Absolute and Convective Instabilities in Parallel Flows
Lecture 1

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1 Overview and Workplan

The idea of these four lectures is to study the asymptotic ($t \to \infty$) properties of the following equation from the linear theory of parallel flow instability:

$$\frac{\partial}{\partial t}\nabla^2\phi + U_0(z)\nabla^2\frac{\partial\phi}{\partial x} - U_0''(z)\frac{\partial\phi}{\partial x} = \frac{1}{Re}\nabla^4\phi, \tag{1a}$$

with boundary conditions

$$\phi = \phi_z = 0, \qquad \text{at } z = 0, 1, \tag{1b}$$

and

$$\lim_{|x|\to\infty}\phi = \lim_{|x|\to\infty}\phi_x = 0, \tag{1c}$$

subject to an initial impulsive disturbance

$$\phi(x, z, t = 0) = \delta(x - x_0)\delta(z - z_0). \tag{1d}$$

Physically, $\phi(x, z, t)$ is the streamfunction of a small flow perturbation that is applied and superimposed on the base flow $U_0(z)$. The base flow is directed in the x-direction but varies in the normal (z-direction). For that reason, it is called parallel (Figure 1). It

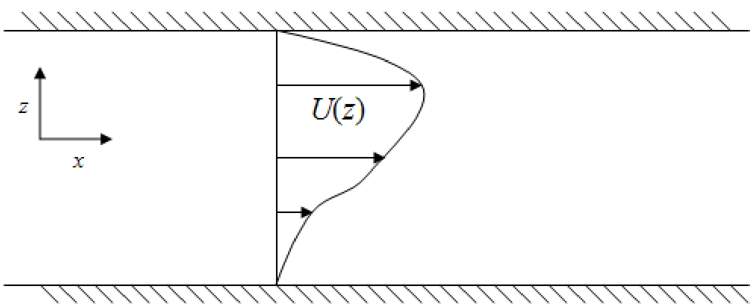

Figure 1: Schematic description of the base parallel flow $U_0(z)$.

is of interest to know whether this perturbation will grow over time or decay. The flow is called **convectively unstable** if

$$\lim_{t\to\infty} |\phi(x,z,t)| = \infty,$$

for **some** x. The flow is called **absolutely unstable** if

$$\lim_{t\to\infty} |\phi(x_0,z_0,t)| = \infty.$$

That is, convectively unstable disturbances grow as they are convected downstream by the base flow $U_0(z)$, while absolutely unstable disturbances grow at the disturbance source.

The problem is first of all approached by studying the theory of Laplace transforms. In the first two lectures, this will be our exclusive focus.

2 Laplace transforms

In this section, let

$$\begin{array}{rcl} F:[0,\infty) & \to & \mathbb{C}, \\ t & \mapsto & F(t) \end{array} \qquad (2)$$

be a complex-valued function of a real variable.

Definition 2.1 *Let The function $F(t)$ is* **at most exponentially diverging** *if there exist real numbers $(\lambda_0, M>0)$ such that*

$$|\mathrm{e}^{-\lambda_0 t}F(t)| \le M, \qquad \text{as } t\to\infty;$$

we call λ_0 the **divergence parameter**.

Definition 2.2 *Let $F(t)$ be at most exponentially diverging, with divergence parameter λ_0.* **Laplace-transform** *of $F(t)$ is defined as follows:*

$$\widehat{F}_\lambda \equiv \mathcal{L}(F) := \int_0^\infty \mathrm{e}^{-\lambda t}F(t)\mathrm{d}t, \qquad \Re(\lambda) > \lambda_0.$$

Theorem 2.1 *The Laplace transform is linear, in the sense that*

$$\mathcal{L}(\alpha F(t) + \beta G(t)) = \alpha\mathcal{L}(F) + \beta\mathcal{L}(G),$$

where α and β are complex conostants and the functions F and G are functions of type (2) whose Laplace transforms exist.

Examples

1. Let $F(t) = \mathrm{e}^{kt}$, with $k > 0$ real. We have

$$\begin{aligned} \widehat{F}_\lambda &= \int_0^\infty \mathrm{e}^{(k-\lambda)t}\mathrm{d}t, \\ &= \lim_{L\to\infty}\left[\frac{1}{k-\lambda}\left(\mathrm{e}^{(k-\lambda)L}-1\right)\right]. \end{aligned} \tag{3}$$

Obviously, we need $\Re(\lambda) > k$ for this integral to exist, hence

$$\widehat{F}_\lambda = \frac{1}{\lambda - k}, \qquad \Re(\lambda) > k.$$

The transform has a simple pole at $\lambda = k$, which is connected to the failure of the integral (3) to exist for $\Re(\lambda)$ sufficiently small. See Figure 2 for a sketch of the λ-domain where $\mathcal{L}(\mathrm{e}^{kt})$ is well-defined.

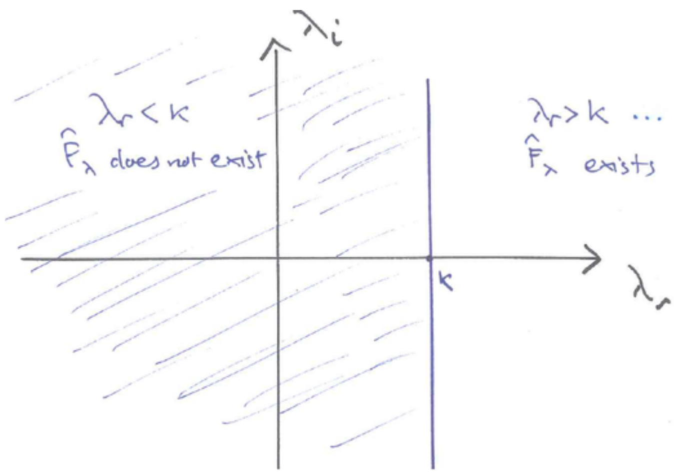

Figure 2: Domain of existence of the complex Laplace transform of e^{kt}.

2. Let $F(t) = \sinh kt$, with $k > 0$ real. We compute

$$\begin{aligned} \mathcal{L}(\mathrm{e}^{kt}) &= \int_0^\infty \mathrm{e}^{(k-\lambda)t}, \\ &= \frac{1}{\lambda-k}, \qquad \Re(\lambda) > k. \end{aligned}$$

Also,

$$\begin{aligned} \mathcal{L}(\mathrm{e}^{-kt}) &= \int_0^\infty \mathrm{e}^{(-k-\lambda)t}, \\ &= \frac{1}{\lambda+k}, \qquad \Re(\lambda) > -k. \end{aligned}$$

By linearity,

$$\mathcal{L}(\sinh kt) = \tfrac{1}{2}\left(\frac{1}{\lambda-k} - \frac{1}{\lambda+k}\right), \qquad \Re(\lambda) > k,$$

where the first inequality trumps the second one. Finally,

$$\mathcal{L}(\sinh kt) = \frac{k}{\lambda^2 - k^2}, \qquad \Re(\lambda) > k.$$

3. Let $F(t) = \delta(t - t_0)$, with $t_0 > 0$. We have

$$\widehat{F}_\lambda = \int_0^\infty \mathrm{e}^{\lambda t}\delta(t - t_0) = \mathrm{e}^{\lambda t_0}.$$

We take $t_0 \downarrow 0$ and define

$$\mathcal{L}(\delta(t)) = 1.$$

3 Inverting Laplace transforms

Let

$$\begin{array}{rcl} F : [0, \infty) & \to & \mathbb{C}, \\ t & \mapsto & F(t) \end{array}$$

be a complex-valued function of a real variable, and moreover, let $F(t)$ be at worst exponentially diverging, with exponential parameter λ_0. We re-write $F(t)$ as

$$F(t) = \mathrm{e}^{\gamma t} G(t),$$

where $\lim_{t\to\infty} G(t) = 0$. Such a G-function exists; we take

$$G(t) = F(t)\mathrm{e}^{-(\lambda_0+\epsilon)t},$$

for ϵ arbitrary and positive (hence, $\gamma = \lambda_0 + \epsilon$). We have

$$\begin{array}{rcl} |G(t)| & = & |F(t)|\mathrm{e}^{-\lambda_0 t - \epsilon t}, \\ & \leq & M\mathrm{e}^{\lambda_0 t}\mathrm{e}^{-\lambda_0 t - \epsilon t}, \qquad \text{as } t \to \infty, \\ & \leq & M\mathrm{e}^{-\epsilon t}, \\ & \to & 0, \qquad \text{as } t \to \infty. \end{array}$$

Also, define $G(t) = 0$ for $t < 0$. It follows that G is L^2 square integrable. Subject to the usual further conditions on G (i.e. piecewise differentiable for $t \in \mathbb{R}$), G can be written in Fourier transform notation:

$$\begin{array}{rcl} G(t) & = & \displaystyle\int_{-\infty}^\infty \frac{\mathrm{d}\omega}{2\pi} \mathrm{e}^{\mathrm{i}\omega t} \widehat{G}_\omega, \\ & = & \displaystyle\int_{-\infty}^\infty \frac{\mathrm{d}\omega}{2\pi} \mathrm{e}^{\mathrm{i}\omega t} \left[\int_{-\infty}^\infty \mathrm{d}s\, \mathrm{e}^{-\mathrm{i}\omega s} G(s)\right] \end{array}$$

Multiply across by $\mathrm{e}^{\gamma t}$:

$$\begin{array}{rcl} \mathrm{e}^{\gamma t} G(t) & = & \displaystyle\frac{\mathrm{e}^{\gamma t}}{2\pi} \int_{-\infty}^\infty \mathrm{d}\omega\, \mathrm{e}^{\mathrm{i}\omega t} \left[\int_{-\infty}^\infty \mathrm{d}s\, \mathrm{e}^{-\mathrm{i}\omega s} G(s)\right], \\ F(t) & = & \displaystyle\frac{\mathrm{e}^{\gamma t}}{2\pi} \int_{-\infty}^\infty \mathrm{d}\omega\, \mathrm{e}^{\mathrm{i}\omega t} \left[\int_0^\infty \mathrm{d}s\, \mathrm{e}^{-\mathrm{i}\omega s} F(s)\mathrm{e}^{-\gamma s}\right], \\ & = & \displaystyle\frac{\mathrm{e}^{\gamma t}}{2\pi} \int_{-\infty}^\infty \mathrm{d}\omega\, \mathrm{e}^{\mathrm{i}\omega t} \underbrace{\left[\int_0^\infty \mathrm{d}s\, \mathrm{e}^{-\lambda s} F(s)\right]}_{=\widehat{F}_\lambda}. \end{array}$$

Let $\lambda = \gamma + \mathrm{i}\omega$, hence $\omega = (\lambda - \gamma)/\mathrm{i}$.

$$F(t) = \frac{\mathrm{e}^{\gamma t}}{2\pi} \int_{-\infty}^{\infty} \left(\mathrm{d}\omega\, \mathrm{e}^{\mathrm{i}\omega t}\right)_{\omega = \frac{\lambda-\gamma}{\mathrm{i}}} \widehat{F}_\lambda.$$

Effecting the change of variables, this is

$$\begin{aligned} F(t) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \left(\mathrm{d}\omega\, \mathrm{e}^{(\gamma+\mathrm{i}\omega)t}\right)_{\omega = \frac{\lambda-\gamma}{\mathrm{i}}} \widehat{F}_\lambda, \\ &= \frac{1}{2\pi\mathrm{i}} \int_{\gamma-\mathrm{i}\infty}^{\gamma+\mathrm{i}\infty} \mathrm{d}\lambda\, \mathrm{e}^{\lambda t} \widehat{F}_\lambda. \end{aligned}$$

The contour

$$\mathcal{B} = \{z \in \mathbb{C} | z = \gamma + \mathrm{i}y, y \in \mathbb{R}\}$$

is called the **Bromwich contour**. It is sketched in Figure 3.

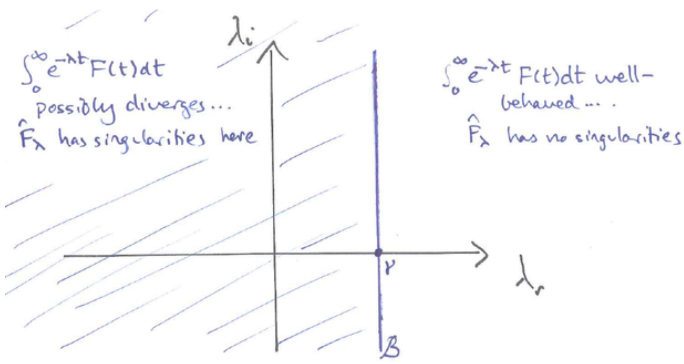

Figure 3: Definition sketch – the Bromwich contour

Suppose now that

$$\lim_{\lambda\to\infty} |\mathrm{e}^{\lambda t}\widehat{F}_\lambda| = 0, \qquad t > 0$$

and consider the contour $\mathcal{C} + \mathcal{B}$ in Figure 4. For now, we consider the case where the singularities of $\mathrm{e}^{\lambda t}\widehat{F}_\lambda$ are poles; branch-cut singularities are considered on a case-by-case basis in the examples to follow. Also, we use the notation $\mathcal{C}$ to denote the limiting countour associated with a semi-circle of radius R centred at $(\gamma, 0)$, with $R \to \infty$. In this limit, the semi-circle encloses all of the singularities (poles) of $\widehat{F}_\lambda$. Also, $\int_\mathcal{C} \mathrm{e}^{\lambda t}\widehat{F}_\lambda \mathrm{d}\lambda = 0$. Hence,

$$\begin{aligned} \frac{1}{2\pi\mathrm{i}} \int_{\mathcal{C}+\mathcal{B}} \mathrm{e}^{\lambda t}\widehat{F}_\lambda &= \sum \text{enclosed residues}, \\ &= \frac{1}{2\pi\mathrm{i}} \left(\int_\mathcal{C} \mathrm{d}\lambda + \int_\mathcal{B} \mathrm{d}\lambda \right) \mathrm{e}^{\lambda t}\widehat{F}_\lambda, \\ &= \frac{1}{2\pi\mathrm{i}} \left(0 + \int_\mathcal{B} \mathrm{d}\lambda \right) \mathrm{e}^{\lambda t}\widehat{F}_\lambda. \end{aligned}$$

Hence,

$$F(t) = \sum \text{enclosed residues}, \tag{4}$$

where 'residues' refers to the residues of $\mathrm{e}^{\lambda t}\widehat{F}_\lambda$ in the half-plane to the left of the line $\Re(\lambda) = \gamma$.

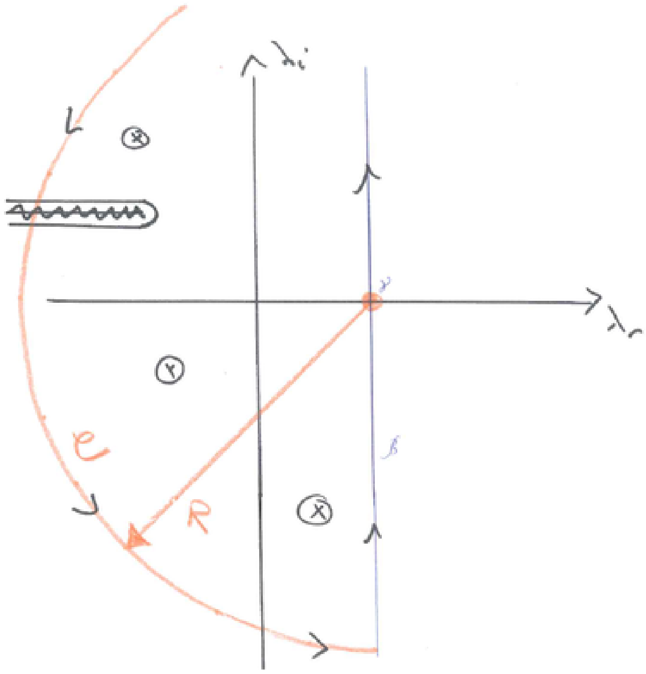

Figure 4: Integration along the Bromwich contour using the Residue Theorem

Example

Let $f(\lambda) = k/(\lambda^2 - k^2)$, with $k > 0$ real. If $f(\lambda)$ is a Laplace transform, compute the generating function of the transofrm.

We compute

$$F(t) = \frac{1}{2\pi\mathrm{i}} \int_{\mathcal{B}} \frac{k\mathrm{e}^{\lambda t}}{\lambda^2 - k^2} \mathrm{d}\lambda,$$

where $\mathcal{B}$ is the Bromiwch contours: it is a straight line parallel to the imaginary axis to the right of the singularities of the integrand

$$\frac{k\mathrm{e}^{\lambda t}}{\lambda^2 - k^2}. \tag{5}$$

Since the singularites of Equation (5) are $\lambda = \pm k$, the Bromwich contour is

$$\mathcal{B} = \{z \in \mathcal{C} | z = (k + \epsilon) + \mathrm{i}y,\, y \in \mathbb{R},\, \epsilon > 0\}.$$

Using the residue theorem, we have

$$\begin{aligned} F(t) &= \mathrm{Res}\left(\frac{k\mathrm{e}^{\lambda t}}{\lambda^2 - k^2}, k\right) + \mathrm{Res}\left(\frac{k\mathrm{e}^{\lambda t}}{\lambda^2 - k^2}, -k\right) \\ &= \lim_{\lambda \to k}\left[(\lambda - k)\frac{k\mathrm{e}^{\lambda t}}{\lambda^2 - k^2}\right] + \lim_{\lambda \to -k}\left[(\lambda + k)\frac{k\mathrm{e}^{\lambda t}}{\lambda^2 - k^2}\right] \\ &= \tfrac{1}{2}\left(\mathrm{e}^{kt} - \mathrm{e}^{-kt}\right) = \sinh(kt), \end{aligned}$$

in agreement with Example 2 in Section 2.

4 Laplace transforms – properties

Throughout this section, let $(F(t), \widehat{F}_\lambda)$ be a valid Laplace-transform pair:

$$\widehat{F}_\lambda = \int_0^\infty F(t)\mathrm{e}^{-\lambda t}\mathrm{d}t, \qquad F(t) = \frac{1}{2\pi\mathrm{i}}\int_{\mathcal{B}} \widehat{F}_\lambda \mathrm{e}^{\lambda t},$$

where $\mathcal{B}$ is the Bromwich contour.

Theorem 4.1 (Substitution) *Let $a \in \mathbb{C}$, and let $f(\lambda) := \widehat{F}_\lambda$ denote the Laplace tranform of the functioni F. Then*

$$f(\lambda - a) = \mathcal{L}\left(\mathrm{e}^{at}F(t)\right).$$

Proof: By direct calculation we have

$$\begin{aligned} f(\lambda - a) &= \widehat{F}_{\lambda-a}, \\ &= \int_0^\infty \mathrm{e}^{-(\lambda-a)t}F(t)\mathrm{d}t, \\ &= \int_0^\infty \mathrm{e}^{-\lambda t}\left[\mathrm{e}^{at}F(t)\right]\mathrm{d}t, \\ &= \mathcal{L}\left(\mathrm{e}^{at}F(t)\right). \end{aligned}$$

Theorem 4.2 (Translation) *Let a be a real positive number and let $f(\lambda) := \widehat{F}_\lambda$. Then*

$$\mathrm{e}^{-b\lambda}f(\lambda) = \int_0^\infty \mathrm{e}^{-\lambda t}F(t-b)H(t-b)\mathrm{d}t,$$

where $H(\cdot)$ is the unit step function,

$$H(x) = \begin{cases} 1, & x > 0, \\ 0, & x < 0. \end{cases}$$

Proof: We have

$$\begin{aligned} \mathrm{e}^{-b\lambda}f(\lambda) &= \int_0^\infty \mathrm{e}^{-b\lambda}\mathrm{e}^{-\lambda t}F(t)\,\mathrm{d}t, \\ &= \int_0^\infty \mathrm{e}^{-(b+t)\lambda}F(t)\,\mathrm{d}t. \end{aligned}$$

Let $\tau = b + t$, with $\tau_{\mathrm{lw}} = b$ and $\tau_{\mathrm{up}} = \infty$. Hence,

$$\mathrm{e}^{-b\lambda}f(\lambda) = \int_b^\infty \mathrm{e}^{-\lambda\tau}F(\tau - b)\,\mathrm{d}\tau.$$

However, consider

$$F(\tau - b)H(\tau - b) = \begin{cases} F(\tau - b), & \tau > b \\ 0, & \tau < b. \end{cases}$$

Hence,

$$
\begin{aligned}
\mathrm{e}^{-b\lambda} f(\lambda) = 0 \times \int_0^b \mathrm{e}^{-\lambda\tau} F(\tau - b)\,\mathrm{d}\tau + 1 \times \int_b^\infty \mathrm{e}^{-\lambda\tau} F(\tau - b)\,\mathrm{d}\tau \\
= \int_0^\infty \mathrm{e}^{-\lambda\tau} F(\tau - b) H(\tau - b)\,\mathrm{d}\tau.
\end{aligned}
$$

Theorem 4.3 (Differentiation in real space) *$F(t)$ be a C^1 function of t, with F and its derivative at worst exponentially diverging. Then $\widehat{(dF/dt)}_\lambda$ exists and*

$$
\widehat{\left(\frac{dF}{dt}\right)}_\lambda = \int_0^\infty \lambda \mathrm{e}^{-\lambda t} F(t)\mathrm{d}t - F(0).
$$

Proof: By assumption, dF/dt is at worst exponentially diverging, and its Laplace transform exists, at least for appropriate λ-values. Also by definition,

$$
\begin{aligned}
\widehat{\left(\frac{dF}{dt}\right)}_\lambda &= \int_0^\infty \mathrm{e}^{-\lambda t} \frac{dF}{dt} \mathrm{d}t, \\
&= \int_0^\infty \left[\frac{d}{dt}\left(\mathrm{e}^{-\lambda t} F\right) + \lambda \mathrm{e}^{-\lambda t} F \right] \mathrm{d}t, \\
&= \lim_{L\to\infty} \mathrm{e}^{-\lambda L} F(L) - F(0) + \int_0^\infty \lambda \mathrm{e}^{-\lambda t} F(t)\mathrm{d}t.
\end{aligned}
$$

For $\Re(\lambda)$ sufficiently large and positive, the limiting boundary term vanishes, and

$$
\widehat{\left(\frac{dF}{dt}\right)}_\lambda \int_0^\infty \lambda \mathrm{e}^{-\lambda t} F(t)\mathrm{d}t - F(0),
$$

as required.

Theorem 4.4 (Differentiation in transform space) *$F(t)$ be piecewise differentiable with respect to t. Then $f(\lambda) := \widehat{F}_\lambda$ is differentiable with respect to λ and, moreover,*

$$
f'(\lambda) = \mathcal{L}\left(-tF(t)\right).
$$

Proof: For suitable λ, the integral

$$
f(\lambda) = \int_0^\infty \mathrm{e}^{-\lambda t} F(t)\mathrm{d}t
$$

is well-defined and is uniformly convergent and may be differentiated under the integral sign with respect to λ. We compute:

$$
\begin{aligned}
f'(\lambda) &= \frac{d}{d\lambda} \int_0^\infty \mathrm{e}^{-\lambda t} F(t)\mathrm{d}t, \\
&= \int_0^\infty \left[\frac{\partial}{\partial\lambda} \mathrm{e}^{-\lambda t} \right] F(t)\mathrm{d}\lambda, \\
&= \int_0^\infty \mathrm{e}^{-\lambda t} \left[-tF(t)\right] \mathrm{d}t, \\
&= \mathcal{L}\left(-tF(t)\right).
\end{aligned}
$$

Definition 4.1 (Convolution) *Let $F(t)$ and $G(t)$ be at-worst exponentially diverging. The convolution of F and G is defined as*

$$(F * G)(t) = \int_0^t F_1(t-\tau)F_2(\tau)\mathrm{d}\tau.$$

Theorem 4.5 (by Faltung) *Let $F(t)$ and $G(t)$ be at-worst exponentially diverging, with Laplace transforms $\widehat{F}_\lambda$ and $\widehat{G}_\lambda$ respectively. Then*

$$\widehat{F}_\lambda \widehat{G}_\lambda = \mathcal{L}\left[(F * G)(t)\right]$$

Proof: Given in Lecture 2.

Absolute and Convective Instabilities in Parallel Flows
Lecture 2

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1 Laplace transforms, examples

In this section, let

$$\begin{array}{rcl} F:[0,\infty) & \rightarrow & \mathbb{C}, \\ t & \mapsto & F(t) \end{array} \tag{1}$$

be a complex-valued function of a real variable, such that

$$|\mathrm{e}^{-\lambda_0 t}F(t)| \leq M, \qquad \text{as } t\rightarrow\infty;$$

The Laplace tranform of $F(t)$ is defined as follows:

$$\widehat{F}_\lambda \equiv \mathcal{L}(F) := \int_0^\infty \mathrm{e}^{-\lambda t}F(t)\mathrm{d}t, \qquad \Re(\lambda) > \lambda_0.$$

Given that $f(\lambda) = \lambda^{-1/2}$ is the Laplace transform of a function, find the generating function.

We must ascribe an unambiguous meaning to $f(z) = z^{1/2}$. We have two possibilities:

$$f(z) = |z|^{1/2}\begin{cases} \cos(\theta/2)+\mathrm{i}\sin(\theta/2), \\ \cos(\pi+\theta/2)+\mathrm{i}\sin(\pi+\theta/2), \end{cases}$$

where $\theta = \arg(z)$. The standard choice is to take $f(z) = |z|^{1/2}\left[\cos(\theta/2)+\mathrm{i}\sin(\theta/2)\right]$. This introduces a discontinuity in $f(z)$ across the half-line $x>0$, since $f(x, 0+\epsilon) = \sqrt{x}$, while $f(x, 2\pi-\epsilon) = \sqrt{x}\cos(\pi)$, and

$$f(x,0+\epsilon) - f(x,2\pi-\epsilon) = 2\sqrt{x}, \qquad x>0.$$

However, we are interested in a situation where the discontinuity should occur across the half-line $x<0$. We therefore unambiguously define

$$f(z) = |z|^{1/2}\begin{cases} \cos(\theta/2)+\mathrm{i}\sin(\theta/2), & 0\leq\theta<\pi, \\ \cos(\pi+\theta/2)+\mathrm{i}\sin(\pi+\theta/2), & \pi<\theta\leq 2\pi \end{cases} := |z|^{1/2}\Theta(\theta).$$

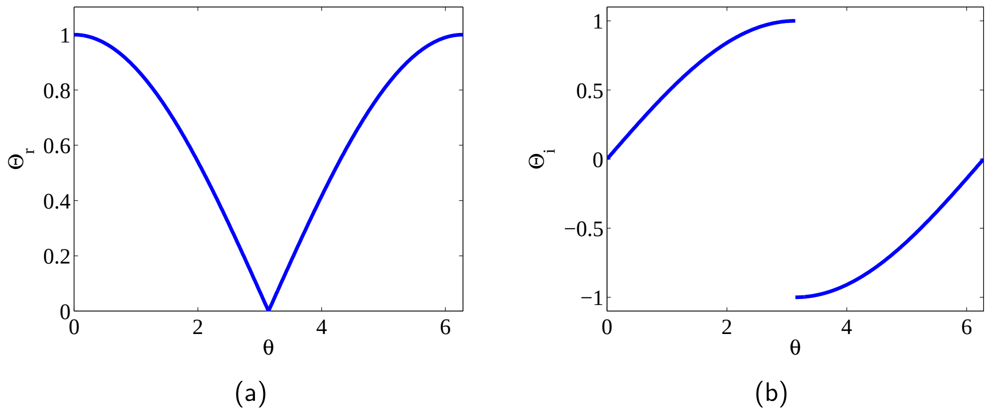

Figure 1: Real and imaginary parts of the complex function $\Theta(\theta)$ showing the jump / cusp at $\theta = \pi$.

The result is plotted in Figure 1. There is a jump / cusp at $\theta = \pi$, meaning that the function $f(z)$ so defined has a branch cut along the half-line

$$\{z = x + 0\mathrm{i}y | x \leq 0\}.$$

Consider now the closed contour $\mathcal{C}$ shown in Figure 2. Since $\mathcal{C}$ encloses no singularities, we have

$$\int_{\mathcal{C}} \frac{\mathrm{e}^{\lambda t}}{\lambda^{1/2}} \mathrm{d}\lambda = 0.$$

Moreover, the contour $\mathcal{C}$ can be regarded as being made up of many parts:

- The Bromwich contour;
- A small semi-circle of radius ϵ centred at zero.
- The lines surrounding the branch cut.
- Semi-circular parts (centred at zero) of radius R, with $R \to \infty$.
- Small linear parts with $z = x \pm \mathrm{i}R$, and $x \in [0, 2\epsilon]$ (say).

We consider these parts separately now, starting with the semi-circle of radius ϵ. This evaluates to

$$\int_{-\pi/2}^{\pi/2} (\epsilon\,\mathrm{i}\,\mathrm{d}\theta)\, \frac{\mathrm{e}^{\epsilon t\cos\theta + \mathrm{i}\epsilon t\sin\theta}}{\epsilon^{1/2}\Theta(\theta)},$$

which vanishes as $\epsilon^{1/2}$ as $\epsilon \to 0$. Also, the semi-circular parts of radius R contain contributions such as

$$\int (R\,\mathrm{i}\,\mathrm{d}\theta)\, \frac{\mathrm{e}^{Rt\cos\theta + \mathrm{i}Rt\sin\theta}}{R^{1/2}\Theta(\theta)}.$$

The limits of integration are unspecified; however, they are in the second and third quadrants where $\cos\theta < 0$. Thus, these contributions vanish as

$$R^{1/2}\mathrm{e}^{-R\alpha}, \qquad \alpha \in \mathbb{R}^+,$$

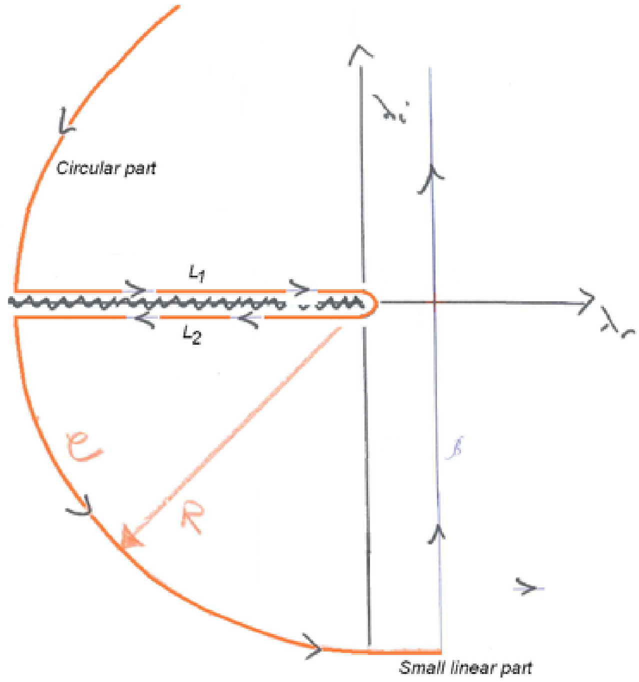

Figure 2: Integration along the Bromwich contour for a function with branch cut along the negative real axis

as $R \to \infty$ (we take $t > 0$). The linear parts vanish similarly. It follows then that

$$F(t) = \frac{1}{2\pi \mathrm{i}} \int_{\mathcal{B}} \frac{\mathrm{e}^{\lambda t}}{\lambda^{1/2}} \mathrm{d}\lambda = -\frac{1}{2\pi \mathrm{i}} \left(\int_{L_1} \mathrm{d}\lambda + \int_{L_2} \mathrm{d}\lambda \right) \frac{\mathrm{e}^{\lambda t}}{\lambda^{1/2}}, \tag{2}$$

where L_1 and L_2 are the contributions from the linear contours surrounding the branch cut.

Consider the integral along L_1. With $\lambda = x + \mathrm{i}\epsilon$, we have

$$\int_{L_1} \frac{\mathrm{e}^{\lambda t}}{\lambda^{1/2}} \mathrm{d}\lambda = \int_{-\infty}^{0} \frac{\mathrm{e}^{(x+\mathrm{i}\epsilon)t}}{(x+\mathrm{i}\epsilon)^{1/2}} \mathrm{d}x.$$

Calling $z = x + \mathrm{i}\epsilon$, we have $\tan\theta = \epsilon/x$, hence

$$\begin{aligned} \sin\theta &= \frac{\epsilon}{x^2+\epsilon^2} \to 0 \text{ as } \epsilon \to 0, \\ \cos\theta &= \frac{x}{x^2+\epsilon^2} \to -1 \text{ as } \epsilon \to 0, \end{aligned}$$

since $x < 0$. Hence,

$$\sin(\theta/2) \to 1, \qquad \cos(\theta/2) \to 0,$$

as $\epsilon \to 0$. Since L_1 is in the upper-half-plane $\theta < \pi$, we take $\lambda^{1/2} = |\lambda|^{1/2}\Theta(\theta)$, with $\Theta(\theta) = \cos(\theta/2) + \mathrm{i}\sin(\theta/2)$. Hence,

$$(x+\mathrm{i}\epsilon)^{1/2} = |x+\mathrm{i}\epsilon|^{1/2}\left[\cos(\theta/2) + \mathrm{i}\sin(\theta/2)\right] \to \mathrm{i}|x|^{1/2},$$

as $\epsilon \downarrow 0$. Thus, we have the following string of relations:

$$
\begin{aligned}
\int_{L_1} \frac{\mathrm{e}^{\lambda t}}{\lambda^{1/2}} \mathrm{d}\lambda &= \int_{-\infty}^{0} \frac{\mathrm{e}^{(x+\mathrm{i}\epsilon)t}}{(x+\mathrm{i}\epsilon)^{1/2}} \mathrm{d}x, \\
&\overset{\epsilon\downarrow 0}{=} \int_{-\infty}^{0} \frac{\mathrm{e}^{xt}}{\mathrm{i}|x|^{1/2}} \mathrm{d}x, \\
&= \int_{-\infty}^{0} \frac{\mathrm{e}^{xt}}{\mathrm{i}(-x)^{1/2}} \mathrm{d}x, \\
&\overset{y=-x}{=} \frac{1}{\mathrm{i}} \int_{0}^{\infty} \frac{\mathrm{e}^{-yt}}{y^{1/2}} \mathrm{d}y, \\
&\overset{X=(ty)^{1/2}}{=} \frac{2}{\mathrm{i}} \int_{0}^{\infty} \mathrm{e}^{-X^2} \mathrm{d}X, \\
&= \frac{2}{\mathrm{i}} \left(\tfrac{1}{2}\sqrt{\pi/t}\right), \\
&= \frac{1}{\mathrm{i}} \sqrt{\pi/t}.
\end{aligned}
$$

We make similar arguments for the second linear contouor. We write $\lambda = x - \mathrm{i}\epsilon$, such that

$$
\int_{L_2} \frac{\mathrm{e}^{\lambda t}}{\lambda^{1/2}} \mathrm{d}\lambda = -\int_{-\infty}^{0} \frac{\mathrm{e}^{(x-\mathrm{i}\epsilon)t}}{(x-\mathrm{i}\epsilon)^{1/2}} \mathrm{d}x.
$$

Calling $z = x - \mathrm{i}\epsilon$, we have $\tan\theta = \epsilon/x$, hence

$$
\begin{aligned}
\sin\theta &= \frac{\epsilon}{\sqrt{x^2+\epsilon^2}} \to 0 \text{ as } \epsilon \to 0, \\
\cos\theta &= \frac{x}{\sqrt{x^2+\epsilon^2}} \to -1 \text{ as } \epsilon \to 0,
\end{aligned}
$$

since $x < 0$. Since L_2 is in the lower-half-plane $\theta < \pi$, we take $\Theta(\theta) = \cos(\pi + \theta/2) + \mathrm{i}\sin(\pi + \theta/2)$, hence

$$
\sin(\pi + \theta/2) \to -1, \qquad \cos(\pi + \theta/2) \to 0,
$$

as $\epsilon \to 0$. Again taking $\Theta(\theta) = \cos(\pi + \theta/2) + \mathrm{i}\sin(\pi + \theta/2)$, we have

$$
(x - \mathrm{i}\epsilon)^{1/2} = |x - \mathrm{i}\epsilon|^{1/2} \left[\cos(\pi + \theta/2) + \mathrm{i}\sin(\pi + \theta/2)\right] \to -\mathrm{i}|x|^{1/2},
$$

as $\epsilon \downarrow 0$. Then, as before, we consider the following string of relations:

$$
\begin{aligned}
\int_{L_2} \frac{\mathrm{e}^{\lambda t}}{\lambda^{1/2}} \mathrm{d}\lambda &= -\int_{-\infty}^{0} \frac{\mathrm{e}^{(x-\mathrm{i}\epsilon)t}}{(x-\mathrm{i}\epsilon)^{1/2}} \mathrm{d}x, \\
&\overset{\epsilon\downarrow 0}{=} -\int_{-\infty}^{0} \frac{\mathrm{e}^{xt}}{-\mathrm{i}|x|^{1/2}} \mathrm{d}x, \\
&= +\int_{-\infty}^{0} \frac{\mathrm{e}^{xt}}{|x|^{1/2}} \mathrm{d}x, \\
&= \frac{1}{\mathrm{i}} \sqrt{\pi/t}.
\end{aligned}
$$

From Equation (2), we have

$$
F(t) = -\frac{1}{2\pi\mathrm{i}} \left(\int_{L_1} \mathrm{d}\lambda + \int_{L_2} \mathrm{d}\lambda \right) \frac{\mathrm{e}^{\lambda t}}{\lambda^{1/2}} = -\frac{1}{2\pi\mathrm{i}} \left(\frac{2}{\mathrm{i}} \sqrt{\pi/t} \right) = \frac{1}{\sqrt{\pi t}}.
$$

2 Laplace transforms – properties

Definition 2.1 (Convolution) *Let $F(t)$ and $G(t)$ be at-worst exponentially diverging. The convolution of F and G is defined as*

$$(F * G)(t) = \int_0^t F_1(t-\tau)F_2(\tau)\mathrm{d}\tau.$$

Theorem 2.1 (by Faltung) *Let $F(t)$ and $G(t)$ be at-worst exponentially diverging, with Laplace transforms $\widehat{F}_\lambda$ and $\widehat{G}_\lambda$ respectively. Then*

$$\widehat{F}_\lambda \widehat{G}_\lambda = \mathcal{L}\left[(F * G)(t)\right]$$

Proof: By direct computation, we have

$$\widehat{F}_\lambda \widehat{G}_\lambda = \int_0^\infty \mathrm{e}^{-\lambda t} F(t)\,\mathrm{d}t \int_0^\infty \mathrm{e}^{-\lambda s} G(s)\,\mathrm{d}s.$$

We first of all re-write the integral as follows:

$$\widehat{F}_\lambda \widehat{G}_\lambda = \lim_{L\to\infty} \int_0^L \mathrm{e}^{-\lambda t} F(t)\mathrm{d}t \int_0^L \mathrm{e}^{-\lambda s} G(s)\,\mathrm{d}s.$$

The **trick** is to re-write this further as

$$\widehat{F}_\lambda \widehat{G}_\lambda = \lim_{L\to\infty} \int_0^L \mathrm{e}^{-\lambda t} F(t)\,\mathrm{d}t \int_0^{L-t} \mathrm{e}^{-\lambda s} G(s)\,\mathrm{d}s.$$

In fact, we have changed the region of integration from an $L \times L$ square to a triangle with vertices at $(0,0)$, $(0,L)$, and $(L,0)$. However, leaving out half the domain of integration does not matter, as the omitted region is 'filled in' as $L \to \infty$ (e.g. Figure 3). Now,

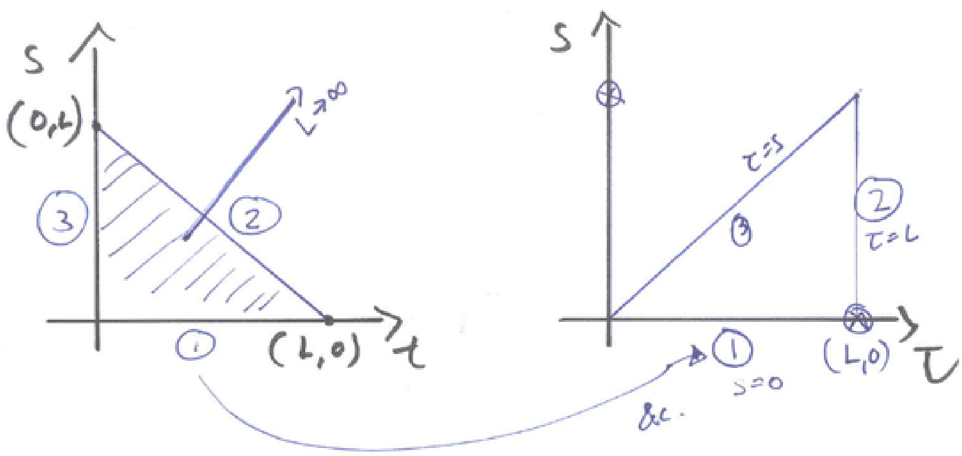

Figure 3: Sketch for the change-of-variables in the Convolution Theorem

we proceed by direct calculation. We want only one free variable in the exponential argument. We do not modify the variable s; instead we define

$$t + s = \tau \implies t = s - \tau$$

Again referring to Figure 3, we have

- Line Segment 1 ($s=0$) is mapped to $s=0$;
- Line Segment 2 ($s=L-t$) implies that $\tau=t+(L-t)=L$ (constant); hence line-segment 2 is mapped to a vertical line segment passing through $\tau=L$.
- The condition on Line Segment 3 ($t=0$) implies $s=\tau$, hence line segment 3 is mapped to the straight line of slope 45^{o} passing through the origin.

Also, consider the transformation, expressed correctly here as

$$\begin{aligned}\tau &= t+s,\\ s' &= s,\end{aligned}$$

with inverse

$$\begin{aligned}t &= \tau-s',\\ s &= s'.\end{aligned}$$

We have

$$\mathrm{d}t\,\mathrm{d}s=\underbrace{\left|\begin{array}{cc}\frac{\partial t}{\partial\tau} & \frac{\partial t}{\partial s'}\\ \frac{\partial s}{\partial\tau} & \frac{\partial s'}{\partial s}\end{array}\right|}_{=J}\mathrm{d}\tau\,\mathrm{d}s'.$$

$$J=\left|\begin{array}{cc}1 & -1\\ 0 & 1\end{array}\right|=1,$$

hence

$$\mathrm{d}t\,\mathrm{d}s=\mathrm{d}\tau\,\mathrm{d}s'.$$

Putting it all together, we have

$$\begin{aligned}
\widehat{F}_\lambda\widehat{G}_\lambda &= \lim_{L\to\infty}\int_0^L \mathrm{e}^{-\lambda t}F(t)\,\mathrm{d}t\int_0^{L-t}\mathrm{e}^{-\lambda s}G(s)\,\mathrm{d}s,\\
&= \lim_{L\to\infty}\int_0^L \mathrm{d}t\int_0^{L-t}\mathrm{d}s\,\mathrm{e}^{-\lambda t}F(t)\mathrm{e}^{-\lambda s}G(s),\\
&= \lim_{L\to\infty}\int_0^L \mathrm{d}\tau\int_0^{\tau}\mathrm{d}s\,F(\tau-s)\mathrm{e}^{-\lambda(\tau-s)}G(s)\mathrm{e}^{-\lambda s},\\
&= \lim_{L\to\infty}\int_0^L \mathrm{d}\tau\,\mathrm{e}^{-\lambda\tau}\int_0^{\tau}\mathrm{d}s\,F(\tau-s)G(s),\\
&= \int_0^\infty \mathrm{d}\tau\,\mathrm{e}^{-\lambda\tau}\left[\int_0^{\tau}\mathrm{d}s\,F(\tau-s)G(s)\right],\\
&= \int_0^\infty \mathrm{d}\tau\,\mathrm{e}^{-\lambda\tau}(F*G)(\tau),\\
&= \mathcal{L}[(F*G)(\tau)].
\end{aligned}$$

Example

Compute the inverse transform of

$$f(\lambda) = \frac{1 - \mathrm{e}^{-a\lambda}}{\lambda}, \qquad a \in \mathbb{R}^+.$$

We break it up into two parts. Consider

$$I_1 = \frac{1}{2\pi\mathrm{i}} \int_{\mathcal{B}} \frac{\mathrm{e}^{\lambda t}}{\lambda} \mathrm{d}\lambda.$$

The Bromwich contour is a straight line parallel to the imaginary axis passing through $z = 0 + \mathrm{i}\epsilon$, with $\epsilon \downarrow 0$. The integrand has a single simple pole at $\lambda = 0$, with

$$\mathrm{Res}\left(\frac{\mathrm{e}^{\lambda t}}{\lambda}, 0\right) = 1.$$

Hence,

$$I_1 = 1, \qquad t > 0.$$

On the other hand, if $t < 0$, to get a convergent integral we would have to close the contour by forming a semi-circle on the **right** of the Bromwich line. However, such a contour encloses no singularities, hence

$$I_1 = 0, \qquad t < 0.$$

We do the second integral by considering

$$I_2 = \frac{1}{2\pi\mathrm{i}} \int_{\mathcal{B}} \frac{\mathrm{e}^{(t-a)\lambda}}{\lambda} \mathrm{d}\lambda.$$

The integrand is

$$\frac{\mathrm{e}^{\lambda_{\mathrm{r}}(t-a)}\mathrm{e}^{\lambda_{\mathrm{i}}(t-a)}}{\lambda}$$

The Bromwich contour is the same as before. For the $\mathcal{B}$-contour given there are two possibilities:

1. $t - a > 0$ – chose $\lambda_{\mathrm{r}} < 0$ – close the contour on the left. Thus, a contribution to the integral is picked up from the pole at $\lambda = 0$.

2. $t - a < 0$ – chose $\lambda_{\mathrm{r}} > 0$ – close the contour on the right. Thus, there are no pole-contributions to the integral and the integal vanishes.

In other words,

$$I_2 = \begin{cases} 1, & \text{if } t > a, \\ 0, & \text{if } t < 1. \end{cases}$$

Finally, the answer is

$$F(t) = H(t) - H(t - a).$$

However, from a sketch, this can be seen to be a top-hat function:

$$F(t) = \begin{cases} 0, & \text{if } t < 0, \\ 1, & \text{if } 0 < t < a, \\ 0, & \text{if } t > a. \end{cases}$$

There is another way of getting at the second integral I_2. From the translation theorem, we have

$$\mathrm{e}^{-a\lambda}\widehat{\phi}_\lambda = \int_0^\infty \mathrm{e}^{-\lambda t}\phi(t)H(t-a)\mathrm{d}t,$$

Taking $\phi(t) = H(t)$, with $\widehat{\phi}_\lambda = 1/\lambda$, we have

$$\begin{aligned} \frac{\mathrm{e}^{-a\lambda}}{\lambda} &= \int_0^\infty \mathrm{e}^{-\lambda t}H(t)H(t-a)\mathrm{d}t, \\ &= \int_0^\infty \mathrm{e}^{-\lambda t}H(t-a)\mathrm{d}t. \end{aligned}$$

Hence, the Laplace transform of $H(t-a)$ is $\mathrm{e}^{-a\lambda}/\lambda$, hence

$$\frac{1}{2\pi\mathrm{i}}\int_{\mathcal{B}}\left(\frac{\mathrm{e}^{-a\lambda}}{\lambda}\right)\mathrm{e}^{\lambda t}\mathrm{d}\lambda = H(t-a),$$

as computed already, using a direect approach.

Absolute and Convective Instabilities in Parallel Flows
Tutorial 1

Dr Lennon Ó Náraigh

1 Bateman Equations

Solve

$$
\begin{aligned}
\frac{dN_1}{dt} &= -\lambda_1 N_1, \\
\frac{dN_2}{dt} &= -\lambda_2 N_2 + \lambda_1 N_1, \\
\frac{dN_3}{dt} &= \lambda_2 N_2,
\end{aligned}
$$

subject to $N_1(0) = n_1 > 0$, $N_2(0) = n_2 \geq 0$, and $N_3(0) = n_3 \geq 0$, where $(\lambda_1, \lambda_2, \lambda_3)$ are positive constants.

2 Poles other than simple poles

Solve

$$
\frac{d^2x}{dt^2} - 2\frac{dx}{dt} + x = 0,
$$

subject to $x(t = 0) = x_0$ and $\dot{}(t = 0) = v_0$.

Absolute and Convective Instabilities in Parallel Flows
Lecture 3

Dr Lennon Ó Náraigh

1 Background

We are interested in the following Cauchy problem:

$$\frac{\partial}{\partial t}\nabla^2\phi + U_0(z)\nabla^2\frac{\partial\phi}{\partial x} - U_0''(z)\frac{\partial\phi}{\partial x} = \frac{1}{Re}\nabla^4\phi, \tag{1a}$$

with boundary conditions

$$\phi = \phi_z = 0, \qquad \text{at } z = 0, 1, \tag{1b}$$

and

$$\lim_{|x|\to\infty}\phi = \lim_{|x|\to\infty}\phi_x = 0, \tag{1c}$$

subject to an initial impulsive disturbance

$$\phi(x, z, t = 0) = \delta(x - x_0)\delta(z - z_0). \tag{1d}$$

2 The solution

Equation (1a) is rewritten in Orr–Sommerfeld operator form as follows:

$$\frac{\partial}{\partial t}\mathcal{M}_{OS}[\partial_x, \partial_z]\psi(x, z, t) = \mathcal{L}_{OS}[\partial_x, \partial_z]\psi(x, z, t) + F(x, z, t), \qquad \psi(x, z, t = 0) = 0. \tag{2}$$

Here, $F(x, z, t)$ represents the momentum source; this can either be continuous-in-time, or be an initial impulse imposed on the system. In the impulsive case, $S(x, z, t) = \delta(t)F(x, z)$, with Laplace transform $S(x, z, \lambda) = F(x, z)$. The solution can be written in terms of Fourier transforms as follows:

$$\psi(x, z, t) = \int_{-\infty}^{\infty}\frac{\mathrm{d}\alpha}{2\pi}\mathrm{e}^{\mathrm{i}\alpha x}\widetilde{\psi}_\alpha(z, t),$$

where the Fourier coefficients $\widetilde{\psi}_\alpha(z, t)$ satisfy

$$\frac{\partial}{\partial t}\mathcal{M}_{OS}[\mathrm{i}\alpha, \partial_z]\widetilde{\psi}_\alpha(z, t) = \mathcal{L}_{OS}[\mathrm{i}\alpha, \partial_z]\widetilde{\psi}_\alpha(z, t) + \delta(t)\widetilde{F}_\alpha(z, t).$$

Each Fourier component $\widetilde{\psi}_\alpha(z,t)$ can be decomposed further via an inverse Laplace tranform:

$$\widetilde{\psi}_\alpha(z,t) = \int_B \mathrm{d}\lambda\, \mathrm{e}^{\lambda t}\widetilde{\psi}_{\alpha\lambda}(z),$$

where B is the Bromwich contour. The components $\widetilde{\psi}_{\alpha\lambda}(z)$ of the inverse Laplace tranform in turn satisfy

$$\lambda\mathcal{M}_{OS}(\mathrm{i}\alpha, z)\widetilde{\psi}_{\alpha\lambda}(z) = \mathcal{L}_{OS}(\mathrm{i}\alpha, \partial_z)\widetilde{\psi}_{\alpha\lambda}(z) + \widetilde{F}_\alpha(z,\lambda), \tag{3}$$

where the Laplace transform of the force function S has been taken with respect to time. This is the **Orr–Sommerfeld eigenvalue problem**.

The (formal) solution to Equation (3) reads

$$\widetilde{\psi}_{\alpha\lambda}(z) = \left[\mathcal{L}_{OS} - \lambda\mathcal{M}_{OS}\right]^{-1}\widetilde{F}_\alpha(z,\lambda)$$

This purely formal solution is understood as follows. We write the solution of Equation (3) as $\widetilde{\psi}_{\alpha\lambda}(z) = \sum_n a_n\phi_{\alpha n}(z)$. Here, the $\phi_{\alpha n}$'s are the eigenfunctions of the Orr–Sommerfeld equation at wavenumber α. Equation (3) is therefore re-written as

$$\lambda\sum_n a_n\mathcal{M}_{OS}\phi_{\alpha n}(z) = \sum_n a_n\mathcal{L}_{OS}\phi_{\alpha n}(z) + \widetilde{F}_\alpha(z,\lambda). \tag{4}$$

The eigenfunctions $\phi^+_{\alpha m}(z)$ of the adjoint OS problem satisfy

$$\int \mathrm{d}z\,\left[\phi^+_{\alpha m}(z)\right]^*\mathcal{M}_{OS}\phi_{\alpha n}(z) = \delta_{nm}.$$

We multiply both sides of Equation (3) by $\left[\phi^+_m(z)\right]^*$ and integrate with respect to z; the result is

$$a_m = \frac{1}{\lambda - \lambda_m}\int \mathrm{d}z\,\left[\phi^+_{\alpha m}(z)\right]^*\widetilde{F}_\alpha(z,\lambda),$$

and

$$\psi_{\alpha\lambda}(z) = \left[\mathcal{L}_{OS} - \lambda\mathcal{M}_{OS}\right]^{-1}\widetilde{F}_\alpha(z,\lambda) = \sum_n \frac{\phi_{\alpha n}(z)}{\lambda-\lambda_n}\int \mathrm{d}z\,\left[\phi^+_{\alpha n}(z)\right]^*\widetilde{F}_\alpha(z,\lambda) := \sum_n \frac{\phi_{\alpha n}(z)F_{\alpha n}(\lambda)}{\lambda-\lambda_n}.$$

Thus, the solution to the Cauchy problem (2) becomes

$$\psi(x,z,t) = \int_{-\infty}^{\infty}\frac{\mathrm{d}\alpha}{2\pi}\mathrm{e}^{\mathrm{i}\alpha x}\int_B \mathrm{d}\lambda\,\mathrm{e}^{\lambda t}\sum_n \frac{\phi_{\alpha n}(z)F_{\alpha n}(\lambda)}{\lambda-\lambda_n}, \tag{5}$$

where the Bromwich contour C is a straight line parallel to the imaginary axis, to the right of all the eigenvalues $\{\lambda_n\}$ of the Orr–Sommerfeld equation. A key property of Equation (5) is the absence of any contributions from a continuous spectrum: for a bounded domain $z \in [0,1]$, the spectrum of the Orr–Sommerfeld equation is entirely discrete.

To make progress, we make a simplifying assumption, without any loss of generality:

Assumption 2.1 *The spectrum of the OS operator is non-degenerate:*

$$\lambda_m = \lambda_n \implies m = n.$$

Then, each pole in the the innermost integral in Equation (5) is first-order. Using the theory of residues, one obtains for the innermost integral

$$\psi(x,z,t) = \sum_n \int_{-\infty}^{\infty} \frac{\mathrm{d}\alpha}{2\pi} \mathrm{e}^{\mathrm{i}\alpha x + \lambda_n t} F_{\alpha n}(\lambda_n)\phi_{\alpha n}(z). \tag{6}$$

The outermost (α-) integral can be computed in certain special cases. This is discussed in the next two sections.

3 Explicit asymptotic solutions

3.1 Monochromatic forcing

For monochromatic, impulsive forcing, $F(x,z,t) = \mathrm{e}^{\mathrm{i}\alpha_0 x}\delta(t)f(z)$, $\widetilde{F}(z,\lambda) = 2\pi f(z)\delta(\alpha - \alpha_0)$, and

$$F_{\alpha n}(\lambda) = \delta(\alpha - \alpha_0)\int \mathrm{d}z \left[\phi^+_{\alpha_0 n}(z)\right]^* f(z) := \delta(\alpha-\alpha_0) f_{\alpha_0 n}.$$

From Equation (6),

$$\psi(x,z,t) = \sum_n \mathrm{e}^{\mathrm{i}\alpha_0 x + \lambda_n(\alpha_0)t} f_{\alpha_0 n}\phi_{\alpha_0 n}(z). \tag{7}$$

Thus,

$$\lim_{t\to\infty} \psi(x,z,t) = \left[f_{\alpha_0 n_{\max}} \mathrm{e}^{\mathrm{i}\alpha_0 x + \lambda_{n_{\max}}(\alpha_0)}\right] \phi_{\alpha_0 n_{\max}}(z),$$

where $n_{\max}$ is that eigenvalue whose real part is maximal over the entire spectrum $\{\lambda_n(\alpha_0)\}$. Note also,

$$\lim_{t\to\infty} \|\psi\|_2(t) = \|\phi_{\alpha_0 n_{\max}}\|_2 \left|f_{\alpha_0 n_{\max}}\right| \mathrm{e}^{\Re[\lambda_{n_{\max}}(\alpha_0)]t},$$

where $\|\cdot\|_2(t)$ is the transient L^2 norm; for a function $\Phi(x,z,t)$,

$$\|\Phi\|_2(t) := \left(\iint \mathrm{d}x\mathrm{d}z |\Phi(x,z,t)|^2\right)^{1/2}.$$

Thus, as $t\to\infty$, the disturbance grows exponentially fast, at a rate

$$\Re[\lambda_{n_{\max}}(\alpha_0)].$$

This is called the **most-dangerous mode**. Obviously, if $\Re[\lambda_{n_{\max}}(\alpha_0)] > 0$ the system is (convectively) unstable. The system is completely linearly stable if $\Re[\lambda_{n_{\max}}(\alpha_0)] < 0$.

3.2 Localized impulsive forcing

For localized impulsive forcing, $F(x,z,t)=\delta(x)\delta(z-z_0)\delta(t)$, with $\widetilde{F}_\alpha(z,\lambda)=\delta(z-z_0)$, and

$$F_{\alpha n}(\lambda)=[\phi^+_{\alpha n}(z_0)]^*$$

Thus,

$$\psi(x,z,t)=\sum_n\int_{-\infty}^{\infty}\frac{\mathrm{d}\alpha}{2\pi}\mathrm{e}^{\mathrm{i}\alpha x+\lambda_n t}[\phi^+_{\alpha n}(z_0)]^*\phi_{\alpha n}(z). \tag{8}$$

This integral can be regarded as being in the form

$$\psi(x,z,t)=\frac{1}{2\pi}\sum_n\int_{-\infty}^{\infty}\mathrm{d}\alpha\,\mathcal{F}_n(\alpha)\mathrm{e}^{\lambda_n(\alpha)t},$$

where

$$\mathcal{F}_n(\alpha)=\mathrm{e}^{\mathrm{i}\alpha x}[\phi^+_{\alpha n}(z_0)]^*\phi_{\alpha n}(z)$$

This integral is now in a form where the saddle-point method can be applied. We need the following additional assumptions:

Assumption 3.1

1. *The phase function $\lambda_n(\alpha)$ has a single dominant saddle point;*
2. *The saddle point is not degenerate.*
3. *If the phase functions or the $\mathcal{F}_n(\alpha)$'s do have singularities, they are located 'far away' from the saddle point, in the sense that they do not prevent us from deflecting the contour $\alpha\in(-\infty,\infty)$ to pass through the dominant saddle point.*

For the dominant saddle point, we compute

1. Saddle-point location: α_0, such that $(d\lambda_n/d\alpha)_{\alpha_0}=0$.
2. The value $\lambda_n(\alpha_0)$,
3. The derivative $\lambda_n''(\alpha_0)$
4. The phase $\varphi_n=\frac{1}{2}\pi-\frac{1}{2}\arg(\lambda_n''(\alpha_0))$
5. $\mathcal{F}_n(\alpha_0)$.

Applying the saddle-point method to the integral (8), we get

$$\psi(x,z,t)\sim\frac{1}{2\pi}\sum_n\frac{\sqrt{2\pi}\mathrm{e}^{\mathrm{i}\alpha_0 x}[\phi^+_{\alpha_0 n}(z_0)]^*\phi_{\alpha_0 n}(z)\mathrm{e}^{-\lambda_n(\alpha_0)t}\mathrm{e}^{\mathrm{i}\varphi_n}}{|t\lambda_n''(\alpha_0)|^{1/2}} \tag{9}$$

We assume a dominant saddle point, taken over the full OS spectrum. We therefore take $\alpha_{0,n_{\max}}$ to be the mode corresponding to

$$\sup_n \Re\left[\lambda(\alpha_{0,n})\right]$$

Then, the limit (9) simplifies further:

$$\lim_{t\to\infty}\psi(x,z,t)=\frac{\mathrm{e}^{\mathrm{i}\varphi_{n_{\max}}}}{\sqrt{2\pi}}\frac{[\phi^{+}_{\alpha_0 n_{\max}}(z_0)]^*\phi_{\alpha_0 n_{\max}}(z)}{\left|t\frac{d^2\lambda_{n_{\max}}}{d\alpha^2}\Big|_{\alpha_0}\right|^{1/2}}\mathrm{e}^{\mathrm{i}\alpha_0 x+\lambda_{n_{\max}}(\alpha_0)t}. \tag{10}$$

Absolute instability

From Equation (10), we see that the instability grows (asymptotically) at the source $x=0$ if $\Re\left[\lambda_{n_{\max}}(\alpha_0)\right]>0$. This is the notion of linear absolute instability. Of course, the simplification that leads from Equation (8) to Equation (10) is possible only when the phase function possesses no singularities close to the saddle point, and when the saddle point is non-degenerate.

The pinching criterion

Recall the formal solution to Cauchy problem (2) (Equation (5)):

$$\psi(x,z,t)=\int_{-\infty}^{\infty}\frac{\mathrm{d}\alpha}{2\pi}\mathrm{e}^{\mathrm{i}\alpha x}\int_B \mathrm{d}\lambda\,\mathrm{e}^{\lambda t}\sum_n\frac{\phi_{\alpha n}(z)F_{\alpha n}(\lambda)}{\lambda-\lambda_n}, \tag{11}$$

where the λ-integration is done first. To get a self-consistent answer, it should be possible to reverse the order of integration, doing the α-integral first, to arrive at a result identical to Equation (10). However, only so-called **pinching saddles** satisfy this self-consistency property. A pinching saddle is one where the α-curves of constant ω (in particular, curves of constant ω, with $\omega_{\mathrm{i}}=\omega_{\mathrm{i}}(\alpha_0)$) ramify into different half-planes. See Huerre and Monkewitz [1990].

References

P. Huerre and P. A. Monkewitz. Local and global instability in spatially developing flows. *Ann. Rev. Fluid Mech.*, 22:473–537, 1990.

Absolute and Convective Instabilities in Parallel Flows
Lecture 4

Dr Lennon Ó Náraigh

1 Background

We are interested in the following Cauchy problem:

$$\frac{\partial}{\partial t}\nabla^2\phi + U_0(z)\nabla^2\frac{\partial\phi}{\partial x} - U_0''(z)\frac{\partial\phi}{\partial x} = \frac{1}{Re}\nabla^4\phi, \tag{1a}$$

with boundary conditions

$$\phi = \phi_z = 0, \qquad \text{at } z = -H, H, \tag{1b}$$

and

$$\lim_{|x|\to\infty}\phi = \lim_{|x|\to\infty}\phi_x = 0, \tag{1c}$$

subject to an initial impulsive disturbance

$$\phi(x, z, t = 0) = \delta(x - x_0)\delta(z - z_0). \tag{1d}$$

Under a Fourier–Laplace transform, the relevant equation (1a) reduces to the more familiar Orr–Sommerfeld eigenvalue equation:

$$\mathrm{i}\alpha\left(U_0(z) - c\right)\left(\partial_z^2 - \alpha^2\right)\phi_\alpha(z) - \mathrm{i}\alpha U_0''(z)\phi_\alpha(z) = \frac{1}{Re}\left(\partial_z^2 - \alpha^2\right)^2\phi_\alpha, \tag{2}$$

or in operator form,

$$\lambda\mathcal{M}_{OS}(\mathrm{i}\alpha, z)\phi_\alpha(z) = \mathcal{L}_{OS}(\mathrm{i}\alpha, \partial_z)\phi_\alpha(z), \tag{3}$$

where $\lambda = -\mathrm{i}\alpha c$. We study the model (dimensionless) velocity field

$$U_0(z) = 1 - \Lambda + 2\Lambda\left\{1 + \sinh^{2N}\left[z\sinh^{-1}(1)\right]\right\}^{-1}, \qquad \Lambda < 0, \tag{4}$$

where Λ and N are dimensionless parameters. In this model flow, the z-parameter takes values in the range $-\infty < z < \infty$. However, we introduce artificail confinement, such that z takes values in the range $[-H, H]$, where H is chosen to be large in an appropriate numerical sense.

Equation (4) models the steady wake profile generated by flow past a bluff body. The quantity $\Lambda = (U_\mathrm{c} - U_\mathrm{max})/(U_\mathrm{c} + U_\mathrm{max})$ is the velocity ratio, where U_c is the wake centreline velocity and $U_{\max}$ is the maximum velocity. Furthermore, N is the shape parameter, which controls the ratio between the mixing-layer thickness and the width of the wake. It ranges from $N = 1$ (the 'sech^2 wake') to $N = \infty$, a 'top-hat wake' bounded by two vortex sheets [Monkewitz, 1988]. The base state (4) is known to be absolutely unstable [Monkewitz, 1988]. The aim of this practical session is to confirm this absolute instability using the saddle-point method.

2 Numerical solution

We consider a standard Chebyshev collocation method using Chebyshev polynomials as the basis functions, wherein confinement is introduced at $z = \pm H$, such that $\phi_\alpha(\pm H) = \phi'_\alpha(\pm H) = 0$. A trial solution involving the Chebyshev polynomials $T_j(\cdot)$ is proposed:

$$\phi_\alpha(z) \approx \sum_{j=0}^{M} a_j T_j(x), \qquad x = \frac{z}{H}, \qquad x \in [-1, 1] \tag{5}$$

this reduces the differential equations (2) to a finite-dimensional eigenvalue problem. The variable x is a simple linear transformation of the z-coordinate, whose range is confined to $[-1, 1]$. The trial solution for $\phi_\alpha(z)$ is substituted into the differential equation (2) and evaluated at $M-3$ interior points. This gives $M-3$ equations in $M+1$ unknowns; the system is closed by evaluating the trial functions at the boundaries $z = \pm H$ (4 further equations). In this way, a finite-dimensional analogue of Equation (3) is obtained:

$$\boldsymbol{A}\boldsymbol{v} = \lambda \boldsymbol{B}\boldsymbol{v}, \tag{6}$$

where $\boldsymbol{A}$ and $\boldsymbol{B}$ are $(M+1) \times (M+1)$ complex matrices, and

$$\boldsymbol{v} = (a_0, \cdots, a_M)^T$$

is a complex column-valued column vector. The eigenvalue λ is obtained using a standard eigenvalue solver.

3 Exercise

Download the Orr–Sommerfeld code from the website and run it over a range of complex wavenumbers, picking out the saddle point(s) and thereby determining whether the flow is stable, convectively unstable, or absolutely unstable. Consider the following relevant paramter groups:

$$(Re, \Lambda, N) = (100, -1.1, 5),$$

and

$$(Re, \Lambda, N) = (100, -1.1, 2),$$

Also, take $H = 8$.

The code is in a tar file which has been encrypted. You must first of all decrypt the tar file, using the command

```
gpg --output myfolder.tar --decrypt matlab_code.tar.gpg
```

I can give you the password. Then, extract the folder:

```
tar -xvf tar -xvf myfolder.tar
```

References

P. A. Monkewitz. The absolute and convective nature of instability in two-dimensional wakes at low Reynolds numbers. *Phys. Fluids*, 31:999, 1988.