Analytical Aspects of Droplet-Impact Modelling

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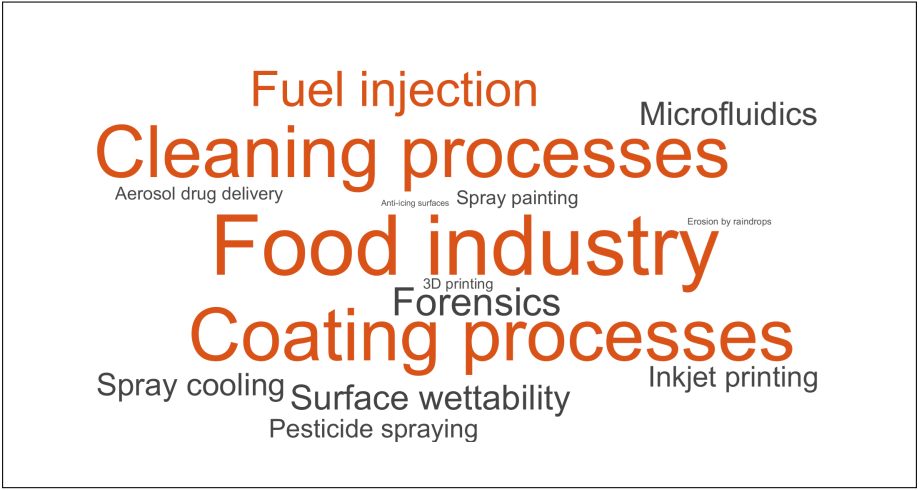
Introduction

- I will look at droplet impact on a smooth surface.
- **Impact, Spread, Retraction**
  In the land of splashes, what the scientist knows as Inertia and Surface Tension are the sculptors in liquids, and fashion from them delicate shapes none the less beautiful because they are too ephemeral for any eye but that of the high-speed camera [Yarin, Annu. Rev. Fluid Mech. (2006)]
- Highlights the importance of **parameters** in such studies; key parameters are the **Weber number**, $\mathrm{We} =$ Inertia/Surface Tension, and the **Reynolds number** – *sculptor has two tools*.

Motivation for studying droplet impact

Wordcloud, weighted by Google Scholar hits on 22/09/2025:



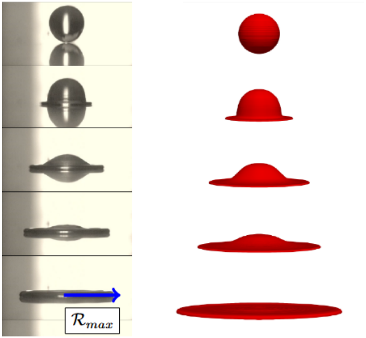
Important to emphasize that **scientific curiosity** is a main motivation here.

Splash Threshold

- Droplet spreading below **splash threshold** (no splash), $K \lesssim 3,000$, where $K = \mathrm{We}\sqrt{\mathrm{Re}}$
- At low We, droplet spreads out into a pancake structure – rim and lamella.
- Of interest is the **maximum spreading radius** $\mathcal{R}_{max}$ and its dependence on We and Re.
- E.g. Roisman's correlation [1]:

$$\frac{\mathcal{R}_{max}}{R_0} = 1.0\mathrm{Re}^{1/5} - 0.37\mathrm{Re}^{2/5}\mathrm{We}^{-1/2},$$

for Re and We based on droplet radius.



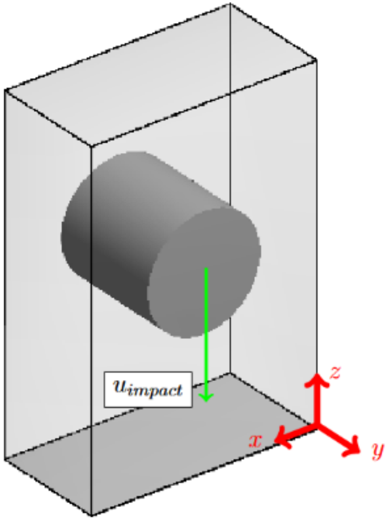
Droplet impact study. Left: high-speed camera. Right: OpenFOAM simulations. Credit: Conor Quigley. Parameters: $\mathrm{Re} = 1700$ and $\mathrm{We} = 20$.

[1] Roisman, I.V., 2009. Inertia dominated drop collisions. II. An analytical solution of the Navier–Stokes equations for a spreading viscous film. Physics of Fluids, 21(5).

Point of Departure: 2D Droplet Impact

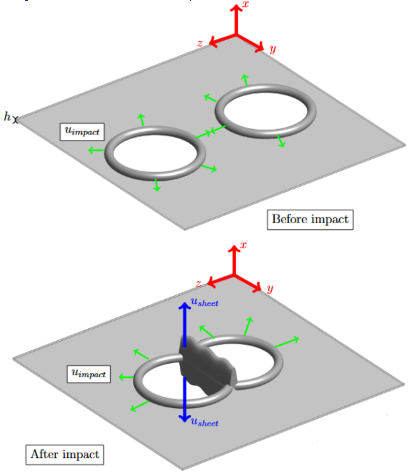
We look at 2D droplet (cylindrical) droplet impact.

- Can't be realised experimentally (?) because of Rayleigh–Plateau instability.
- But some experiments come close.
- Opportunity to look at the RL model in a simplified geometry.
- Theory running a little ahead of the experiments but interesting all the same.

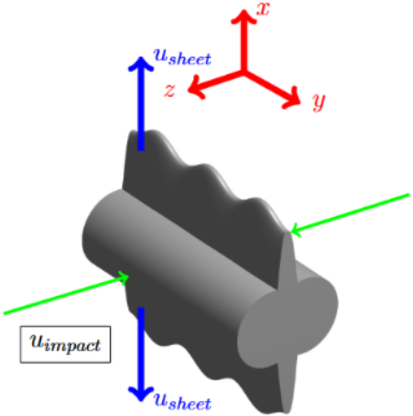


Also an interesting test case in DNS – reported on extensively in literature (2D Cartesian versus 3D axisymmetric).

Experiments / DNS that come close...



Néel *et al.*[1]

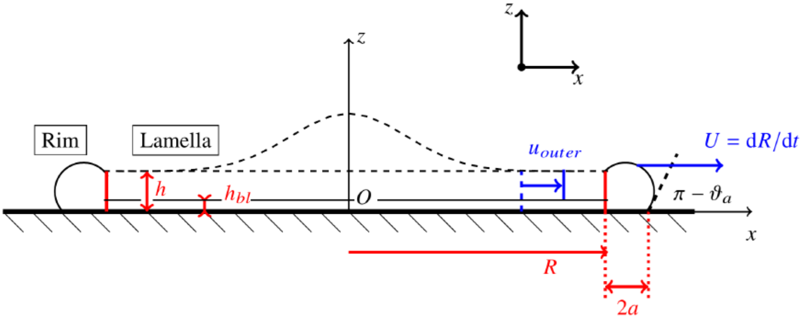


Tang *et al.*[2]

[1]Néel, B., Lhuissier, H. and Villermaux, E., 2020. 'Fines' from the collision of liquid rims. Journal of Fluid Mechanics, 893, p.A16.

[2]Tang, K., Adcock, T.A.A. and Mostert, W., 2024. Fragmentation of colliding liquid rims. Journal of Fluid Mechanics, 987, p.A18.

Methodology: 2D Rim-Lamella Model



- Break up impacting 2D droplet into rim and lamella.
- Hyperbolic flow in lamella, with $u = x/(t + t_0)$ and $w = -z/(t + t_0)$.
- Kinematic condition gives lamella height, $h_t + uh_x = w$ at $z = h$.
- Mass and momentum transfer between the rim and the lamella.

2D Rim-Lamella Model – Inviscid

In the inviscid case, mass and momentum balance yield the following set of equations:

$$
\begin{aligned}
\frac{\mathrm{d}V}{\mathrm{d}t} &= 2\left(u_0 - U\right)h, \qquad \frac{\mathrm{d}R}{\mathrm{d}t} = U, \\
V\frac{\mathrm{d}U}{\mathrm{d}t} &= 2\left(u_0 - U\right)^2 h - \frac{2\gamma}{\rho}\left(1 - \cos\vartheta_a\right).
\end{aligned}
$$

- V is the **area** of the 2D rim.
- $u_0 = R/(t + t_0)$.
- $h \equiv h(R,t) = [(\tau + t_0)/(t + t_0)]h_{init}$, $h(R,t) \sim (t + t_0)^{-1}$.

Key observation: The corresponding rim-lamella model in the 3D axisymmetric case has a geometric factor of $2\pi R$ on the right-hand side. The fact that this factor does not occur in 2D rim-lamella model is important.

2D Rim-Lamella Model – Exact Solution

Velocity defect $\Delta = u_0 - U$. Equation for $V\Delta$:

$$\frac{\mathrm{d}}{\mathrm{d}t}(V\Delta) + \frac{V\Delta}{t+t_0} = 2hc^2 = 2(\gamma/\rho)(1-\cos\vartheta_a).$$

- Exact solution for $V\Delta$.
- Substitute back into $\Delta = [R/(t+t_0)] - (\mathrm{d}R/\mathrm{d}t)$ and solve for R.
- Result is complicated but it is in closed form.
- Crucially, we calculate:

$$\frac{R_{max}}{R_0} \approx \tfrac{1}{27}\frac{V_{tot}^2}{h_{init}U_0^2(\tau+t_0)^2 R_0}\left(\frac{V_{tot}}{2h_{init}R_0}\right)\frac{\mathrm{We}}{1-\cos\vartheta_a}, \qquad \mathrm{We} \gg 1.$$

- $R_{max}/R_0 \sim \mathrm{We}$ is an intrinsically 2D scaling behaviour and was observed by Néel *et al.* in their work on the colliding liquid cylinders.

2D Rim-Lamella Model – Viscous

We proceed by analogy to the 3D axisymmetric case[3] and reason out the rim-lamella model for the 2D Cartesian (viscous) case:

$$\begin{aligned} \frac{\mathrm{d}V}{\mathrm{d}t} &= 2\left(\overline{u}-U\right)h, \qquad \frac{\mathrm{d}R}{\mathrm{d}t} = U, \\ V\frac{\mathrm{d}U}{\mathrm{d}t} &= 2\left[\left(\overline{u}-U\right)^2 - c^2\right]h, \end{aligned}$$

where

$$\overline{u} = \frac{R}{t+t_0}\left(1-\frac{h_{bl}}{h}\right),$$

and

$$h_{bl}(t) = \alpha\sqrt{\nu\left(t+t_1\right)}, \qquad \nu = \text{Kinematic viscosity},$$

and where α and t_1 are free parameters.

[3]Bustamante, M.D. and Ó Náraigh, L., 2025. Bounds on the spreading radius in droplet impact: the viscous case. In Proceedings A (Vol. 481, No. 2313, p. 20240791). The Royal Society.

2D Rim-Lamella Model – Bounds in the Viscous Case

2D viscous model contains a boundary-layer correction, which means it is no longer exactly solvable. E.g. Δ equation becomes:

$$\frac{\mathrm{d}\Delta}{\mathrm{d}t} + \frac{\Delta}{t+t_0}\left(1 - \frac{h_{bl}}{h}\right) = -2(\Delta^2 - c^2)(h/V) - \frac{u_0}{t+t_0}\frac{h_{bl}}{h}\underbrace{\left[1 - \frac{h_{bl}}{h} + \tfrac{1}{2}\frac{t+t_0}{t+t_1}\right]}_{\geq 0}.$$

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- Provided the 'hard terms' have a definite sign, we can ignore them
- Pay a price: the $=$ is replaced with an inequality.
- Using comparison theorems, we can place bounds on the maximum spreading radius.

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$$k_1 \mathrm{Re}^{1/3} - k_2(1 - \cos\vartheta_a)^{1/2}(\mathrm{Re}/\mathrm{We})^{1/2} \leq \mathcal{R}_{max}/R_0 \leq k_1 \mathrm{Re}^{1/3}.$$

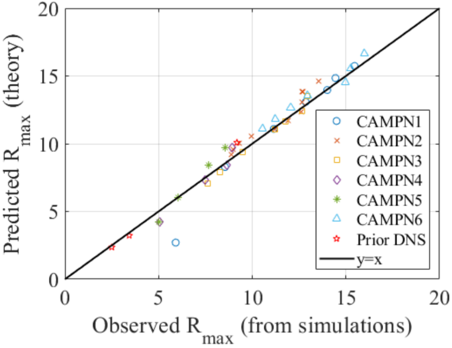
Importance of **dimensionality**: $\mathcal{R}_{max}/R_0 \sim k_1 \mathrm{Re}^{1/3}$ in 2D, as opposed to $\mathrm{Re}^{1/5}$.

2D Rim-Lamella Model – Viscous – Compare with DNS

Motivated by the bounds, we propose a correlation

$$\mathcal{R}_{max}/R_0 = k_1 \mathrm{Re}^{1/3} - k_0(1-\cos\vartheta_a)^{1/2}(\mathrm{Re}/\mathrm{We})^{1/2}.$$

- Validate correlation over repeated campaigns of simulations, varying Re, We, and ϑ_a systematically.
- Results using `interFoam` and a constant contact-angle model.
- A further campaign using a Diffuse Interface Method.
- Results of prior simulations (different authors) also considered.



Conclusions and keynote

- Investigated Rim-Lamella models in 2D Cartesian geometry.
  - Equivalent to cylindrical droplets, which are inherently unstable.
  - But close surrogates exist in the experimental literature.
  - Obtained exact solution of the rim-lamella model in the inviscid case.
  - Again obtained bounds in the viscous case.
  - Good agreement between theory, numerical simulation, and experiments.
- **Keynote:** Theory of **a priori bounds** has proved very fruitful – finds wider use in Fluid Mechanics: turbulence, mixing, spreading, and who knows where else **...** ?

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