Some new mathematical insights into the rim-lamella model for droplet spreading

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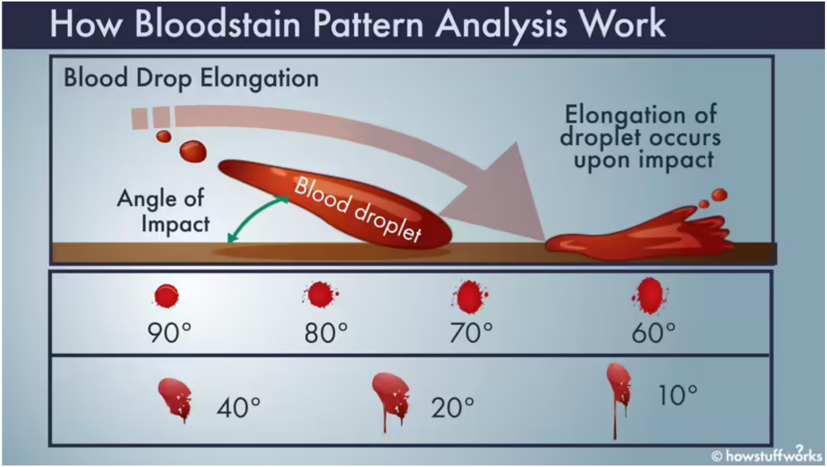
Introduction

- I will look at droplet impact on a smooth surface.
- **Impact, Spread, Retraction**
  In the land of splashes, what the scientist knows as Inertia and Surface Tension are the sculptors in liquids, and fashion from them delicate shapes none the less beautiful because they are too ephemeral for any eye but that of the high-speed camera [Yarin, Annu. Rev. Fluid Mech. (2006)]
- Highlights the importance of **parameters** in such studies; key parameters are the **Weber number**, $\mathrm{We} =$ Inertia/Surface Tension, and the **Reynolds number** – *sculptor has two tools*.

Motivation

One particular application in Bloodstain Pattern analysis:



Bloodstain Pattern Analysis – Brief Tutorial

Six parameters needed to trace back to the original source of the bloodstain:

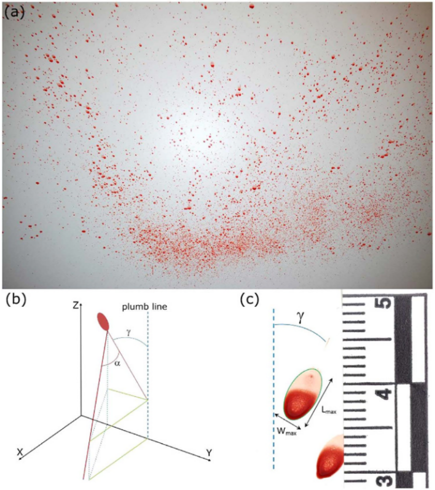
- x, y, z
- α, γ,
- **Impact speed** U_0

Parameters x, y, z, and γ can be measured directly.

Parameter α can be inferred from $\sin\alpha = W_{max}/L_{max}$.

This leaves the impact speed.

Theory due to Laan et al.[1]



[1] Laan, N., de Bruin, K.G., Slenter, D., Wilhelm, J., Jermy, M. and Bonn, D., 2015. Bloodstain pattern analysis: implementation of a fluid dynamic model for position determination of victims. Scientific reports, 5(1), p.11461.

Bloodstain Pattern Analysis – Brief Tutorial

- Impact speed can be estimated from a correlation:

$$\frac{W_{max}}{D_0} = a_0 \mathrm{Re}^{1/5} \frac{P^{1/2}}{a_1 + P^{1/2}(\sin\alpha)^{4/5}}, \qquad P = \mathrm{We}\,\mathrm{Re}^{-2/5}.$$

- a_0 and a_1 are fitting parameters.
- W_{max} can be measured, D_0 can be inferred through the volume of a bloodstain.
- $\mathrm{We} \propto U_0^2$ and $\mathrm{Re} \propto U_0$ – equation can be solved to yield U_0.

Spreading radius

- Normal impact, $\alpha = \pi/2$:

$$\frac{D_{max}}{D_0} = a_0 \mathrm{Re}^{1/5} \frac{P^{1/2}}{a_1 + P^{1/2}}$$

- A classical problem in droplet impact – D_{max}/D_0 is the **spreading ratio**.
- For large P (binomial approximation) we obtain:

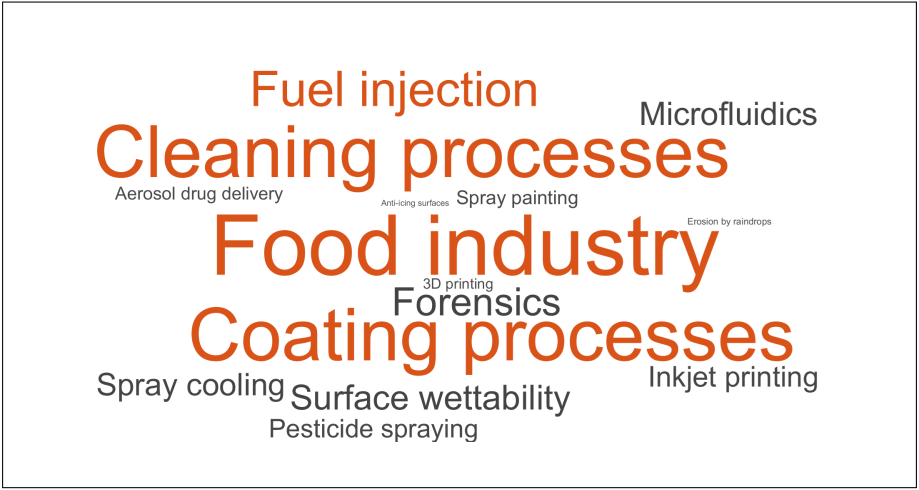
$$\frac{D_{max}}{D_0} = a_0 \mathrm{Re}^{1/5} - a_0 \cdot a_1 \, \mathrm{We}^{-1/2} \mathrm{Re}^{2/5},$$

which is the semi-empirical correlation obtained by Roisman[2].

[2] Roisman, I.V., 2009. Inertia dominated drop collisions. II. An analytical solution of the Navier–Stokes equations for a spreading viscous film. Physics of Fluids, 21(5).

Other applications

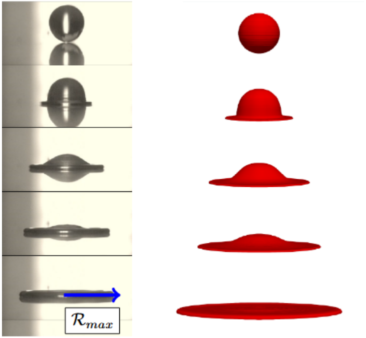
Wordcloud, weighted by Google Scholar hits on 22/09/2025:



Important to emphasize that **scientific curiosity** is a main motivation here.

Splash Threshold

- Droplet spreading below **splash threshold** (no splash), $K \lesssim 3,000$, where $K = \mathrm{We}\sqrt{\mathrm{Re}}$
- At low We, droplet spreads out into a pancake structure – rim and lamella.
- We work in this regime, to figure out the dependence of the spreading ratio on We and Re.



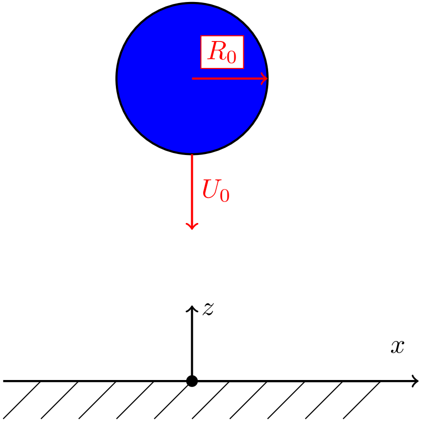
Droplet impact study. Left: high-speed camera. Right: OpenFOAM simulations. Credit: Conor Quigley. Parameters: $\mathrm{Re} = 1700$ and $\mathrm{We} = 20$.

For the avoidance of doubt...

We use the following definitions for the Reynolds and Weber numbers:

$$\mathrm{Re} = \frac{\rho U_0 R_0}{\mu},$$
$$\mathrm{We} = \frac{\rho U_0^2 R_0}{\gamma},$$

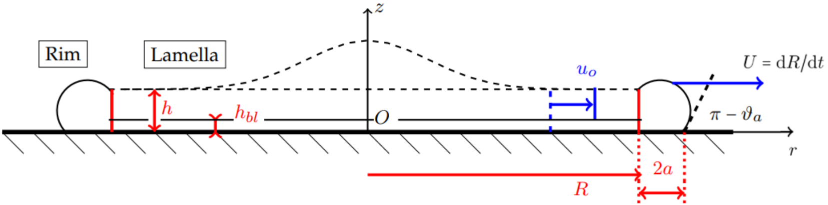
where ρ is the fluid density, μ the viscosity, and γ the surface tension.



Rim-Lamella Models

Below the splash threshold, we may use a rim-lamella model to describe the spreading ratio.

- General model for describing dynamics of rim-lamella structure.
- Mass and momentum equations for the rim.
- Driven by fluxes from the lamella into the rim.
- Balanced by the tendency of surface tension to promote retraction.



- Key variables are rim position R, rim velocity U, rim volume V, and lamella height h.

Context

Using a rim-lamella model as a fundamental assumption, we have proved rigorous results:

Inviscid case:

$$\frac{\mathcal{R}_{max}}{R_0} = k\mathrm{We}^{1/2}, \qquad \mathrm{Re} = \infty,$$

Viscous case:

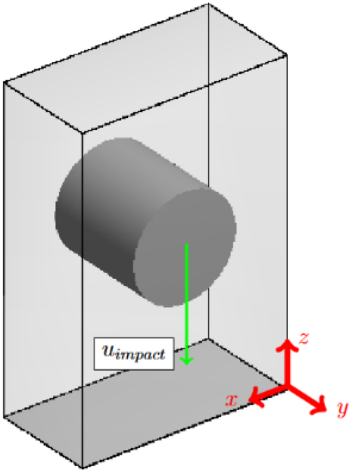
$$k_1\mathrm{Re}^{1/5} - k_2(1-\cos\vartheta_a)^{1/2}\mathrm{Re}^{2/5}\mathrm{We}^{-1/2} \le \frac{\mathcal{R}_{max}}{R_0} \le k_1\mathrm{Re}^{1/5}, \qquad \mathrm{Re} < \infty.$$

Here, k, k_1, and k_2 are constant.

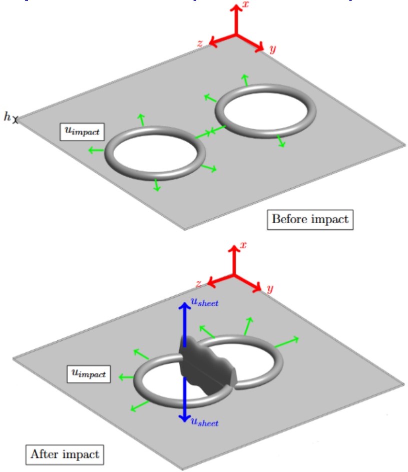
Point of Departure: 2D Droplet Impact

Droplet spreading is fundamentally 3D (axisymmetric). Mathematically, 2D droplets are possible, **so why not study them**?

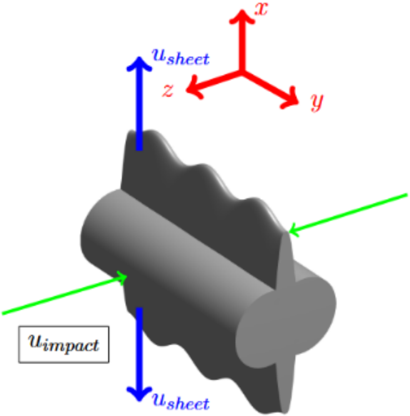
- Can't be realised experimentally (?) because of Rayleigh–Plateau instability.
- But some experiments come close.
- **Aim of this talk**: to formulate 2D model and get results:
  - 2D model way more tractable analytically.
  - Exact solutions possible in the inviscid case.
- **Terminology:** 2D Cartesian Droplet



Experiments (and DNS) that come close...



Néel *et al.*[3]



Tang *et al.*[4]

[3] Néel, B., Lhuissier, H. and Villermaux, E., 2020. 'Fines' from the collision of liquid rims. Journal of Fluid Mechanics, 893, p.A16.

[4] Tang, K., Adcock, T.A.A. and Mostert, W., 2024. Fragmentation of colliding liquid rims. Journal of Fluid Mechanics, 987, p.A18.

Collaborators

Faculty	Alidad Amirfazli (York University Toronto) Miguel Bustamante (UCD) *Conceptualization, Analysis*
PhD Student	Yating Hu (York University Toronto) *Analysis*
Master's Student	Juan Mairal (UCD) *CFD*
Final-Year Project Students	Conor Quigley (UCD), Joseph Anderson (UCD), Patrick Murray (UCD) *High-speed camera work, CFD*
Summer Research Student	Nicola Young (UCD) *Analysis, CFD*

Rim-Lamella Modelling – Inviscid Case

After impact, a rim-lamella structure forms. Pressure rapidly decays to zero, such that 1D Euler equations (inviscid) applies:

$$
\begin{aligned}
\frac{\partial h}{\partial t} + \frac{\partial}{\partial x}(uh) &= 0, \\
\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} &= 0.
\end{aligned}
$$

- Valid for $t \geq \tau$ and $r \in (0, R)$.
- R marks the end of the lamella and the start of the rim.
- Exact solution:

$$
u = \frac{x}{t+t_0}, \qquad h = (t+t_0)^{-1} f\left(\frac{r}{t+t_0}\right).
$$

- The function f is not specified in this analysis (method of characteristics).

2D Rim-Lamella Model – Inviscid

Mass and momentum balance in the rim are described by the following ordinary differential equations, valid in the inviscid limit:

$$\begin{aligned} \frac{\mathrm{d}V}{\mathrm{d}t} &= 2\left(u_0 - U\right)h, \qquad \frac{\mathrm{d}R}{\mathrm{d}t} = U, \\ V\frac{\mathrm{d}U}{\mathrm{d}t} &= 2\left(u_0 - U\right)^2 h - \frac{2\gamma}{\rho}\left(1 - \cos\vartheta_a\right). \end{aligned}$$

- V is the **area** of the 2D rim.
- $u_0 = R/(t + t_0)$.
- $h \equiv h(R,t) = [(\tau + t_0)/(t + t_0)]h_{init}$, $h(R,t) \sim (t + t_0)^{-1}$.

Key observation: The corresponding rim-lamella model in the 3D axisymmetric case has a geometric factor of $2\pi R$ on the right-hand side. The fact that this factor does not occur in 2D rim-lamella model is important.

2D Rim-Lamella Model – Exact Solution

Velocity defect $\Delta = u_0 - U$. Equation for $V\Delta$:

$$\frac{\mathrm{d}}{\mathrm{d}t}(V\Delta) + \frac{V\Delta}{t+t_0} = 2hc^2 = 2(\gamma/\rho)(1-\cos\vartheta_a).$$

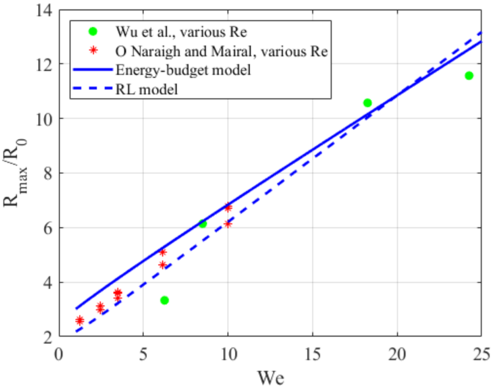
- Exact solution for $V\Delta$.
- Substitute back into $\Delta = [R/(t+t_0)] - (\mathrm{d}R/\mathrm{d}t)$ and solve for R.
- Result is complicated but it is in closed form.
- Crucially, we calculate:

$$\frac{R_{max}}{R_0} \approx \tfrac{1}{27}\frac{V_{tot}^2}{h_{init}U_0^2(\tau+t_0)^2 R_0}\left(\frac{V_{tot}}{2h_{init}R_0}\right)\frac{\mathrm{We}}{1-\cos\vartheta_a}, \qquad \mathrm{We} \gg 1.$$

- $R_{max}/R_0 \sim \mathrm{We}$ is an intrinsically 2D scaling behaviour and was observed by Néel *et al.* in their work on the colliding liquid cylinders.

2D Rim-Lamella Model – Compare with CFD

- Head-on collision of cylindrical droplets a common setup in CFD tests.
- Equivalent to impact on hard surface with $\vartheta_a = 90°$, once Re is sufficiently large (no boundary layer).
- Opportunity to compare RL model with CFD.



Rim-Lamella model: Viscous Case

Previous description is modified by the presence of a viscous **boundary layer**.

- BL theory gives $h_{bl} \sim \sqrt{\nu x / u_o}$
- u_o is the **outer flow**, $u_o = x/(t + t_0)$
- Combination gives $h_{bl} \sim \sqrt{\nu(t + t_0)}$, independent of r.
- Introduce extra degrees of freedom to get $h_{bl} = \xi \sqrt{\nu(t + t_1)}$.

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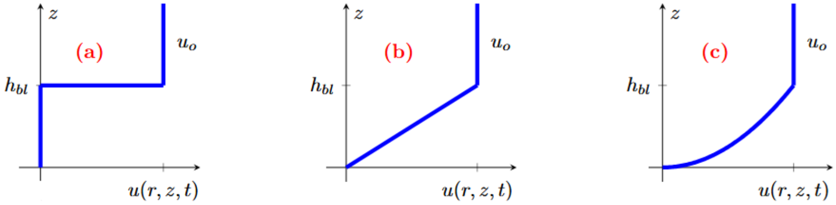
Flow transitions rapidly to zero across BL:



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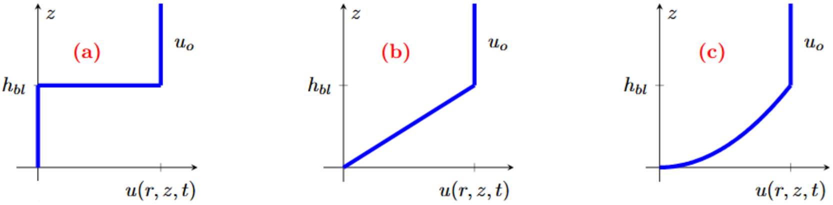


Option (c) is the Kármán's Momentum Integral Theory, but Option (a) is more analytically tractable for our purposes.

Kinematic Condition

We use Option 1 – kinematic condition (h-equation) becomes:

$$\frac{\partial h}{\partial t} + u_o(x,t)\frac{\partial h}{\partial x} = -\frac{2}{t+t_0}(h - h_{bl}),$$

valid for $t > \tau$, $h < h_{bl}$.

- By the method of characteristics:

$$h(x,t) = \frac{1}{(t+t_0)} g\left(\frac{x}{t+t_0}\right) + h_{PI}(t).$$

- Solution at lamella extremity $x = R$ depends only on t.

2D Rim-Lamella Model – Viscous

Systematic mass and momentum balances lead to a simple final answer for the 2D rim-lamella model:

$$\begin{aligned} \frac{\mathrm{d}V}{\mathrm{d}t} &= 2\left(\overline{u} - U\right)h, \qquad \frac{\mathrm{d}R}{\mathrm{d}t} = U, \\ V\frac{\mathrm{d}U}{\mathrm{d}t} &= 2\left[\left(\overline{u} - U\right)^2 - c^2\right]h. \end{aligned}$$

Here $\overline{u}$ is the depth-averaged velocity, evaluated at the lamella extremity:

$$\overline{u} = u_o(R,t)\left(1 - \frac{h_{bl}}{h}\right), \qquad u_o(R,t) = \frac{R}{t + t_0}.$$

Gronwall's Inequality

Viscous model does not have exact solutions but we can use Gronwall's Inequality to provide bounds.

Differential form [edit]

Let I denote an interval of the real line of the form $[a, \infty)$ or $[a, b]$ or $[a, b)$ with $a < b$. Let β and u be real-valued continuous functions defined on I. If u is differentiable in the interior I° of I (the interval I without the end points a and possibly b) and satisfies the differential inequality

$$u'(t) \le \beta(t)\, u(t), \qquad t \in I^\circ,$$

then u is bounded by the solution of the corresponding differential *equation* $v'(t) = \beta(t)\, v(t)$:

$$u(t) \le u(a) \exp\left(\int_a^t \beta(s)\, \mathrm{d}s \right)$$

for all $t \in I$.

Remark: There are no assumptions on the signs of the functions β and u.

Results

Proceeding as in the 3D axisymmetric case[5], we place bounds on the maximum spreading radius:

$$k_1 \mathrm{Re}^{1/3} - k_2 (1-\cos\vartheta_a)^{1/2} (\mathrm{Re}/\mathrm{We})^{1/2} \le \mathcal{R}_{max}/R_0 \le k_1 \mathrm{Re}^{1/3}.$$

- Importance of dimensionality: $\mathcal{R}_{max}/R_0 \sim k_1 \mathrm{Re}^{1/3}$ in 2D, as opposed to $\mathrm{Re}^{1/5}$.
- Due to dimensional analysis, as final film thickness is governed by viscous boundary layer.

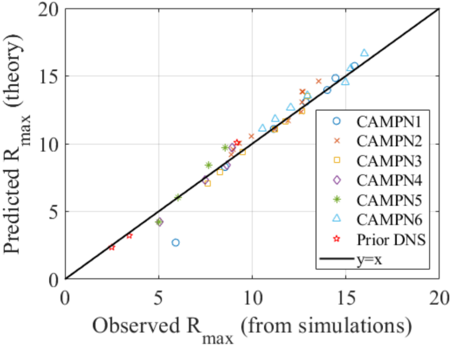
[5] Bustamante, M.D. and Ó Náraigh, L., 2025. Bounds on the spreading radius in droplet impact: the viscous case. Proceedings of the Royal Society A: Mathematical, Physical and Engineering Sciences, 481(2313).

2D Rim-Lamella Model – Viscous – Compare with DNS

Motivated by the bounds, we propose a correlation

$$\mathcal{R}_{max}/R_0 = k_1 \mathrm{Re}^{1/3} - k_0(1-\cos\vartheta_a)^{1/2}(\mathrm{Re}/\mathrm{We})^{1/2}.$$

- Validate correlation over repeated campaigns of simulations, varying Re, We, and ϑ_a systematically.
- Results using `interFoam` and a constant contact-angle model.
- A further campaign using a Diffuse Interface Method.
- Results of prior simulations (different authors) also considered.



Summary and Conclusions

- Looked at Rim-Lamella models in 2D Cartesian geometry. ☑
  - Equivalent to cylindrical droplets, which are inherently unstable.
  - But close surrogates exist in the experimental literature.
  - Obtained exact solution of the rim-lamella model in the inviscid case. ☑
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  - Obtained bounds in the viscous case. ☑
- Excellent agreement between theory, numerical simulation, and experiments.
- Theory of **a priori bounds** has proved very fruitful – could find wider use in Fluid Mechanics: turbulence, mixing, spreading, **...**.

Acknowledgments

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