropriate region results in the unknot). For example, Tanaka [2008] uses ω to treat the case of Whitehead doubles of knots, which all have unknotting number 1.

Alternatively, ω can be used to simplify knots with larger unknotting numbers, provided that these take the form of consecutive twist regions of two strands. For example, the twist knots are

[45] The resulting link component consists of a number of circles that link with R_k and a number of disjoint circles that do not link with R_k. The point is that resolving crossings in the right part of the diagram either results in decreasing the number of strands passing through R_k by 2 or doesn't have any effect.

Alternatively, one could argue that the portion of the diagram passing through R_k remains unchanged when resolving crossings on the right. So all of the resulting uncrossed diagrams have the same number of strands passing through R_k. Then applying any isotopy preserves the number of strands intersecting the spanning disc transversally, as the mod 2 intersection is homotopy invariant.

treated by Masbaum [2003] and double twist knots by Lauridsen [2010].

The knot $T(3,4)$ has unknotting number 3 and it is not of the above particularly straightforward form, so it is not amenable to simplification via ω, at least with the approaches we have been implementing so far. Since we don't even have cancellation of the infinite sum above, we cannot take into account the additional terms either.

Some directions the project could take are

1. Compute the colored Jones polynomial of $T(3,4)$ and pretzel knots via diagrammatic calculus only. However, this has basically already been done (see Walsh [2013]). It is also not obvious how to move from such an expression to the cyclotomic expansion of the colored Jones polynomial.

2. Compute the colored Jones polynomial of a family of knots with unknotting number 1, or of compelexity type given by twist regions of two strands.

3. Compute the colored Jones polynomial of $T(3,4)$ and/or the family $T(3,2^t)$ via other techniques (braid representations, quantum groups, etc.)

4. Compute the cyclotomic expansion of the colored Jones polynomial of the $T(2,2k+1)$ family via skein-theoretic techniques. I believe Robert mentioned this hasn't been done yet.

Final Presentation Slides

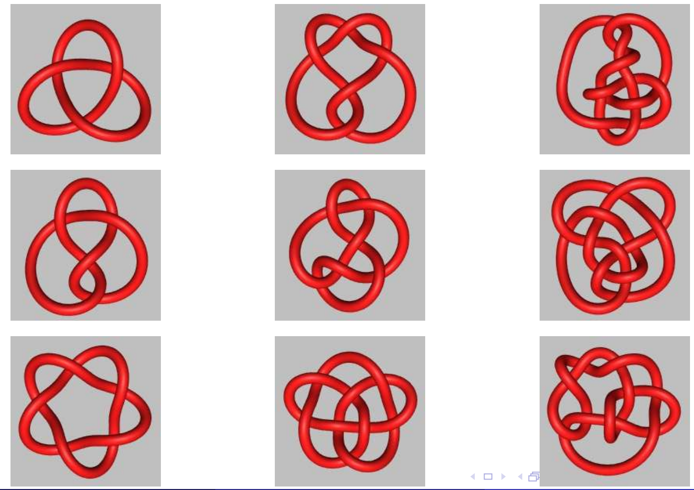
Quantum Invariants and Modularity

Fields Institute, Summer 2019

Colin Bijaoui, Beckham Myers, Aaron Tronsgard, Shaoyang Zhou
Professors Hans Boden, Robert Osburn, Will Rushworth

September 1, 2019

Knots

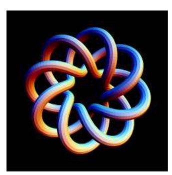


Introduction

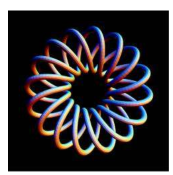
We are studying the *colored Jones polynomial* of the infinite family of torus knots $T(3, 2^t)$:

$T(3,2)$

$T(3,4)$



$T(3,8)$



$T(3,16)$

... and its number-theoretic properties.

Project outline

Given a knot K, we can consider the schematic:

K

Project outline

Given a knot K, we can consider the schematic:

$$\begin{tikzcd}
 & \underbrace{J_N(K^*;q)}_{\text{cyclotomic}} \\
K \arrow[ur] \arrow[dr] & \\
 & \underbrace{J_N(K;q)}_{\text{non-cyclotomic}}
\end{tikzcd}$$

Project outline

Given a knot K, we can consider the schematic:

$$\begin{tikzcd}
 & \underbrace{J_N(K^*;q)}_{\text{cyclotomic}} \arrow[r, squiggly, "x=-1", "q=\zeta_N"'] & U(x;q) \\
K \arrow[ur] \arrow[dr] & & \\
 & \underbrace{J_N(K;q)}_{\text{non-cyclotomic}} \arrow[r, squiggly, "q=\zeta_N"] & F(q)
\end{tikzcd}$$

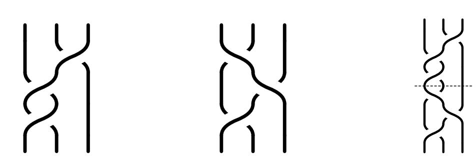
Project outline

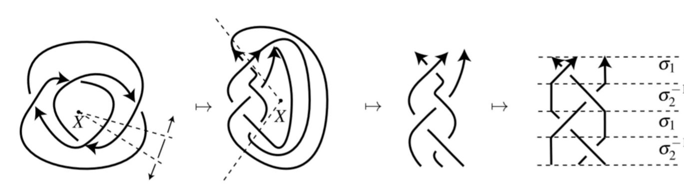
Given a knot K, we can consider the schematic:

$$\begin{tikzcd}
 & \underbrace{J_N(K^*;q)}_{\text{cyclotomic}} \arrow[r, squiggly, "x=-1", "q=\zeta_N"'] & U(x;q) \arrow[dd, dashed, leftrightarrow, "q\mapsto q^{-1}"] \\
K \arrow[ur] \arrow[dr] & & \\
 & \underbrace{J_N(K;q)}_{\text{non-cyclotomic}} \arrow[r, squiggly, "q=\zeta_N"] & F(q)
\end{tikzcd}$$

The Jones polynomial via braid group representations

Let $\mathfrak{B}_n$ be the ***braid group on n strands***.





The Jones polynomial via braid group representations

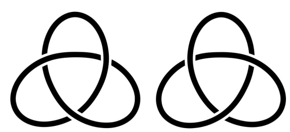
We can define a representation $\rho : \mathfrak{B}_n \to GL(V^{\otimes n})$.

Definition [Jones, 1984]

Let K be a knot, and let $B \in \mathfrak{B}_n$ be a braid such that $K = \overline{B}$. The ***Jones polynomial*** of K is defined

$$J(K) = q^{-3\omega(K)/4} \cdot \mathrm{Tr}(\rho(B) \circ \mu^{\otimes n})$$

The Jones polynomial is powerful!



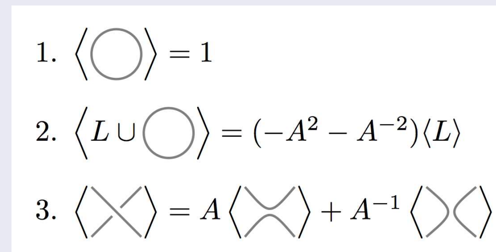
The Jones polynomial via the Kauffman bracket

Theorem [Kauffman, 1987]

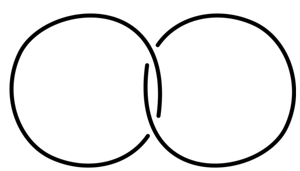
The Jones polynomial can be computed

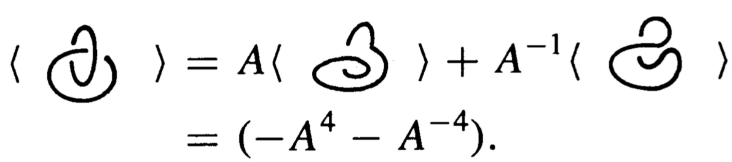
$$J(K) = \Big[(-A^3)^{-\omega(K)}\langle K\rangle\Big]_{q=A^4}$$

where $\langle K\rangle$ is the ***Kauffman bracket*** of K, defined by the local relations



The Jones polynomial of the Hopf link





The colored Jones polynomial

The ***Nth colored Jones polynomial*** is obtained from a similar braid representation $\rho : \mathfrak{B}_n \to GL(V^{\otimes n})$, where $\dim V = N$.

Example

We have $J_2(K) = J(K)$.

Theorem

The Nth colored Jones polynomial can be computed

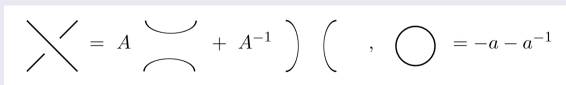
$$J_N(K) = (-1)^{N-1}\langle K(e_{N-1})\rangle$$

where $K(e_{N-1})$ is the knot K cabled with the element e_{N-1}.

Skein theory

Definition

The ***Kauffman bracket skein algebra*** of a 3-manifold M, denoted $\mathcal{K}(M)$, is the free $\mathbb{Z}[A, A^{-1}]$-algebra generated by isotopy classes of embeddings $\coprod S^1 \hookrightarrow M$ modulo the local relation



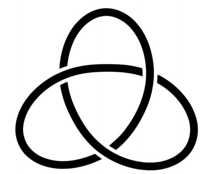
There are distinguished elements $e_0, e_1, \ldots \in \mathcal{K}(S^1 \times D^2)$ defined by the recursive relations

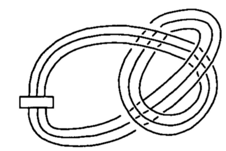
$$\begin{aligned} e_0 &= 1 \\ e_1 &= z \\ e_i &= ze_{i-1} - e_{i-2}. \end{aligned}$$

These are the ***Jones-Wenzl Idempotents***.

Skein theory

To find $J_N(K)$, we must then evaluate the bracket $\langle K(e_{N-1})\rangle$.

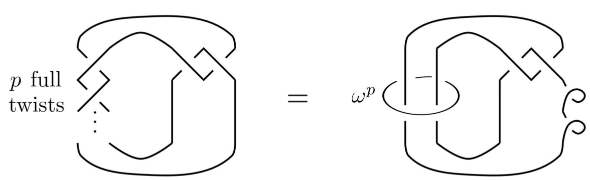




Computing the colored Jones polynomial

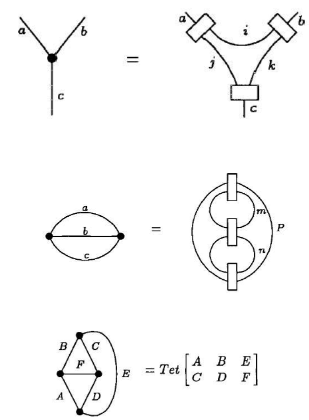
We need to compute the $\mathcal{U}_t$ series for $T(3, 2^t)$.

- Tools: *skein theory* and *graphical calculus*.
- Use a special element ω in the *skein algebra* to resolve twist regions in the knot. The resulting diagram is computationally simpler.



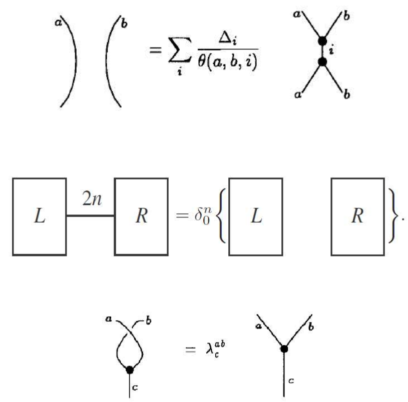
Graphical Calculus

Basic elements in graphical calculus: trivalent graphs, θ-net, and the tetrahedron net.



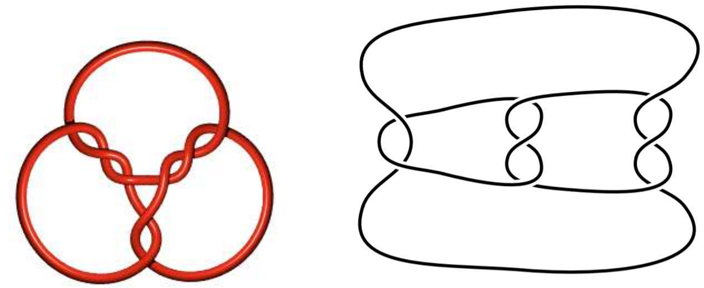
Graphical calculus

Common techniques in graphical calculus: *fusion*, *splitting*, and *resolving half-twists*



Computation of T(3,4)

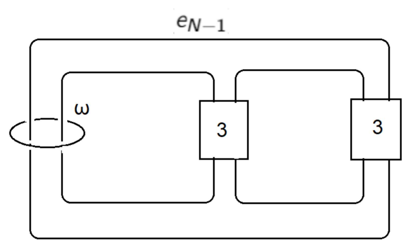
We write $T(3,4)$ as the pretzel knot $T(-2,3,3)$.



$P(-2,3,3)$

Computation of T(3,4)

First, we spot a
full twist and apply ω,



The Kauffmann bracket of this configuration is
$\langle K(e_{N-1})\rangle = \langle \omega, \widetilde{K}(e_{N-1})\rangle$.

Computation of T(3,4)

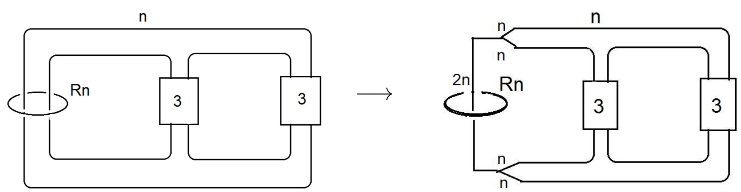
- According to the existing literature,

$$\begin{aligned}\langle \omega, \widetilde{K}(e_{N-1})\rangle &= \sum_{n=0}^{N-1}\sum_{k=0}^{\infty}(-1)^{N-1-n}\begin{bmatrix} N+n \\ N-1-n\end{bmatrix} c_k\langle R_k, \tilde{K}(R_n)\rangle \\ &= \sum_{n=0}^{N-1}(-1)^{N-1-n}\begin{bmatrix} N+n \\ N-1-n\end{bmatrix} c_k\langle {\color{red}R_n}, \tilde{K}(R_n)\rangle \\ &= \sum_{n=0}^{N-1}(-1)^{N-1-n}\begin{bmatrix} N+n \\ N-1-n\end{bmatrix} c_k\langle R_n, \tilde{K}(e_n)\rangle.\end{aligned}$$

- Notice the cancellation property in the second line.
- Such an expression gives the desired cyclotomic expansion.

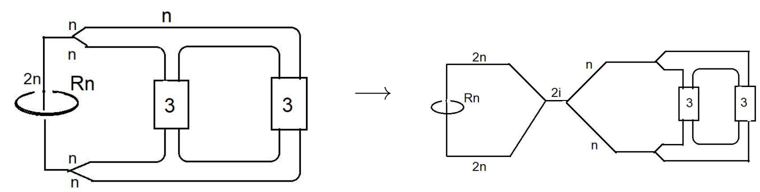
Computation of T(3,4)

For $\langle R_n, \widetilde{K}(e_n)\rangle$, we apply fusion to the following diagram:



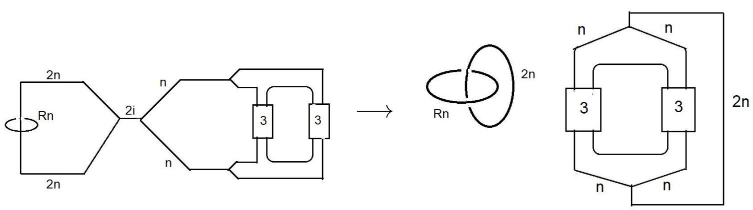
Computation of T(3,4)

We apply another fusion:



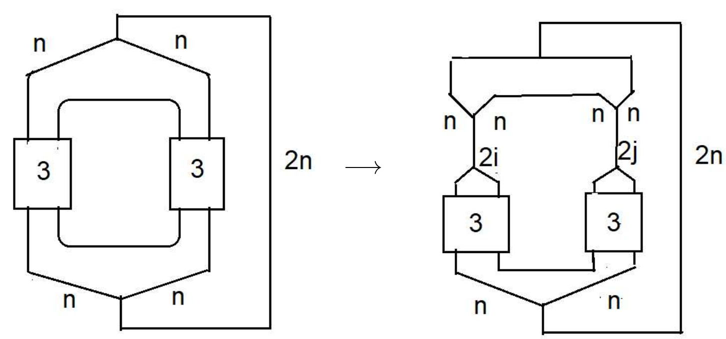
Computation of T(3,4)

And split the diagrams into two parts:



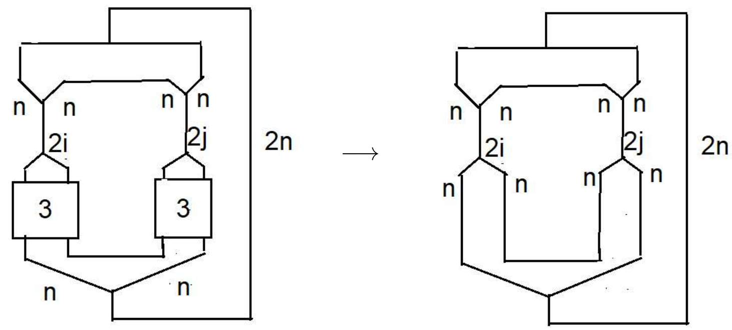
Computation of T(3,4)

Apply two more fusions:



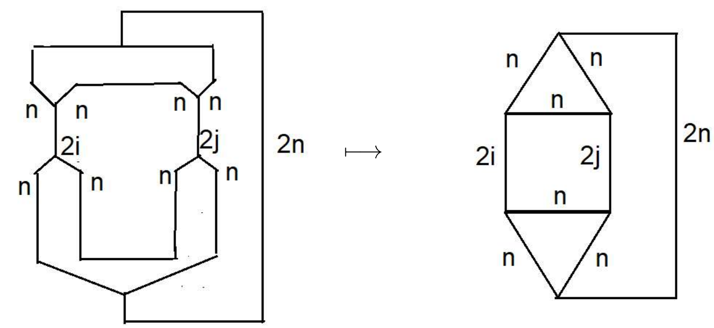
Computation of T(3,4)

Then resolve all the half-twists:



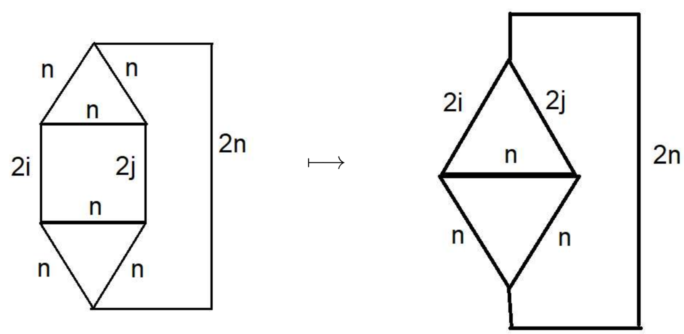
Computation of T(3,4)

A graph isomorphism yields the following:



Computation of T(3,4)

The relation between the following two diagrams can be found in the existing literature:



Computation of T(3,4)

By putting every piece together, we have

$$\langle R_n, \tilde{K}(e_n)\rangle = \frac{\langle R_n, e_{2n}\rangle}{\langle e_{2n}\rangle} \sum_{i=0}^{n} \sum_{j=n-i}^{n} \frac{\langle e_{2j}\rangle\langle e_{2i}\rangle\theta(2n,2i,2j)}{\theta(n,n,2i)\theta(n,n,2j)}$$
$$\times \gamma(2i;n,n)^{3}\gamma(2j;n,n)^{3}\left(\frac{[i]![j]!}{[n]![i+j-n]!}\right)^2.$$

The exponents in red are the number of half twists we have resolved.

Analysis of computation

- Our formula of the colored Jones polynomial for T(3,4) does not match with the correct one in Mathematica's Knot Data package.
- The original literature has a *hidden* assumption that the component of the knot that is encircled by ω should be isotopic to the unknot. This is the only situation in which the cancellation property holds.
- We can still write down a correct formula, but it is not yet clear how to obtain the desired cyclotomic expansion.

The $\mathcal{F}_t$ series and number theory

There are two main results regarding the $\mathcal{F}$-series:

1. $\mathcal{F}_t$ satisfies the 'strange' identity

$$\mathcal{F}_t(q) \text{ ``$=$'' } -\frac{1}{2}\sum_{n\geq 0} n\chi_{3\cdot 2^{t+1}}(n) q^{\frac{n^2-(3-2^{t+1})^2}{3\cdot 2^{t+2}}}$$

   where χ is some periodic function.

2. The strange identity implies certain congruence results for the coefficients of $\mathcal{F}_t(1-q) = \sum_{n\geq 0} \gamma^{(t)}(n) q^n$.

For example, we will expect to see congruences of the form $\gamma^{(t)}(5n+4) \equiv 0 \pmod 5$ for all $n \in \mathbb{N}$.

The bigger picture here is that congruences of this form are symptomatic of certain modular properties.

Non-cyclotomic colored Jones polynomial

We have an expression for the colored Jones polynomial of $T_{(3,2^t)}$:

$$J_N(T_{(3,2^t)}, q) = (-1)^{h''(t)} q^{2^t-1-N} \sum_{n\geq 0} (q^{1-N})_n q^{-Nnm(t)}$$
$$\times \sum_{3\sum_{l=1}^{m(t)-1} j_l l \equiv 1[m(t)]} (-q^{-N})^{\sum_{l=1}^{m(t)-1} j_l} q^{\frac{-a(t)-h'(t)m(t)+\sum_{l=1}^{m(t)-1} j_l l}{m(t)} + \sum_{l=1}^{m(t)-1}\binom{j_l}{2}}$$
$$\times \sum_{k=0}^{m(t)-1} q^{-kN} \prod_{l=1}^{m(t)-1} \begin{bmatrix} n+\chi(l\leq k) \\ j_l \end{bmatrix}$$

where $a(t), m(t), h(t)$ are natural numbers depending on t.

The series $\mathcal{F}_t$

We define our series $\mathcal{F}_t$ to be:

$$\mathcal{F}_t(q) := (-1)^{h''(t)} \sum_{n\geq 0} (q)_n \sum_{3\sum_{l=1}^{m(t)-1} j_l l \equiv 1[m(t)]} (-1)^{\sum_{l=1}^{m(t)-1} j_l}$$
$$\cdot\, q^{\frac{-a(t)-h'(t)m(t)+\sum_{l=1}^{m(t)-1} j_l l}{m(t)} + \sum_{l=1}^{m(t)-1}\binom{j_l}{2}} \sum_{k=0}^{m(t)-1} \prod_{l=1}^{m(t)-1} \begin{bmatrix} n+\chi(l\leq k) \\ j_l \end{bmatrix}$$

Note that $\mathcal{F}_t$ is not well defined as a function, or even as a power series in q. It is only defined for q a root of unity, or as a power series when evaluated at $1-q$.

Proving the strange identity

The idea is to write down a well-defined power series that agrees with $\mathcal{F}_t$, and the RHS of the strange identity in the limit $q \to \zeta$.

- The first obstacle is showing that this new series is related to both sides of the strange identity.
- Then, algebraic manipulation of the series and some formal differentiation yields a useful lemma in the form of a q-series identity.

There is one assumption remaining regarding the convergence of an infinite series of polynomials.

A simple q-series identity

Lemma*

$$\frac{1}{2}\sum_{n\geq 0} n\chi_{3\cdot 2^{t+1}}(n) q^{\frac{n^2-(2^{t+1}-3)^2}{3\cdot 2^{t+1}}} - \frac{2^{t+1}-3}{2}(q^{2^{t+1}}, q^{2^t-1}, q^{2^t+1}; q^{2^{t+1}})_\infty (q^{2^{t+2}-2}, q^2; q^{2^{t+2}})_\infty$$

$$= -(q)_\infty \left(\sum_{i=1}^{\infty} \frac{q^i}{1-q^i}\right)(-1)^{h''(t)} \sum_{n\geq 0} b_n^{(t)}(q) + (q)_\infty (-1)^{h''(t)} \sum_{n\geq 0}(n-h(t)) b_n^{(t)}(q)$$

$$- (-1)^{h''(t)} \sum_{n\geq 0} [(q)_n - (q)_\infty] \sum_{3\sum_{l=1}^{m(t)-1} j_l l \equiv 1[m(t)]} (-1)^{\sum_{l=1}^{m(t)-1} j_l}$$

$$\times q^{\frac{-a(t)-h'(t)m(t)+\sum_{l=1}^{m(t)-1} j_l l}{m(t)} + \sum_{l=1}^{m(t)-1}\binom{j_l}{2}} \sum_{k=0}^{m(t)-1} \prod_{l=1}^{m(t)-1} \begin{bmatrix} n+\chi(l\leq k) \\ j_l \end{bmatrix}$$

And now, taking $q \to \zeta$ on both sides of the lemma gives the strange identity.

Congruences

It turns out that functions satisfying strange-like identities have a very specific property:

- We can make a divisibility statement on the polynomials appearing in the s-dissection of $\mathcal{F}_t(1-q)$.
- Using properties of binomial coefficients, the theorems of Lucas and Kummer, we can show the desired congruence result.

Main result

Theorem*

Let $t \geq 2$ be an integer and write $\mathcal{F}_t(1-q) = \sum_{n\geq 0} \gamma^{(t)}(n) q^n$

Let $p \geq 5$ be a prime. If $j \in \{1, 2, ..., p-1-\max S_t(p)\}$, then

$$\gamma^{(t)}(p^r n - j) \equiv 0 \pmod{p^r}$$

For all r, $n \in \mathbb{N}$.

* We still have the assumption that was required for the Lemma.

And again, the presence of these congruences points to modular properties of the $\mathcal{F}_t$ series.

Examples

For example, here are some of the sets $S_3(p)$:

- $S_3(5) = \{0, 1, 2\}$
- $S_3(7) = \{0, 2, 3, 4\}$
- $S_3(13) = \{0, 2, 5, 6, 7, 8, 11\}$

Which yield the congruences:

- $\gamma^{(3)}(5n-1) \equiv \gamma^{(3)}(5n-2) \equiv 0 \pmod 5$
- $\gamma^{(3)}(7n-1) \equiv \gamma^{(3)}(7n-2) \equiv 0 \pmod 7$
- $\gamma^{(3)}(13n-1) \equiv 0 \pmod{13}$

Numerical evidence of congruences

We wanted to obtain numerical evidence of the congruences in the coefficients of $F_t(1-q)$ for $t > 2$.

Mathematica

- Computation took a long time.
- Was only able to compute first 21 terms.

Python

- Used Simpy & Multiprocessing to try to compute more terms.
- Slow and used a very large amount of memory.

Numerical evidence of congruences

C++ with NTL library

- The NTL library has data structures and algorithms for arbitrary length signed integers and polynomials with arbitrary length signed integer coefficients.
- We were dealing with large integers: the coefficient of q^{50} in $F_3(1-q)$ was 964156696491543336297427280538550742787805588409492484118 15764941235391015641348121186
- NTL's polynomial arithmetic is very fast and its data structures allow for efficient storage of large polynomials.
- Coupled with minimizing creation of temporary variables in loops we were able to produce a program which uses very little memory.

Numerical evidence of congruences

- The outer sum of $F_3(1-q)$ was split into m threads and run in parallel on one of Google's N2 Servers.
- The program calculated the first 51 terms of $F_3(1-q)$ and tested for the presence of prime and prime power congruences.

Evidence of prime power congruences:

Expected results:

- 5n+3, 5n+4
- 7n+5, 7n+6
- 11n+8, 11n+9, 11n+10
- 13n+12
- 25n+23, 25n+24

Numerical results:

- 5n+3, 5n+4
- 7n+4, 7n+5, 7n+6
- 11n+8, 11n+9, 11n+10
- 13n+12
- 25n+23, 25n+24

Conclusion

Given a knot K, we can consider the schematic

$$\begin{tikzcd}
 & \underbrace{J_N(K^*;q)}_{\text{cyclotomic}} \arrow[r, squiggly, "x=-1", "q=\zeta_N"'] & U(x;q) \arrow[r, squiggly] & \text{'mock modular form'} \arrow[d, dashed, leftrightarrow, "q\mapsto q^{-1}"] \\
K \arrow[ur] \arrow[dr] & & & \\
 & \underbrace{J_N(K;q)}_{\text{non-cyclotomic}} \arrow[r, squiggly, "q=\zeta_N"] & F(q) \arrow[r, squiggly] & \text{'quantum modular form'}
\end{tikzcd}$$

Thank you!

Project Report - Numerical Computations

Colin reports the following results.

Summary

We wanted to obtain numerical evidence of prime power congruences in the coefficients of $F_t(1-q)$ for $t > 2$. Targeting $t = 3$, we tried to compute terms of $F_3(1-q)$ using Mathematica to test for prime power congruences. Computation took a long time and we were only able to compute the first 21 terms.

We used Simpy (a Python package) & Multiprocessing to try to compute more terms. Unfortunately, this was slow and used a very large amount of memory.

In order to further investigate the presence of prime power congruences for the F series, a program was written in C++ using the NTL library. The NTL library has data structures and algorithms for arbitrary length signed integers and polynomials with arbitrary length signed integer coefficients. This was necessary as we were dealing with large integers: the coefficient of q^{50} in $F_3(1-q)$ was

$$\begin{gathered}9641566964915433362974272805385507427878055884094 \\ 924841181576494123235391015641348121 86\end{gathered}$$

NTLs polynomial arithmetic is very fast and its data structures allow for efficient storage of large polynomials. Coupled with minimizing creation of temporary variables in loops we were able to produce a program which uses very little memory. The outer sum of $F_3(1-q)$ was split into m threads and run in parallel on one of Google's N2 Servers. The program calculated the first 51 terms of $F_3(1-q)$ and tested for the presence of prime and prime power congruences.

Expected results:

- 5n+3, 5n+4
- 7n+5, 7n+6
- 11n+8, 11n+9, 11n+10
- 13n+12
- 25n+23, 25n+24

Numerical results:

- 5n+3, 5n+4
- 7n+4, 7n+5, 7n+6
- 11n+8, 11n+9, 11n+10
- 13n+12

- 25n+23, 25n+24

We were pleased to see very close agreement between expected results and numerical results.

Project Report - Prime Power Congruences

Introduction

Recently a formula was given for the colored Jones polynomial by Konan [3] for an infinite family of Torus knots $T(3,2^t)$, $t \geq 2$. This formula allows for the definition of a q-series which agrees, up to some power of q, with it at roots of unity. This so-called $\mathcal{F}$-series generalizes the Kontsevich-Zagier strange series:

$$F(q) := \sum_{n\geq 0}(q)_n$$

Which appears similarly from the colored Jones polynomial of the Trefoil.
$(q)_n$ denotes as usual the q-Pochhamer symbol.

In particular, we have from [3] that the colored Jones polynomial corresponding to $T(3,2^t)$ is:

$$\begin{aligned} J_N(T_{(3,2^t)},q) =&(-1)^{h''(t)}q^{2^t-1-N}\sum_{n\geq 0}(q^{1-N})_n q^{-Nnm(t)} \\ &\times \sum_{3\sum_{l=1}^{m(t)-1} j_l l \equiv 1[m(t)]} (-q^{-N})^{\sum_{l=1}^{m(t)-1} j_l} q^{\frac{-a(t)-h'(t)m(t)+\sum_{l=1}^{m(t)-1} j_l l}{m(t)} + \sum_{l=1}^{m(t)-1}\binom{j_l}{2}} \\ &\times \sum_{k=0}^{m(t)-1} q^{-kN} \prod_{l=1}^{m(t)-1} \begin{bmatrix} n+\chi(l\leq k) \\ j_l \end{bmatrix} \end{aligned} \tag{1}$$

We define the corresponding $\mathcal{F}$-series to be:

$$\begin{aligned} \mathcal{F}_t(q) :=&(-1)^{h''(t)}\sum_{n\geq 0}(q)_n \sum_{3\sum_{l=1}^{m(t)-1} j_l l \equiv 1[m(t)]} (-1)^{\sum_{l=1}^{m(t)-1} j_l} \\ &\cdot q^{\frac{-a(t)-h'(t)m(t)+\sum_{l=1}^{m(t)-1} j_l l}{m(t)} + \sum_{l=1}^{m(t)-1}\binom{j_l}{2}} \sum_{k=0}^{m(t)-1} \prod_{l=1}^{m(t)-1} \begin{bmatrix} n+\chi(l\leq k) \\ j_l \end{bmatrix} \end{aligned} \tag{2}$$

Clearly, we have the equality $\mathcal{F}_t(\zeta_N) = \zeta_N^{2^t-1} J_N(T_{(3,2^t)},\zeta_N)$ as desired, where ζ_N is an N^{th} root of unity. It is worth pointing out, however, that just as with the Kontsevich-Zagier series, $\mathcal{F}_t(q)$ is not defined as a formal power series in q anywhere other than at roots of unity. When q is a root of unity the sum in fact terminates and is finite. We also have that $\mathcal{F}_t(1-q)$ is a formal power series in q.

The goal of the following report is to prove congruence results on the coefficients $\gamma(n)$ of the power series expansion $\mathcal{F}_t(1-q) = \sum_{n\geq 0}\gamma(n)q^n$. This result is in analogy to the case of the Kontsevich-Zagier series and the trefoil.

The general idea of the proof will be to establish the so-called strange identity, and use a result of Ahlgren-Kim-Lovejoy [1] along with the ideas of Straub [4] to prove the congruences.

The Function $H_t(x)$

The motivation of this section is to lay the groundwork for the derivation of the strange identity that follows. In particular we want to use a theorem proved by Ahlgren, Kim, and Lovejoy in [1].

let $F(q)$ be a function of the form $F(q) = \sum_{n\geq 0} (q)_n f_n(q)$, where the $f_n(q)$ are polynomials.

And let $P^{(\nu)}_{a,b,\chi}(q) = \sum_{n\geq 0} n^\nu \chi(n) q^{\frac{n^2-a}{b}}$ where $\nu \in \{0,1\}$, $a \geq 0$ and $b > 0$ are integers, and $\chi : \mathbb{Z} \to \mathbb{C}$ is a function satisfying the following two properties

1. $\chi(n) \neq 0$ only if $\frac{n^2-a}{b} \in \mathbb{Z}$
2. For every root of unity ζ, the function $n \mapsto \chi(n)\zeta^{\frac{n^2-a}{b}}$ is periodic with mean value 0

Consider the partial sums $F(q,N) = \sum_{n=0}^{N} (q)_n f_n(q)$

And their s-dissections: $F(q,N) = \sum_{n=0}^{s-1} q^n A_{F,s}(N,i,q^s)$

Finally, define the set $S_{a,b,\chi}(s) := \{\frac{n^2-a}{b} \pmod{s} : \chi(n) \neq 0\}$. Then the theorem is as follows:

Theorem. *Suppose that for each root of unity ζ we have the asymptotic expansion*

$$P^{(\nu)}_{a,b,\chi}(\zeta e^{-t})\ F(\zeta e^{-t} \text{ as } t \to 0^+$$

Suppose that s and N are positive integers and that $i \notin S_{a,b,\chi}(s)$. Then we have

$$(q)_{\lambda(N,s)} | A_{F,s}(N,i,q)$$

Where $\lambda(N,s) = \lfloor \frac{N+1}{s} \rfloor$

With this in mind we are able to define our function

$$H_t(x) = H_t(x,q) := \sum_{n\geq 0} \chi_{3\cdot 2^{t+1}} q^{\frac{n^2-(2^{t+1}-3)^2}{3\cdot 2^{t+2}}} x^{\frac{n-(2^{t+1}-3)}{2}} \tag{3}$$

Where $\chi_{3,2^{t+1}}(n) = \begin{cases} 1 & n \equiv 3-2^{t+1}, 2^{t+3}-3 \pmod{3\cdot 2^{t+1}} \\ -1 & n \equiv 2^{t+2}-3, 2^{t+1}+3 \pmod{3\cdot 2^{t+1}} \\ 0 & \text{Otherwise} \end{cases}$

The function $\chi(n)$

First, I will verify that $\chi_{3\cdot 2^t}(n)$ satisfies the conditions required by Theorem for $t \geq 3$.

We need to verify that the corresponding power of q, $\frac{n^2-(3-2^t)^2}{3\cdot 2^{t+1}}$, is an integer if $\chi_{3\cdot 2^t}(n) \neq 0$.

(i) Consider the case $n \equiv 3 - 2^t \pmod{3 \cdot 2^t}$. Write $n = 3 - 2^t + 3m2^t$.
Then we have: $\frac{n^2-(3-2^t)^2}{3\cdot 2^{t+1}} = \frac{(3-2^t+3m2^t)^2-(3-2^t)^2}{3\cdot 2^{t+1}} = 3 \cdot 2^{t-1}m^2 + m(3-2^t) \in \mathbb{Z}$

(ii) If $n \equiv 2^{t+2} - 3$, write $n = 2^{t+2} - 3 + 3m2^t$. Then we have:
$\frac{n^2-(3-2^t)^2}{3\cdot 2^{t+1}} = \frac{(2^{t+2}-3+3m2^t)^2-(3-2^{t+1})^2}{3\cdot 2^{t+1}} = 3 \cdot 2^{t-1}m^2 + m(2^{t+2}-3) + 5 \cdot 2^{t-1} - 3 \in \mathbb{Z}$

(iii) If $n \equiv 2^{t+1} - 3$, write $n = 2^{t+1} - 3 + 3m2^t$. Then we have:
$\frac{n^2-(3-2^t)^2}{3\cdot 2^{t+1}} = \frac{(2^{t+1}-3+3m2^t)^2-(3-2^t)^2}{3\cdot 2^{t+1}} = 3 \cdot 2^{t-1}m^2 + m(2^{t+1}-3) + 2^{t-1} - 1 \in \mathbb{Z}$

(iv) If $n \equiv 2^t + 3$, write $n = 2^t + 3 + 3m2^t$. Then we have:
$\frac{n^2-(3-2^t)^2}{3\cdot 2^{t+1}} = \frac{(2^t+3+3m2^t)^2-(3-2^t)^2}{3\cdot 2^{t+1}} = 3 \cdot 2^{t-1}m^2 + m(2^t+3) + 2 \in \mathbb{Z}$

So we see that the first condition is satisfied.

Now, let $\varphi(n) := \chi_{3\cdot 2^t}(n)\zeta^{\frac{n^2-(3-2^t)^2}{3\cdot 2^{t+1}}}$, where ζ is a primitive N^{th} root of unity. What remains is to check that ϕ is periodic with mean value 0.
Periodicity is obvious. $\chi_{3\cdot 2^t}(n)$ is by definition periodic with period $3 \cdot 2^t$, and the function $n \to \zeta^{\frac{n^2-(3-2^t)^2}{3\cdot 2^{t+1}}}$ is periodic with period $3 \cdot 2^t N$. To see the latter, consider
$\zeta^{\frac{(n+3\cdot 2^t)^2-(3-2^t)^2}{3\cdot 2^{t+1}}} = \zeta^{\frac{n^2-(3-2^t)^2+3\cdot 2^{t+1}N+9\cdot 2^{2t}N^2}{3\cdot 2^{t+1}}} = \zeta^{\frac{n^2-(3-2^t)^2}{3\cdot 2^{t+1}}}\zeta^{\frac{3\cdot 2^{t+1}N(1+3\cdot 2^{t-1}N)}{3\cdot 2^{t+1}}} = \zeta^{\frac{n^2-(3-2^t)^2}{3\cdot 2^{t+1}}}\zeta^{N(1+3\cdot 2^{t-1}N)} = \zeta^{\frac{n^2-(3-2^t)^2}{3\cdot 2^{t+1}}}$.
It then follows that $\varphi(n)$ is periodic with period $3 \cdot 2^t N$.

It just remains to show that φ has mean value 0. That is, $\sum\limits_{n=0}^{3\cdot 2^t N} \varphi(n) = 0$.
Consider first the case where N is even. Then we have
$\chi_{3\cdot 2^t}(n + 3 \cdot 2^{t-1}N) = \chi_{3\cdot 2^t}(n)$.
Then, noting that φ is supported only on odd integers, we may assume that n is odd. Consider
$\zeta^{\frac{(n+3\cdot 2^{t-1}N)^2-(3-2^t)^2}{3\cdot 2^{t+1}}} = \zeta^{\frac{n^2-(3-2^t)^2}{3\cdot 2^{t+1}}}\zeta^{\frac{3\cdot 2^t Nn+9\cdot 2^{2t-2}N}{3\cdot 2^{t+1}}} = \zeta^{\frac{n^2-(3-2^t)^2}{3\cdot 2^{t+1}}}\zeta^{\frac{Nn}{2}}\zeta^{3\cdot 2^{t-3}N^2} = \zeta^{\frac{n^2-(3-2^t)^2}{3\cdot 2^{t+1}}}\zeta^{\frac{Nn}{2}}$. Then note that since n is odd, $\zeta^{Nn}2 = -1$. Thus we have:
$\zeta^{\frac{(n+3\cdot 2^{t-1}N)^2-(3-2^t)^2}{3\cdot 2^{t+1}}} = -\zeta^{\frac{n^2-(3-2^t)^2}{3\cdot 2^{t+1}}}$.
Finally, this gives us $\varphi(n + 3 \cdot 2^{t-1}N) = -\varphi(n)$. And this is sufficient for the desired result, $\sum\limits_{n=0}^{3\cdot 2^t N} \varphi(n) = 0$.

We must now consider the case where N is odd. The sum we are interested in is $\sum\limits_{n=0}^{3\cdot 2^t N} \varphi(n)$. This can be broken up into 4 sums, each of which sums over a congruence class (mod $3 \cdot 2^t$). Write

$$\begin{aligned}\sum_{n=0}^{3\cdot 2^t N} \varphi(n) = \sum_{m=0}^{N-1} \zeta^{3\cdot 2^{t-1}m^2+(3-2^t)m} - \sum_{m=0}^{N-1} \zeta^{3\cdot 2^{t-1}m^2+(2^{t+1}-3)m+2^{t-1}-1} \\ + \sum_{m=0}^{N-1} \zeta^{3\cdot 2^{t-1}m^2+(2^{t+2}-3)m+5\cdot 2^{t-1}-3} - \sum_{m=0}^{N-1} \zeta^{3\cdot 2^{t-1}m^2+(2^t+3)m+2}\end{aligned}$$

Claim: a) $\sum_{m=0}^{N-1} \zeta^{3\cdot 2^{t-1}m^2+(3-2^t)m} = \sum_{m=0}^{N-1} \zeta^{3\cdot 2^{t-1}m^2+(2^{t+1}-3)m+2^{t-1}-1}$

b) $\sum_{m=0}^{N-1} \zeta^{3\cdot 2^{t-1}m^2+(2^{t+2}-3)m+5\cdot 2^{t-1}-3} = \sum_{m=0}^{N-1} \zeta^{3\cdot 2^{t-1}m^2+(2^t+3)m+2}$

Proof: a) The key observation to make is that since we are summing over a full period, it doesn't actually matter that we sum from $m = 0$ to $m = N - 1$, so long as we hit N consecutive values of m. The idea is to shift the index of one of the sums so that we get the same power of ζ in the two sums we wish to be equal. The shift we are looking for is of the form $m \to m + i$ for some i, and we note that we actually only care about the power of ζ mod N, since ζ is an N^{th} root of unity. Let's look at the exponent in the second sum under such a shift.

We have : $3 \cdot 2^{t-1}(m + i)^2 + (2^{t+1} - 3)m + 2^{t-1} - 1$.
Collecting terms by degree in m gives:
$3 \cdot 2^{t-1}m^2 + (3 \cdot 2^t i + 2^{t+1} - 3)m + 3 \cdot 2^{t-1}i^2 + (2^{t+1} - 3)i + 2^{t-1} - 1$
We want this to be congruent to the exponent in the first sum (mod N). The coefficients on m^2 are equal, and so obviously congruent for every i. Equating the coefficients of m gives the congruence equation:
$3 \cdot 2^t i + 2^{t+1} - 3 \equiv 3 - 2^t \pmod{N}$. Rearranging yields
$6(2^{t-1}i + 2^{t-1} - 1) \equiv 0 \pmod{N}$.
This clearly has a solution when $2^{t-1}i \equiv 1 - 2^{t-1} \pmod{N}$. And since we know that N is odd, we have $(N, 2^{t-1}) = 1$, so this equation has a solution, call it i_N. It remains to verify that i_N also satisfies the desired congruence for the constant term. We want $3 \cdot 2^{t-1}i^2 + (2^{t+1} - 3)i + 2^{t-1} - 1 \equiv 0 \pmod{N}$.
Substituting i_N gives:

$$\begin{aligned}
3i_N \cdot 2^{t-1}i_n &+ 4 \cdot 2^{t-1}i_N - 3i_N + 2^{t-1} - 1 \\
&\equiv 3i_N(1 - 2^{t-1}) + 4(1 - 2^{t-1}) - 3i_N + 2^{t-1} - 1 \\
&= -3 \cdot 2^{t-1}i_N + 3 - 3 \cdot 2^{t-1} \\
&\equiv -3(1 - 2^{t-1}) + 3 - 3 \cdot 2^{t-1} = 0
\end{aligned}$$

Where all of the equivalences are taken mod N. So we have the desired result, and shifting the second sum by i_N makes it explicitly equal to the first.

b) The second part of the claim follows very similarly to the first. We again are looking for a shift of the form $m \to m + i$. So let's look at the exponent in the second sum under such a shift. Following similar steps to above we arrive at the congruence equation $6(2^{t-1}i + 1 - 2^{t-1}) \equiv 0 \pmod{N}$. And this clearly holds for i_N solution to $2^{t-1}i \equiv 2^{t-1} - 1 \pmod{N}$. The existence of such a solution is guaranteed by the fact that $(N, 2^{t-1}) = 1$.
Again, it remains to verify that i_N satisfies the desired congruence in the constant term. We want $3 \cdot 2^{t-1}i^2 + (2^t + 3)i + 2 \equiv 5 \cdot 2^{t-1} - 3$. Substituting in i_N into the LHS gives:

$$\begin{aligned}
3i_N \cdot 2^{t-1}i_N &+ 2 \cdot 2^{t-1}i_N + 3i_N + 2 \\
&\equiv 3i_N(2^{t-1} - 1) + 2(2^{t-1} - 1) + 3i_N + 2 \\
&\equiv 3(2^{t-1} - 1) + 2 \cdot 2^{t-1} \\
&= 5 \cdot 2^{t-1} - 3
\end{aligned}$$

And we see that the shift by i_N explicitly makes the two sums equal. $\square$

From this claim we see directly that $\varphi(n)$ has mean value 0 for N odd. So we've shown that $\varphi(n)$ has mean value 0 for any choice of $N \in \mathbb{N}$.
Thus, $\chi_{3\cdot 2^t}(n)$ has all of the desired properties.

A proposition

We need to establish a connection between our function $H_t(x)$ and the series $\mathcal{F}_t(q)$ that we wish to derive the strange identity for. This is done by the following proposition.

Proposition.

$$\begin{aligned} H_t(x) = &(-1)^{-h''(t)} q^{-h'(t)} x^{-h(t)} \sum_{n\geq 0} (x)_{n+1} x^{nm(t)} \\ &\times \sum_{3\sum_{l=1}^{m(t)-1} j_l l \equiv 1[m(t)]} (-x)^{\sum_{l=1}^{m(t)-1} j_l} q^{\frac{-a(t)+\sum_{l=1}^{m(t)-1} j_l l}{m(t)} + \sum_{l=1}^{m(t)-1} \binom{j_l}{2}} \\ &\times \sum_{k=0}^{m(t)-1} x^k \prod_{l=1}^{m(t)-1} \begin{bmatrix} n + \chi(l \leq k \\ j_l \end{bmatrix} \end{aligned}$$

Note that the χ appearing in the product on the RHS is not to be confused with $\chi_{3,2^{t+1}}$, instead it is a logical indicator function, taking the value 1 if its input is true, and 0 otherwise.
Also, note that the RHS of the proposition is the RHS of (3.3.26) in [3].

Proof of Proposition: Konan showed that the RHS satisfies the difference equation

$$f(x) = 1 - q^2 x^3 - q^{2^t-1} x^{2^t} + q^{3+2^t} x^{3+2^t} + q^{5\cdot 2^t - 3} x^{3\cdot 2^t} f(q^2 x)$$

It is thus sufficient to show that our function in Equation (3) satisfies the same equation.

First I want to find the canonical representatives of the congruence classes on which $\chi_{3,2^{t+1}}(n)$ depends. The representatives given for $\chi_{3,2^{t+1}}(n) = -1$ are already canonical, that is the given representatives lie between 0 and $3\cdot 2^{t+1}$. We can however rewrite the conditions for when $\chi_{3,2^{t+1}}(n) = 1$.

If $n \equiv 3 - 2^{t+1}$, we note that it is in the congruence class with canonical representative $3 + 2^{t+2}$. And we also note that the canonical representative of the class containing $2^{t+3} - 3$ is $2^{t+1} - 3$.

Now, we have $H_{3,2^t}(x) = \sum_{n\geq 0} \chi_{3,2^{t+1}}(n) q^{\frac{n^2-(2^{t+1}-3)^2}{3\cdot 2^{t+2}}} x^{\frac{n-(2^{t+1}-3)}{2}}$. Let's pull out the non-zero terms occurring in the interval $0 \leq n < 3\cdot 2^{t+1}$. These are precisely the 4 terms obtained by taking n equal to the canonical representatives of the above congruence classes.

(i) For $n = 2^{t+1} - 3$, we get the term: $1 \cdot q^0 \cdot x^0 = 1$

(ii) For $n = 2^{t+1}+3$, we get the term: $-q^{\frac{(2^{t+1}+3)^2-(t^{t+1}-3)^2}{3\cdot 2^{t+2}}} x^{\frac{2^{t+1}+3-2^{t+1}+3}{2}} = -q^{\frac{3\cdot 2^{t+3}}{3\cdot 2^{t+2}}} x^3 = -q^2x^3$

(iii) For $n = 2^{t+2}-3$, we get the term: $-q^{\frac{(2^{t+2}-3)^2-(2^{t+1}-3)^2}{3\cdot 2^{t+2}}} x^{\frac{2^{t+2}-3-2^{t+1}+3}{2}} = -q^{\frac{3\cdot 2^{2t+2}-3\cdot 2^{t+2}}{3\cdot 2^{t+2}}} x^{2^t} = -q^{2^t-1}x^{2^t}$

(iv) For $n = 2^{t+2}+3$, we get the term: $q^{\frac{(2^{t+2}+3)^2-(2^{t+1}-3)^2}{3\cdot 2^{t+2}}} x^{\frac{2^{t+2}+3-2^{t+1}+3}{2}} = q^{\frac{3\cdot 2^{2t+2}+9\cdot 2^{t+2}}{3\cdot 2^{t+2}}} x^{3+2^t} = q^{3+2^t}x^{3+2^t}$.

We can thus write:
$H_{3,2^t}(x) = 1 - q^2x^3 - q^{2^t-1}x^{2^t} + q^{3+2^t}x^{3+2^t} + \sum_{n\geq 3\cdot 2^{t+1}} \chi_{3,2^{t+1}}(n) q^{\frac{n^2-(t^{t+1}-3)^2}{3\cdot 2^{t+2}}} x^{\frac{n-(2^{t+1}-3)}{2}}$.
Looking now at the sum on the RHS, shift the index $n \to n + 3\cdot 2^{t+2}$, and noting that $\chi_{3,2^{t+1}}(n + 3\cdot 2^{t+1}) = \chi_{3,2^{t+1}}(n)$, we write the sum:
$\sum_{n\geq 0} \chi_{3,2^{t+1}}(n) q^{\frac{(n+3\cdot 2^{t+1})^2-(2^{t+1}-3)^2}{3\cdot 2^{t+2}}} x^{\frac{n+3\cdot 2^{t+1}-(2^{t+1}-3)}{2}}$.
By expanding and splitting the powers we see this is equal to:
$x^{3\cdot 2^t} \sum_{n\geq 0} \chi_{3,2^{t+1}}(n) q^{\frac{n^2-(2^{t+1}-3)^2}{3\cdot 2^{t+2}}} q^{\frac{3\cdot 2^{t+2}n+9\cdot 2^{2t+2}}{3\cdot 2^{t+2}}} x^{\frac{n-(2^{t+1}-3)}{2}}$.

Examining the 2nd power on q in this sum we see:
$\frac{3\cdot 2^{t+2}n+9\cdot 2^{2t+2}}{3\cdot 2^{t+2}} = n + 3\cdot 2^t = n + 2^{t+1} + 2^t = n - 2^{t+1} + 3 + 2^{t+2} + 2^t - 3 = n - (2^{t+1}-3) + 5\cdot 2^t - 3$.
We can thus write our sum as:
$q^{5\cdot 2^t-3}x^{3\cdot 2^t} \sum_{n\geq 0} \chi_{3,2^{t+1}}(n) q^{\frac{n^2-(2^{t+1}-3)^2}{3\cdot 2^{t+2}}} q^{n-(2^{t+1}-3)} x^{\frac{n-(2^{t+1}-3)}{2}} = q^{5\cdot 2^t-3}x^{3\cdot 2^t} H_{3,2^t}(q^2x)$.

We have thus the desired difference equation

$$H_{3,2^t}(x) = 1 - q^2x^3 - q^{2^t-1}x^{2^t} + q^{3+2^t}x^{3+2^t} + q^{5\cdot 2^t-3}x^{3\cdot 2^t} H_{3,2^t}(q^2x)$$

And this proves the proposition □

The Strange Identity

The goal here is to get an identity like the one used by Hikami and Kirillov in the derivation of the strange identity for the knot $T(3,4)$.

First, we rewrite the RHS of Proposition () as:

$$\begin{aligned}(-1)^{-h''(t)}(1-x)\sum_{n\geq 0}\left[(qx)_n - (qx)_\infty\right] x^{nm(t)} &\sum_{3\sum_{l=1}^{m(t)-1} j_l l \equiv 1[m(t)]} (-1)^{\sum_{l=1}^{m(t)-1} j_l} \\ &\cdot x^{-h(t)+\sum_{l=1}^{m(t)-1} j_l} \cdot q^{\frac{-a(t)-h'(t)m(t)+\sum_{l=1}^{m(t)-1} j_l l}{m(t)} + \sum_{l=1}^{m(t)-1}\binom{j_l}{2}} \\ &\cdot \sum_{k=0}^{m(t)-1} x^k \prod_{l=1}^{m(t)-1} \begin{bmatrix} n + \chi(l\leq k) \\ j_l \end{bmatrix} + (x)_\infty (-1)^{h''(t)} x^{-h(t)} M_t(x,q)\end{aligned} \tag{4}$$

Where

$$M_t(x,q)=\sum_{n\geq 0}x^{nm(t)}\sum_{3\sum_{l=1}^{m(t)-1}j_l l\equiv 1[m(t)]}(-1)^{\sum_{l=1}^{m(t)-1}j_l}x^{\sum_{l=1}^{m(t)-1}j_l}.$$

$$\cdot q^{\frac{-a(t)-h'(t)m(t)+\sum_{l=1}^{m(t)-1}j_l l}{m(t)}+\sum_{l=1}^{m(t)-1}\binom{j_l}{2}}\sum_{k=0}^{m(t)-1}x^k\prod_{l=1}^{m(t)-1}\begin{bmatrix}n+\chi(l\leq k)\\ j_l\end{bmatrix}$$

Our approach is to find $G_t(x,q)\in\mathbb{Z}[[x,q]]$ satisfying $M_t(x,q)=\frac{1}{1-x}G_t(x,q)$. This factor of $\frac{1}{1-x}$ will cancel with the $(1-x)$ in $(x)_\infty$ and allow us to proceed as in Hikami-Kirillov, bypassing the q-binomial theorem. To give a flavor for the general case, I will first consider the case of $T(3,4)$.

The case T(3,4)

Setting $t=2$, we have
$M_2(x,q)=\sum\limits_{n\geq 0}x^{2n}\sum\limits_{k\geq 0}x^{2k-1}q^{2k(k+1)}\begin{bmatrix}n\\ 2k+1\end{bmatrix}+\sum\limits_{n\geq 0}x^{2n}\sum\limits_{k\geq 0}x^{2k}q^{2k(k+1)}\begin{bmatrix}n+1\\ 2k+1\end{bmatrix}$. Note that the first sum includes all odd powers of x, and the second includes all even powers of x. We can write
$M_2(x,q)=\sum\limits_{n\geq 0}\sum\limits_{k\geq 0}x^{2(n+k)-1}q^{2k(k+1)}\begin{bmatrix}n\\ 2k+1\end{bmatrix}+\sum\limits_{n\geq 0}\sum\limits_{k\geq 0}x^{2(n+k)}q^{2k(k+1)}\begin{bmatrix}n+1\\ 2k+1\end{bmatrix}$. And we can collect the coefficients on x^i by looking at all pairs n,k s.t. $n+k=i$. We thus have:
$M_2(x,q)=\sum\limits_{n\geq 0}x^{2n-1}\sum\limits_{k\geq 0}q^{2k(k+1)}\begin{bmatrix}n-k\\ 2k+1\end{bmatrix}+\sum\limits_{n\geq 0}x^{2n}\sum\limits_{k\geq 0}q^{2k(k+1)}\begin{bmatrix}n+1-k\\ 2k+1\end{bmatrix}$. And note that the sum over k is in fact finite, due to the presence of the q-binomial coefficient. And finally, we can rewrite this as:

$$M_2(x,q)=\sum_{\text{n odd}}x^n\sum_{k\geq 0}q^{2k(k+1)}\begin{bmatrix}\frac{n+1}{2}-k\\ 2k+1\end{bmatrix}+\sum_{\text{n even}}x^n\sum_{k\geq 0}q^{2k(k+1)}\begin{bmatrix}\frac{n}{2}+1-k\\ 2k+1\end{bmatrix}\tag{5}$$

Now, we divide $M_2(x,q)$ by $\frac{1}{1-x}$. The result is:

$$\begin{aligned}G_2(x,q)=&\sum_{\text{n odd}}x^n\left[\sum_{k\geq 0}q^{2k(k+1)}\begin{bmatrix}\frac{n+1}{2}-k\\ 2k+1\end{bmatrix}-\sum_{k\geq 0}q^{2k(k+1)}\begin{bmatrix}\frac{n-1}{2}+1-k\\ 2k+1\end{bmatrix}\right]\\ &+\sum_{\text{n even}}x^n\left[\sum_{k\geq 0}q^{2k(k+1)}\begin{bmatrix}\frac{n}{2}+1-k\\ 2k+1\end{bmatrix}-\sum_{k\geq 0}q^{2k(k+1)}\begin{bmatrix}\frac{n}{2}-k\\ 2k+1\end{bmatrix}\right]\end{aligned}$$

And we thus have

$$G_2(x,q)=\sum_{\text{n even}}x^n\left[\sum_{k\geq 0}q^{2k(k+1)}\begin{bmatrix}\frac{n}{2}+1-k\\ 2k+1\end{bmatrix}-\sum_{k\geq 0}q^{2k(k+1)}\begin{bmatrix}\frac{n}{2}-k\\ 2k+1\end{bmatrix}\right]$$

Importantly for the derivation of the strange identity is that in the limit $x\to 1$, $G(x,q)$ approaches a power series in q. Also important to note is that the piece with odd powers of n vanishes upon multiplying by $(1-x)$. I think that this is the key to obtaining the limiting behavior as $x \to 1$. And this is something that we will see a generalization of in the general case.

We have now the identity

$$\begin{aligned} H_2(x) =& (qx)_\infty \sum_{n\geq 0} x^{2n} \left[\sum_{k\geq 0} q^{2k(k+1)} \begin{bmatrix} n+1-k \\ 2k+1 \end{bmatrix} - \sum_{k\geq 0} q^{2k(k+1)} \begin{bmatrix} n-k \\ 2k+1 \end{bmatrix} \right] \\ &+ (1-x) \sum_{n\geq 0} \left[(qx)_n - (qx)_\infty \right] x^{2n} \sum_{k\geq 0} x^{2k-1} q^{2k(k+1)} \left(\begin{bmatrix} n \\ 2k+1 \end{bmatrix} + \begin{bmatrix} n+1 \\ 2k+1 \end{bmatrix} \right) \end{aligned} \tag{6}$$

Now, we want to differentiate both sides of (6) with respect to x, and then take $x \to 1$.

First, let's look at the LHS.
I will need Watson's quintuple product. Recall:

$$\sum_{k\in\mathbb{Z}} q^{\frac{k(3k-1)}{2}} x^{3k} (1 - xq^k) = (q, x, qx^{-1}; q)_\infty (qx^2, qx^{-2}; q^2)_\infty \tag{7}$$

Now, look at the LHS of (6) and differentiate in x, then set $x = 1$:

$$\frac{1}{2} \sum_{n\geq 0} n\chi_{24}(n) q^{\frac{n^2-25}{48}} - \frac{5}{2} \sum_{n\geq 0} \chi_{24}(n) q^{\frac{n^2-25}{48}} \tag{8}$$

The goal is to write the second sum in (8) in such a way as to apply Watson's quintuple product. As we have done before, we can break this sum into 4 pieces:

$$\begin{aligned} \sum_{n\geq 0} \chi_{24}(n) q^{\frac{n^2-25}{48}} =& \sum_{k\geq 0} (q^{12k^2+5k} - q^{12k^2+13k+3} \\ &+ \sum_{k\geq 0} (q^{12k^2+19k+7} - q^{12k^2+11k+2}) \end{aligned}$$

Now, reindex $k \to k+1$ in the second sum, which we can write as $\sum_{k\geq 1} (q^{12k^2-5k} - q^{12k^2-13k+3}$. This allows us to write

$$\sum_{n\geq 0} \chi_{24}(n) q^{\frac{n^2-25}{48}} = \sum_{k\in\mathbb{Z}} q^{12k^2+5k} - q^{12k^2+13k+3} \tag{9}$$

And if we want to write this in the form of the LHS of Watson's identity, we want the form $\sum_{k\in\mathbb{Z}} y^{\frac{k(3k-1)}{2}} x^{3k} (1 - xy^k)$. Where y and x are powers of q. Following this through, we find that $y = q^8$ and $x = q^3$. We can now write (**??**) in the form

$$\frac{1}{2} \sum_{n\geq 0} n\chi_{24}(n) q^{\frac{n^2-25}{48}} - \frac{5}{2} (q^8, q^3, q^5; q^8)_\infty (q^{14}, q^2; q^{16})_\infty \tag{10}$$

Now, consider the RHS of (6). Differentiate in x, and then send $x \to 1$. We obtain

$$\begin{aligned} &-(q)_\infty \left(\sum_{i=1}^{\infty} \frac{q^i}{1-q^i}\right) \sum_{n\geq 0} \left[\sum_{k\geq 0} q^{2k(k+1)} \begin{bmatrix} n+1-k \\ 2k+1 \end{bmatrix} - \begin{bmatrix} n-k \\ 2k+1 \end{bmatrix}\right] \\ &+(q)_\infty \sum_{n\geq 0} 2n \left[\sum_{k\geq 0} q^{2k(k+1)} \begin{bmatrix} n+1-k \\ 2k+1 \end{bmatrix} - \begin{bmatrix} n-k \\ 2k+1 \end{bmatrix}\right] \\ &-\sum_{n\geq 0} [(q)_n - (q)_\infty] \sum_{k\geq 0} q^{2k(k+1)} \left(\begin{bmatrix} n \\ 2k+1 \end{bmatrix} + \begin{bmatrix} n+1 \\ 2k+1 \end{bmatrix}\right) \end{aligned} \tag{11}$$

And we thus have the q-series identity:

$$\begin{aligned} &\frac{1}{2}\sum_{n\geq 0} n\chi_{24}(n) q^{\frac{n^2-25}{48}} - \frac{5}{2}(q^8, q^3, q^5; q^8)_\infty (q^{14}, q^2; q^{16})_\infty \\ &= -(q)_\infty \left(\sum_{i=1}^{\infty} \frac{q^i}{1-q^i}\right) \sum_{n\geq 0} \left[\sum_{k\geq 0} q^{2k(k+1)} \begin{bmatrix} n+1-k \\ 2k+1 \end{bmatrix} - \begin{bmatrix} n-k \\ 2k+1 \end{bmatrix}\right] \\ &+ (q)_\infty \sum_{n\geq 0} 2n \left[\sum_{k\geq 0} q^{2k(k+1)} \begin{bmatrix} n+1-k \\ 2k+1 \end{bmatrix} - \begin{bmatrix} n-k \\ 2k+1 \end{bmatrix}\right] \\ &-\sum_{n\geq 0} [(q)_n - (q)_\infty] \sum_{k\geq 0} q^{2k(k+1)} \left(\begin{bmatrix} n \\ 2k+1 \end{bmatrix} + \begin{bmatrix} n+1 \\ 2k+1 \end{bmatrix}\right) \end{aligned} \tag{12}$$

I've verified this numerically in Mathematica :)))))) !!! It works!!

And the strange identity follows by setting $q = \zeta e^{-t}$ and taking the limit $t \to 0^+$, noting that all terms with a factor of $(q)_\infty$ vanish in the limit.

The general case

Let's move on to the general case. For brevity I introduce the shorthand $\nu = \frac{-a(t)-h'(t)m(t)+\sum_{l=1}^{m(t)-1} j_l l}{m(t)} + \sum_{l=1}^{m(t)-1} \binom{j_l}{2}$, and $[j_l] = \sum_{l=1}^{m(t)-1} j_l$. Also I adopt the convention that all congruences are made mod $m(t)$. We write:

$$\begin{aligned} M_t(x,q) = \sum_{k=0}^{m(t)-1} \sum_{n\geq 0} \sum_{3\sum_{l=1}^{m(t)-1} j_l l \equiv 1} &(-1)^{[j_l]} x^{nm(t)+k+[j_l]}. \\ &\cdot q^\nu \prod_{l=1}^{m(t)-1} \begin{bmatrix} n+\chi(l\leq k) \\ j_l \end{bmatrix} \end{aligned}$$

Similarly to the case of $t = 2$, the immediate goal is to write this explicitly as a power series in x. I am going to do this by breaking the 3rd sum into 4 cases based on the residue of $\sum_{l=1}^{m(t)-1} j_l \pmod{m(t)}$.

For $0 \leq i < m(t)$, consider the case $[j_l] \equiv i \bmod(m(t))$. Then we have

$$
\begin{aligned}
M_t(x,q) = \sum_{i=0}^{m(t)-1} \sum_{k=0}^{m(t)-1} \sum_{n\geq 0} \sum_{[j_l]\equiv i} & x^{m(t)(n+\frac{[j_l]-i}{m(t)})+i+k}(-1)^{[j_l]}q^{\nu} \prod_{l=1}^{m(t)-1} \begin{bmatrix} n+\chi(l\leq k) \\ j_l \end{bmatrix} \\
= \sum_{k=0}^{m(t)-1} \sum_{n\geq 0} \sum_{[j_l]\equiv i} & x^{nm(t)+i+k}(-1)^{[j_l]}q^{\nu} \prod_{l=1}^{m(t)-1} \begin{bmatrix} n-\frac{[j_l]-i}{m(t)}+\chi(l\leq k) \\ j_l \end{bmatrix} \\
= \sum_{k=0}^{m(t)-1} \sum_{n\equiv i+k} \sum_{[j_l]\equiv i} & x^{n}(-1)^{[j_l]}q^{\nu} \prod_{l=1}^{m(t)-1} \begin{bmatrix} \frac{n-k-[j_l]}{m(t)}+\chi(l\leq k) \\ j_l \end{bmatrix}
\end{aligned}
$$

We are now in a position to write $M_t(x,q) = \sum_{n\geq 0} a_n^{(t)}(q)x^n$, where each of the $a_n^{(t)}(q)$ is a polynomial in q with integral coefficients. The definition of $a_n^{(t)}(q)$ depends on the residue of n mod $m(t)$.

For $n \equiv m \pmod{m(t)}$, $0 \leq m < m(t)$, we have

$$
a_n^{(t)}(q) = \sum_{k=0}^{m(t)-1} \sum_{[j_l]\equiv k} (-1)^{[j_l]}q^{\nu} \prod_{l=1}^{m(t)-1} \begin{bmatrix} \frac{n-\overline{(m-k)}-[j_l]}{m(t)}+\chi(l\leq \overline{m-k}) \\ j_l \end{bmatrix}
$$

Where $\overline{m-k}$ denotes the residue of $m-k$ mod $m(t)$.

Now, what remains is to divide by $\frac{1}{1-x}$ to obtain $G_t(x,q)$. If we write $G_t(x,q) = \sum_{n\geq 0} b_n^{(t)}(q)x^n$, where $b_n^{(t)}(q) = a_n^{(t)}(q) - a_{n-1}^{(t)}(q)$. This is where we see cancellation similar to the $t = 2$ case. In the difference $a_n(q) - a_{n-1}(q)$, where $n \equiv m \pmod{m(t)}$ the $k = m^{th}$ sums in $a_n(q)$ and $a_{n-1}(q)$ are equal and cancel each other out. My hope is that this is enough to see that $G_t(1,q)$ converges as a power series in q, although that isn't entirely clear to me at this point.

I think it is, and the reason is that it allows only a relatively small number of terms to survive. The majority of the positive piece $b_n(q)$ is 'caught' by the negative piece of $b_{n+1}(q)$ and they annihilate each other. What I believe is going on is that this cancellation allows the m^{th} or $(m-1)^{th}$ sum in $b_n(q)$ to survive, which lets the behavior in the limit through.

Supposing that $G_t(1,q) = \sum_{n\geq 0} b_n^{(t)}(q)$ is well defined, which I have verified numerically for $t = 3$ (beyond that the computation becomes too involved for my computer) we have the following identity:

$$H_t(x) =(-1)^{-h''(t)}(1-x)\sum_{n\geq 0}\left[(qx)_n-(qx)_\infty\right]x^{nm(t)}\sum_{3\sum_{l=1}^{m(t)-1}j_l l\equiv 1[m(t)]}(-1)^{\sum_{l=1}^{m(t)-1}j_l}$$
$$\cdot x^{-h(t)+\sum_{l=1}^{m(t)-1}j_l}\cdot q^{\frac{-a(t)-h'(t)m(t)+\sum_{l=1}^{m(t)-1}j_l l}{m(t)}+\sum_{l=1}^{m(t)-1}\binom{j_l}{2}} \tag{13}$$
$$\cdot\sum_{k=0}^{m(t)-1}x^k\prod_{l=1}^{m(t)-1}\begin{bmatrix}n+\chi(l\leq k)\\ j_l\end{bmatrix}+(qx)_\infty(-1)^{h''(t)}x^{-h(t)}\sum_{n\geq 0}b_n^{(t)}(q)x^n$$

This is our analog of Lemma (i) used by Hikami and Kirillov in deriving the strange identity. From here, we want to differentiate both sides in x and then send $x\to 1$.

We proceed exactly in analogy with the case of $T(3,4)$. First, look at the LHS of (13) and differentiate in x, then set $x=1$:

$$\frac{1}{2}\sum_{n\geq 0}n\chi_{3\cdot 2^{t+1}}(n)q^{\frac{n^2-(2^{t+1}-3)^2}{3\cdot 2^{t+1}}}-\frac{2^{t+1}-3}{2}\sum_{n\geq 0}\chi_{3\cdot 2^{t+1}}(n)q^{\frac{n^2-(2^{t+1}-3)^2}{3\cdot 2^{t+2}}} \tag{14}$$

The goal is to write the second sum in (14) in such a way as to apply Watson's quintuple product. As we have done before, we can break this sum into 4 pieces:

$$\sum_{n\geq 0}\chi_{3\cdot 2^{t+1}}(n)q^{\frac{n^2-(2^{t+1}-3)^2}{3\cdot 2^{t+2}}} =\sum_{k\geq 0}(q^{3\cdot 2^tk^2+(2^{t+1}-3)k}-q^{3\cdot 2^tk^2+(2^{t+2}-3)k+2^t-1}$$
$$+\sum_{k\geq 0}(q^{3\cdot 2^tk^2+(2^{t+2}+3)k+2^t+3}-q^{3\cdot 2^tk^2+(2^{t+1}+3)k+2})$$

Now, reindex $k\to k+1$ in the second sum, which we can write as $\sum_{k\geq 1}(q^{3\cdot 2^tk^2+(2^{t+1}-3)k}-q^{3\cdot 2^tk^2+(3-2^{t+2})k+2^t-1}$. This allows us to write

$$\sum_{n\geq 0}\chi_{3\cdot 2^{t+1}}(n)q^{\frac{n^2-(2^{t+1}-3)^2}{3\cdot 2^{t+2}}}=\sum_{k\in\mathbb{Z}}q^{3\cdot 2^tk^2+(2^{t+1}-3)k}-q^{3\cdot 2^tk^2+(2^{t+2}-3)k+2^t-1} \tag{15}$$

And if we want to write this in the form of the LHS of Watson's identity, we want the form $\sum_{k\in\mathbb{Z}}y^{\frac{k(3k-1)}{2}}x^{3k}(1-xy^k)$. Where y and x are powers of q. Following this through, we find that $y=q^{2^{t+1}}$ and $x=q^{2^t-1}$. We can now write (**??**) in the form

$$\frac{1}{2}\sum_{n\geq 0}n\chi_{3\cdot 2^{t+1}}(n)q^{\frac{n^2-(2^{t+1}-3)^2}{3\cdot 2^{t+1}}}-\frac{2^{t+1}-3}{2}(q^{2^{t+1}},q^{2^t-1},q^{2^t+1};q^{2^{t+1}})_\infty(q^{2^{t+2}-2},q^2;q^{2^{t+2}})_\infty \tag{16}$$

Now, let's shift our focus to the RHS of (13). As before we differentiate in x and send $x\to 1$:

$$-(q)_\infty\left(\sum_{i=1}^{\infty}\frac{q^i}{1-q^i}\right)(-1)^{h''(t)}\sum_{n\geq 0}b_n^{(t)}(q)+(q)_\infty(-1)^{h''(t)}\sum_{n\geq 0}(n-h(t))b_n^{(t)}(q)$$
$$-(-1)^{h''(t)}\sum_{n\geq 0}\left[(q)_n-(q)_\infty\right]\sum_{3\sum_{l=1}^{m(t)-1}j_l l\equiv 1[m(t)]}(-1)^{\sum_{l=1}^{m(t)-1}j_l} \tag{17}$$
$$\cdot q^{\frac{-a(t)-h'(t)m(t)+\sum_{l=1}^{m(t)-1}j_l l}{m(t)}+\sum_{l=1}^{m(t)-1}\binom{j_l}{2}}\sum_{k=0}^{m(t)-1}\prod_{l=1}^{m(t)-1}\begin{bmatrix}n+\chi(l\leq k)\\ j_l\end{bmatrix}$$

We thus have the q-series identity:

$$\frac{1}{2}\sum_{n\geq 0} n\chi_{3\cdot 2^{t+1}}(n)q^{\frac{n^2-(2^{t+1}-3)^2}{3\cdot 2^{t+1}}} - \frac{2^{t+1}-3}{2}(q^{2^{t+1}}, q^{2^t-1}, q^{2^t+1}; q^{2^{t+1}})_\infty (q^{2^{t+2}-2}, q^2; q^{2^{t+2}})_\infty$$

$$= -(q)_\infty \left(\sum_{i=1}^{\infty}\frac{q^i}{1-q^i}\right)(-1)^{h''(t)}\sum_{n\geq 0} b_n^{(t)}(q) + (q)_\infty (-1)^{h''(t)}\sum_{n\geq 0}(n-h(t))b_n^{(t)}(q)$$

$$-(-1)^{h''(t)}\sum_{n\geq 0}[(q)_n - (q)_\infty] \sum_{3\sum_{l=1}^{m(t)-1} j_l l \equiv 1[m(t)]} (-1)^{\sum_{l=1}^{m(t)-1} j_l}$$

$$\cdot\, q^{\frac{-a(t)-h'(t)m(t)+\sum_{l=1}^{m(t)-1} j_l l}{m(t)} + \sum_{l=1}^{m(t)-1}\binom{j_l}{2}} \sum_{k=0}^{m(t)-1}\prod_{l=1}^{m(t)-1}\begin{bmatrix} n+\chi(l\leq k) \\ j_l \end{bmatrix}$$

I've managed to verify this numerically for $t = 3$. Higher values of t prove too computationally heavy for my computer.

From here, the strange identity follows from setting $q = \zeta e^{-t}$ and taking the limit as $t \to 0^+$. Thus, we have:

$$-\frac{1}{2}\sum_{n\geq 0} n\chi_{3\cdot 2^{t+1}}(n)q^{\frac{n^2-(2^{t+1}-3)^2}{3\cdot 2^{t+1}}} \text{ `='} \mathcal{F}_t(q) \tag{18}$$

Update

In the previous section I believe to have shown the strange identity up to the assumption on the series $\sum_{n\geq 0} b_n^{(t)}(q)$. I hope here to argue that this series does indeed converge as a formal power series in q.

We have

$$b_n^{(t)}(q) = \sum_{k=0}^{m(t)-1}\sum_{[j_l]\equiv k}(-1)^{[j_l]}q^{\nu}\Bigg(\prod_{l=1}^{m(t)-1}\begin{bmatrix} \frac{n-\overline{(n-k)}-[j_l]}{m(t)} + \chi(l\leq \overline{n-k}) \\ j_l \end{bmatrix} - \prod_{l=1}^{m(t)-1}\begin{bmatrix} \frac{n-\overline{(n-k)}-[j_l]}{m(t)} + \chi(l\leq \overline{n-1-k}) \\ j_l \end{bmatrix}\Bigg)$$

And we can write the series

$$\sum_{n\geq 0} b_n^{(t)}(q) = \sum_{n\geq 0}\sum_{k=0}^{m(t)-1}\sum_{[j_l]\equiv k}(-1)^{[j_l]}q^{\nu}\Bigg(\prod_{l=1}^{m(t)-1}\begin{bmatrix} n-\lfloor\frac{[j_l]}{m(t)}\rfloor + \chi(l\leq \overline{n-k}) \\ j_l \end{bmatrix} - \prod_{l=1}^{m(t)-1}\begin{bmatrix} n-\lfloor\frac{[j_l]}{m(t)}\rfloor + \chi(l\leq \overline{n-1-k}) \\ j_l \end{bmatrix}\Bigg)$$

Where we have written $n = n' m(t) + \bar{n}$ and summed over n' (relabeling it n). Then, using the definition of the q-binomial coefficients:

$$\begin{bmatrix} n \\ m \end{bmatrix} = \frac{(1-q^n)(1-q^{n-1}...(1-q^{n-m+1})}{(1-q)(1-q^2)...(1-q^m)}$$

We can write our sum as

$$\sum_{n\geq 0}\sum_{k=0}^{m(t)-1}\sum_{[j_l]\equiv k}(-1)^{[j_l]}q^{\nu}\left(\prod_{l=1}^{m(t)-1}\frac{1}{\prod_{i=1}^{j_l}(1-q^i)}\right)\left(\prod_{l=1}^{m(t)-1}\prod_{i=0}^{j_l-1}(1-q^{n-\lfloor\frac{[j_l]}{m(t)}\rfloor+\chi(l\leq\overline{n-k})-i})\right.$$
$$\left.-\prod_{l=1}^{m(t)-1}\prod_{i=0}^{j_l-1}(1-q^{n-\lfloor\frac{[j_l]}{m(t)}\rfloor+\chi(l\leq\overline{n-1-k})-i})\right)$$

Notice that the two double products are identical for all choices of $l\neq\overline{n-k}$. We can thus write the difference of the double sums to be

$$\left(\prod_{l=1}^{\overline{n-k}}\prod_{i=0}^{j_l-1}(1-q^{n-\lfloor\frac{[j_l]}{m(t)}\rfloor-i})\right)\left(\prod_{l=\overline{(n-k)}+1}^{m(t)-1}\prod_{i=0}^{j_l-1}(1-q^{n-\lfloor\frac{[j_l]}{m(t)}\rfloor+1-i})\right)$$
$$\times\left(\prod_{i=0}^{j_{\overline{(n-k)}}-1}(1-q^{n-\lfloor\frac{[j_l]}{m(t)}\rfloor+1-i})-\prod_{i=0}^{j_{\overline{(n-k)}}-1}(1-q^{n-\lfloor\frac{[j_l]}{m(t)}\rfloor-i})\right)$$

Then we notice that the two products in the last factor above are identical except for 1 multiplicand each. This factor can be written as

$$\left(\prod_{i=0}^{j_{\overline{(n-k)}}-2}(1-q^{n-\lfloor\frac{[j_l]}{m(t)}\rfloor-i})\right)\left((1-q^{n-\lfloor\frac{[j_l]}{m(t)}\rfloor+1})-(1-q^{n-\lfloor\frac{[j_l]}{m(t)}\rfloor-j_{\overline{n-k}}+1})\right)$$

Finally, we see that the difference of binomial coefficients in each term of our series is $O(q^{n-\lfloor\frac{[j_l]}{m(t)}\rfloor-j_{\overline{n-k}}+1})$.

And recall that this is multiplied by $q^{\frac{-a(t)-h'(t)m(t)+\sum_{l=1}^{m(t)-1}j_l l}{m(t)}+\sum_{l=1}^{m(t)-1}\binom{j_l}{2}}$. So overall every term in the series is $O(q^{n-j_{\overline{(n-k)}}+1-h'(t)+\sum_{l=1}^{m(t)-1}\binom{j_l}{2}})$.

And thus $\sum\limits_{n\geq 0}b_n^{(t)}(q)$ indeed converges as a power series in q. And we are able to remove the last assumption on the proof of the strange identity; the strange identity has been shown to hold for all $t\geq 2$.

Congruences

I want to generalize the methods of Straub [4] to prove prime power congruences for $\mathcal{F}_t(1-q)$. We have the 'strange' identity:

$$\mathcal{F}_t(q)' =' -\frac{1}{2}\sum_{n\geq 0}n\chi_{3\cdot 2^{t+1}}q^{\frac{n^2-(2^{t+1}-3)^2}{3\cdot 2^{t+2}}}$$

If we then consider the s-dissection of $\mathcal{F}_t(q)=\sum\limits_{i=0}^{s-1}q^iA_s(N,i,q^s)$, we can apply Theorem () to get that
for $i\in S_t(s)=\{\frac{n^2-(2^{t+1}-3)^2}{3\cdot 2^{t+2}}(\text{mod s}):\chi_{3\cdot 2^{t+1}}(n)\neq 0\}$:

$$(q)_{\lambda(N,s)}|A_s(N,i,q) \tag{19}$$

Where $\lambda(N,s)=\lfloor\frac{N+1}{s}\rfloor$. We use this to prove the following.

Theorem. *For* $\mathcal{F}_t(1-q) = \sum\limits_{n\geq 0} \gamma^{(t)}(n)q^n$, *we have*

$$\gamma^{(t)}(np^r - j) \equiv 0 \ (mod\ p^r) \tag{20}$$

For every $n, r \geq 0$ *integers and* $j \in \{1, 2, ..., n-1 - max\ S_t(p)\}$

Proof: Consider the truncation $\mathcal{F}_t(q, N)$ and separate it into sums over $S_t(p)$ and its complement.

$$\mathcal{F}_t(1-q, N) = \sum_{i\in S_t(p)} (1-q)^i A_p(N, i, (1-q)^p) + \sum_{i\notin S_t(p)} (1-q)^i A_p(N, i, (1-q)^p)$$

Then, using Equation (19), and the fact that $(1-q^k)^n | (q)_M$ for $M \geq kn$, we can write:

$$\mathcal{F}_t(1-q, N) = \sum_{i\in S_t(p)} (1-q)^i A_p(N, i, (1-q)^p) + (1-(1-q)^p)^n \sum_{i\notin S_t(p)} (1-q)^i f_i(q)$$

For n such that $N \geq np^2 - 1$, and some polynomials $f_i(q) \in \mathbb{Z}[q]$.

Claim: $(1-(1-q)^p)^n \equiv O(q^{np-(r-1)(p-1)})$ (mod p^r)
Proof: Consider $(1-q)^p$. Write $(1-q)^p = 1 + qf(q) - q^p$.
Then $(1-(1-q)^p)^n = (qf(q) - q^p)^n = \sum\limits_{i=0}^{n} \binom{n}{i} (qf(q))^i (-q^p)^{n-i}$. Taking this (mod p^r), we get

$$(1-(1-q)^p)^n \equiv \sum_{i=0}^{r-1} \binom{n}{i} (qf(q))^i (-q^p)^{n-i}$$

And the claim follows. □

We are thus able to write:

$$\mathcal{F}_t(1-q, N) \equiv \sum_{i\in S_t(p)} (1-q)^i A_p(N, i, (1-q)^p) + O(q^{np-(r-1)(p-1)}) \text{ (mod } p^r)$$

The first sum is an integral linear combination of terms of the form $(1-q)^{i+kp}$ for $i \in S_t(p)$. We want to show that

$$\binom{i+kp}{mp^r - j} \equiv 0 \text{ mod}(\ p^r) \tag{21}$$

For every $i \in S_t(p)$ and $j = 1, 2, ..., p-1-\max\{S_t(p)\}$ where $k, m \in \mathbb{N}$.

Consider the p-adic expansions of
$x := i + kp = \sum\limits_{\alpha\geq 0} x_\alpha p^\alpha$, and $y := mp^r - j = \sum\limits_{\alpha \geq 0} y_\alpha p^\alpha$.
By construction, we have that $x_0 = i$, $y_0 = p - j$, and $y_\alpha = p - 1$ for $1 \leq \alpha \leq p^{r-1}$. Since $y_0 > x_0$, we are guaranteed a carry in the addition of the least order p-adic digits of $x - y$ and y by Lucas' theorem. Then it follows that we have at least r carries in the addition of $x - y$ and y since y_α is maximal for $1 \leq \alpha \leq p^{r-1}$. Thus, by Kummer's theorem we have the congruence (20) □

References

[1] Ahlgren, S. Kim, B. Lovejoy, J. *Dissections of strange q-series.* (2018) arXiv:1812.02922

[2] Hikami, K. Kirillov, A. *Hypergeometric generating function of L-function, Slater's identities, and quantum invariant.* (2006) St. Petersburg Mathematical Journal **17**:01 pp. 143-156

[3] Konan, I. *Autour des q-series, des formes modulaires quantiques et des noeuds toriques* (2016) https://www.irif.fr/_media/users/konan/memoire.pdf

[4] Straub, A. *Congruences for Fishburn numbers modulo prime powers.* (2015) International Journal of Number Theory **11**:05 pp. 1679-1690