A Mathematical Framework of Optimal Control in Industrial Drying

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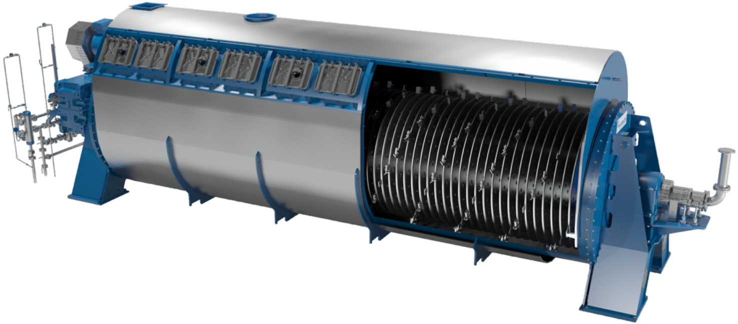
Introduction

Industrial Driers are a key part of the food processing industry:

- A product is heated and the moisture is evaporated.
- Heating the product is expensive. Large amounts of energy are required to overcome the latent heat of the water and to produce evaporation – **energy intensive**.
- The dried product has to meet certain constraints on the moisture content and **temperature** for quality control.
- The drying must therefore be **controlled** to minimize energy consumption while at the same time maximizing product quality.
- This could be a problem for **Optimal Control Theory**

Disc Drier

Although we will work with abstract models, the machine we have in mind is the Disc Drier.



- Drum diameter $\sim 1\,\mathrm{m}$, length $\sim 7\,\mathrm{m}$.
- Moist product in one end, dry product out the other.
- Product heated by contact with heated discs (indirect contact, steam), product pumped along length of drier.
- Rotating discs have 'stirrers' to agitate product (mixing).
- Evaporation of moisture into gas stream and there to a collector.

Strategy of Project / Plan of Talk

1. Develop very simple toy model of industrial heating, hence:
   - Numerical solutions,
   - Analytical solutions,
   - Analytical Optimal Control Theory
   - Numerical Optimal Control Theory
   - Functional Analysis Context
2. Develop a more detailed model of industrial drier... still probably not yet usable, but with:
   - More Physics ('three-equation model'),
   - Evaporation,
   - Numerical Solutions, Optimal Control Theory
3. Optimal Control establishes a 'target'; final part of project is to investigate more practical control algorithms, and determine how close they are to optimal.

Simple Mathematical Model

We start by looking at a simple mathematical model (SMM) of industrial heating, which is a kind of 'Newton's Law of cooling with advection', with T as product temperature:

$$\frac{\partial T}{\partial t} + u_0 \frac{\partial T}{\partial x} = k \left[q(t) - T \right], \qquad x \in (0, \ell], \qquad t > 0.$$

A model of an industrial process with an **inlet**:

$$T(x = 0, t) = T_{inlet}(t)$$

a **treatment** $q(t)$, and an **outlet**, the outlet is computed from $T(x = \ell, t)$. There is also an initial condition, $T(x, t = 0) = T_{init}(x)$.

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The goal will be to choose a treatment $q(t)$ to maintain the outlet temperature $T(x = \ell, t)$ at a target state – hence, **Optimal Control Theory**.

Analytical and Numerical Solutions

We first try to understand the model by looking at solutions.

Analytic solution via Laplace Transforms / Method of Characteristics:

$$T(x,t) = \begin{cases} T_{init}(x - u_0 t)\mathrm{e}^{-kt} + \mathcal{I}(t), & u_0 t < x, \\ \left[T_{inlet}\left(t - \frac{x}{u_0}\right) - \mathcal{I}\left(t - \frac{x}{u_0}\right)\right]\mathrm{e}^{-kx/u_0} + \mathcal{I}(t), & u_0 t > x, \end{cases}$$

with particular integral:

$$\mathcal{I}(t) = \mathrm{e}^{-kt}\int_0^t \mathrm{e}^{kt'}\left[k\,q(t')\right]\mathrm{d}t'.$$

Numerical solution via second-order upwind scheme:

$$\frac{T_i^{n+1} - T_i}{\Delta t} + u_0\left(\frac{3T_i^n - 4T_{i-1}^n + T_{i-2}^n}{2\Delta x}\right) = k\left[q(t^n) - T_i^n\right].$$

Sample Results

Sample forcing term:

$$q(t) = [50 + 5\sin(2\pi t)]\ ^\circ\mathrm{C}.$$

Sample inlet condition:

$$T_{inlet}(t) = [100 + 10\sin(2\pi t)]\ ^\circ\mathrm{C}.$$

Inlet condition: $T_{init}(x) = 100^\circ\mathrm{C}.$

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Parameters:

Parameter	Value
ℓ	$5\,\mathrm{m}$
u_0	$1\,\mathrm{m}\cdot\mathrm{min}^{-1}$
k	$0.5\,\mathrm{min}^{-1}$
N	200
Δt	$10^{-3}\,\mathrm{min}$

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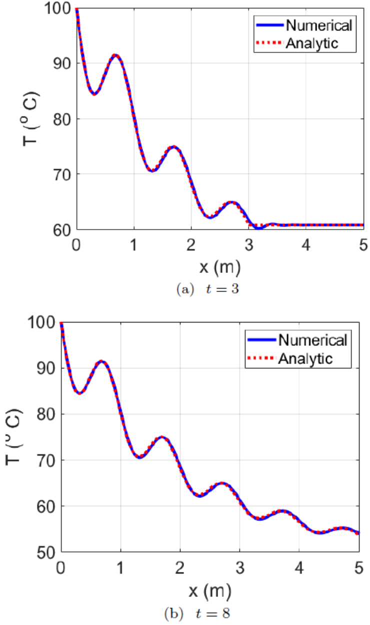

Analytical Optimal Control

Set $T(\ell, t) = T_*$ in the analytical solution. For $t < \ell/u_0$:

$$T_* + \frac{u_0}{k} T'_{init}(\ell - u_0 t)\mathrm{e}^{-kt} + q(0)\mathrm{e}^{-kt} = q(t), \qquad t < \ell/u_0.$$

For $t > \ell/u_0$:

$$T_* - T_{inlet}(t - \ell/u_0)\mathrm{e}^{-k\ell/u_0} - \frac{1}{k} T'_{inlet}(t - \ell/u_0)\mathrm{e}^{-k\ell/u_0} + q(t - \ell/u_0)\mathrm{e}^{-k\ell/u_0} = q(t).$$

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- Unknown quantity $q(0)$.
- If $T_{\mathrm{init}} = T_0 = T_*$, then $q(0)$ can be taken as zero. Then, first equation has has a closed-form solution, and second equation can be solved recursively.
- A possible jump discontinuity at the first residence time $t_0 = \ell/u_0$:

$$\begin{aligned} q(t_0^-) &= T_* + \frac{u_0}{k} T'_{\mathrm{init}}(0)\mathrm{e}^{-kt_0}, \\ q(t_0^+) &= T_* - T_{inlet}(0)\mathrm{e}^{-kt_0} - \frac{1}{k} T'_{inlet}(0)\mathrm{e}^{-kt_0}. \end{aligned}$$

Numerical Optimal Control I

Introduce penalty function:

$$J(T,q)=\tfrac{1}{2}\int_0^\tau \left[T(\ell,t)-T_*\right]^2 \mathrm{d}t.$$

Adjoin to the penalty function the constraint:

$$L(T,q,\psi)=\tfrac{1}{2}\int_0^\tau \left[T(\ell,t)-T_*\right]^2 \mathrm{d}t+\int_0^\tau \mathrm{d}t\int_0^\ell \mathrm{d}x\left[\frac{\partial T}{\partial t}+u_0\frac{\partial T}{\partial x}-k\left(q(t)-T\right)\right]\psi.$$

Here, ψ has the interpretation of the Lagrange multiplier.

Numerical Optimal Control II

Make small variations in temperature T and control q:

$$\delta L = \int_0^\tau [T(\ell,t) - T_*]\,\delta T(\ell,t)\mathrm{d}t - \int_0^\tau \mathrm{d}t \int_0^\ell k\psi\delta q\,\mathrm{d}x$$
$$+ \int_0^\tau \int_0^\ell \left[-\frac{\partial\psi}{\partial t} - u_0\frac{\partial\psi}{\partial x} + k\psi\right]\delta T\,\mathrm{d}x + \int_0^\ell [\psi\delta T]_{t=\tau}\,\mathrm{d}x + \int_0^\tau [u_0\psi\delta T]_{x=\ell}\,\mathrm{d}t.$$

Identify the **adjoint problem**:

$$-\frac{\partial\psi}{\partial t} - u_0\frac{\partial\psi}{\partial x} + k\psi = 0, \qquad x \in [0,\ell),$$

with outlet condition

$$\psi(\ell,t) = -(1/u_0)[T(\ell,t) - T_*],$$

and terminal condition $\psi(x,\tau) = 0$.

Numerical Optimal Control III

Thus,

$$\delta L = -\int_0^T \mathrm{d}t\, \delta q \int_0^\ell k\psi(x,t)\mathrm{d}x.$$

Indeed,

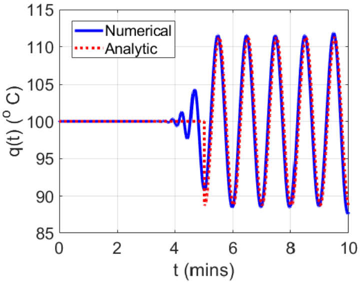
$$\frac{\delta L}{\delta q}(t) = -\int_0^\ell k\psi(x,t)\mathrm{d}x.$$

Suggestive of algorithm:

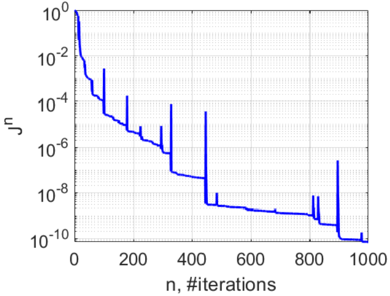
Algorithm 1 Steepest-Descent Algorithm for the SMM

1: Initialize the control using some initial guess $q(t) = q_0(t)$.
2: Solve SMM to the final time τ, compute $T(\ell, t)$ for each $t \in [0, \tau]$.
3: Solve the adjoint problem.
4: Update the control $q(t) \leftarrow q(t) + \lambda \int_0^\ell (k\psi)\, \mathrm{d}x$, where λ is a small positive parameter.
5: Repeat steps 2–4 until L is sufficiently small.

Numerical Optimal Control – Results



Excellent agreement, apart from at jump discontinuity (numerical diffusion).



Plot showing the value of the cost function $J(T, q)$ at each iteration using the **Barzelai–Borwein** method.

Mathematical Setting I

We let $T(x,t) \in Y$ and $q(t) \in U$. We let $e(T,q) = \partial_t T + u_0 \partial_x T - k(q - T) \in Z$. As e is linear in T and q, this can also be identified with a linear problem,

$$e(T,q) = AT + Bq, \qquad A \in \mathcal{L}(Y,Z).$$

Here, Y and Z are appropriate Banach spaces, $\mathcal{L}(Y,Z)$ is the space of linear operators from X to Y, and U is an appropriate Hilbert space.

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where $e : Y \times U \to Z$ is defined with respect to its action on test functions:

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$$\begin{aligned}
\langle \psi, e(T,q) \rangle_{Z^*,Z} = &\int_0^\tau \mathrm{d}x \int_0^\ell (-T\partial_t \psi)\,\mathrm{d}x + \int_0^\tau \mathrm{d}x \int_0^\ell u_0(-T\partial_x \psi)\,\mathrm{d}x \\
&- \int_0^\tau \mathrm{d}x \int_0^\ell k(T-q)\psi\,\mathrm{d}x + \int_0^\tau u_0 \left[T(\ell,t)\psi(\ell,t) - T(0,t)\psi(0,t)\right]\mathrm{d}t \\
&+ \int_0^\ell \left[T(x,\tau)\psi(x,\tau) - T(x,0)\psi(x,0)\right]\mathrm{d}x,
\end{aligned}$$

for all test functions $\psi \in Z^*$.

Mathematical Setting II

Lagrangian: $L : Y \times U \times Z^* \to \mathbb{R}$, where

$$L(T, q, \psi) = J(T, q) + \langle \psi, e(T, q) \rangle_{Z^* Z}$$

(same as before!). The first-order optimality conditions in the direction $(\delta T, \delta\psi, \delta q)$ are:

$$\begin{aligned}
\langle J_T(T,q), \delta T \rangle_{Y^*,Y} + \langle \psi, e_T(T,q)\delta T \rangle_{Z^*,Z} &= 0 \quad \textbf{Adjoint Problem} \\
\langle \delta\psi, e_T(T,q)\delta T \rangle_{Z^*,Z} &= 0 \quad \textbf{Constraint} \\
\langle J_q(T,q), \delta q \rangle_{U^*,U} + \langle \psi, e_q(T,q)\delta q \rangle_{Z^*,Z} &= 0 \quad \textbf{Optimality Condition}, \\
& \qquad \delta J/\delta q = 0
\end{aligned}$$

Mathematical Setting III

If, instead, we view the solution T as a function of the forcing q, $T(q)$, we obtain the reduced cost function

$$\widehat{J}(q) = J(T(q), q) = J(T(q), q) + \langle \psi, e(T(q), q) \rangle_{Z^*, Z} = L(T(q), q, \psi).$$

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We can differentiate w.r.t. q to obtain:

$$\begin{aligned} \langle \widehat{J}'(q), \delta q \rangle_{U^*, U} &= \langle L_q(T(q), q, \psi(q)), \delta q \rangle_{U^*, U}, \\ &= \langle J_q(T(q), q), \delta q \rangle_{U^*, U} + \langle \psi(q), e_q(T(q), q), \delta q \rangle_{U^*, U}, \\ &= \langle J_q(T(q), q), \delta q \rangle_{U^*, U} + \langle e_q(T(q), q)^* \psi(q), \delta q \rangle_{U^*, U}, \end{aligned}$$

which is true for all admissible perturbations $\delta q \in U$.

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Riesz Representation Theorem for U, we identify the vector $\widehat{J}'(q) \in U^* = U$:

$$\widehat{J}'(q) = J_q(T(q), q) + e_q(T(q), q)^* \psi(q),$$

which is precisely $(\delta J / \delta q)(t) = k \int_0^\ell k \psi(x, t) \mathrm{d}x$.

General Result

Theory for a more general cost functiond due to Hinze et al.[1], for:

$$\min_{(T,q)\in Y\times U} J(T,q)+\frac{1}{2}\alpha\|q\|_U^2, \text{ subject to } AT+Bq=0,\ q\in U_{ad}\subset U,\ T\in Y_{ad}\subset Y, \quad (1)$$

The following result holds:

Theorem (Hinze et al.)

Suppose the following statements are true:

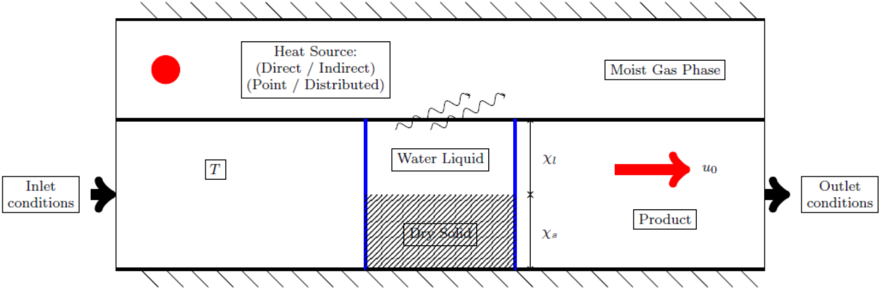
1. $\alpha \geq 0$, $U_{ad} \subset U$ *is convex, and in the case of* $\alpha = 0$, *bounded;*
2. $Y_{ad} \subset Y$ *is convex and closed, such that* $AT + Bq = 0$ *for at least one set of functions* $T \in Y_{ad}$ *and* $q \in U_{ad}$*;*
3. $A \in \mathcal{L}(Y, Z)$ *has a bounded inverse.*

Then, the problem (1) *has an optimal solution* (T_{opt}, q_{opt}). *If* $\alpha > 0$, *then the solution is unique.*

[1] Optimization with PDE Constraints, Hinze, Pinnau, Ulbrich, and Ulbrich (Springer, 2009)

A more detailed model of an industrial drier

We now look at a more detailed model of an industrial drier, based on the following concept:



Schematic diagram showing the generic process calculations for an industrial dryer

Mass Balance

We look at the solid stream only, introduce

$$\epsilon_s = \rho_s X_s, \qquad \epsilon_l = \rho_l X_l,$$

where X_s and X_l are volume fractions, with $X_s + X_l + X_{\text{moist gas}} = 1$. Conservation of mass then yields, simply:

$$\begin{aligned} \frac{\partial \epsilon_s}{\partial t} + \frac{\partial}{\partial x}(u_0 \epsilon_s) &= 0, \\ \frac{\partial \epsilon_w}{\partial t} + \frac{\partial}{\partial x}(u_0 \epsilon_w) &= -\dot{m}, \end{aligned}$$

where $\dot{m}$ is the drying rate, with dimensions of $\mathrm{kg}/(\mathrm{m}^3 \cdot \mathrm{s}^{-1})$, and is a general function of ϵ_s, ϵ_l, and temperature. We take u_0 to be a **constant velocity**.

Also useful: $\overline{\rho} = \epsilon_s + \epsilon_l$, with

$$\frac{\partial \overline{\rho}}{\partial t} + \frac{\partial}{\partial x}(u_0 \overline{\rho}) = -\dot{m}.$$

Hence, **mass fractions** $x_l = \epsilon_l / \overline{\rho}$ and $x_s = \epsilon_s / \overline{\rho}$.

Energy Balance

Internal energy per unit mass (symbol!):

$$e = \left[\sum_{i=s,l} c_{p,i} (\epsilon_i / \overline{\rho}) \right] (T - T_{\text{ref}}) \, ,$$

where

- $c_{p,i}$ is the specific heat per unit mass of the i^{th} component,
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Energy conservation:

$$\frac{\partial \overline{\rho} e}{\partial t} + \frac{\partial}{\partial x}(u_0 \overline{\rho} e) = \dot{q} - \dot{m} h_l.$$

where

- $\dot{q}$ is the power per unit volume in (heating),
- $\dot{m} h_l$ is the power loss per unit volume (evaporation), h_l is the specific latent heat.

Three-Equation Model (TEM)

Re-write energy equation as an advection equation, and summarize **three-equation model**:

$$
\begin{aligned}
\frac{\partial \epsilon_s}{\partial t} + \frac{\partial}{\partial x}(u_0 \epsilon_s) &= 0, \\
\frac{\partial \epsilon_w}{\partial t} + \frac{\partial}{\partial x}(u_0 \epsilon_w) &= -\dot{m}, \\
\frac{\partial T}{\partial t} + u_0 \frac{\partial T}{\partial x} &= \frac{\dot{q} - \dot{m}\left[h_l - c_{p,w}(T - T_{\text{ref}})\right]}{\sum_{i=s,l} c_{p,i}\epsilon_i}.
\end{aligned}
$$

Drying Rate – Constant Drying Rate

Two regimes. Initial regime – **constant drying rate** – limited by conditions on moist gas phase:

$$\dot{m} = (m_0 k_c) a_V \left(Y_*(T) - Y_{air}\right),$$

where

- $m_0 k_c$ are constants, a_V is the surface area exposed per unit volume,
- Y is the absolute humidity of the air (dry gas basis), e.g. $Y_{air} = M_{water\,vapour}/M_{dry\,air}$,
- $Y_*(T)$ is the a.h. of the air at saturation, at the temperature T.

$Y_*(T)$ is computed in terms of pressures:

$$Y_*(T) = 0.62198 \left(\frac{p_{ws}}{p_a - p_{ws}}\right),$$

Here, p_a is the pressure of the moist gas and p_{ws} is the saturation pressure of water, obtainable from Clausius–Clapheron relations or similar, e.g. Antoin's Equation:

$$p_{ws} = 10^{A - B/(C+T)}.$$

Drying Rate – Falling Drying Rate

After surface moisture is evaporated, remaining moisture must be drawn from inside product to surface – diffusion-limited – **falling drying rate** – Lewis Equation:

$$\dot{m} = k_f \epsilon_s \left(X - X_* \right) \text{ OR } \dot{m} = k_f \left(\epsilon_l - X_* \epsilon_w \right).$$

X_* is the **equilibrium moisture level**.

Summarizing,

$$\dot{m} = \begin{cases} (m_0 k_c) a_V \left(Y_*(T) - Y_{air} \right), & X > X_c, \\ k_f \left(\epsilon_l - X_* \epsilon_w \right), & X \leq X_c. \end{cases}$$

where k_c and k_f are rate coefficients and X_c is the changeover moisture level.

> We view $\dot{m}$ as a 'module', we work with the falling drying-rate only for definiteness / mathematical simplicity, other modules can be 'slotted in' as required.

Power / Typical Parameters

Drier power:

$$P = A_\times \int_0^\ell \dot{q} \mathrm{d}x,$$

where $A_\times$ is $\times$-sectional area.

With $\dot{q}$ constant, this gives

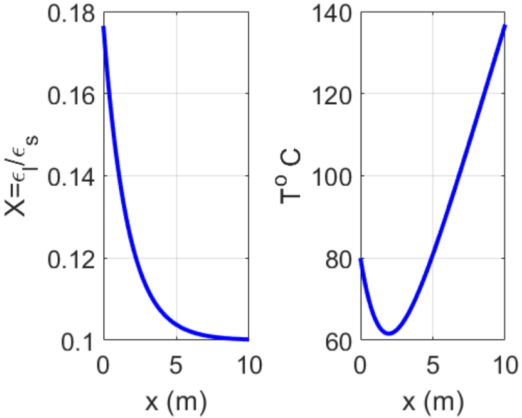
$$\dot{q} = \frac{P}{A_\times \ell}.$$

Typical parameters:

Parameter	Value
P	4×10^4 Watt
ℓ	$10\,\mathrm{m}$
u_0	$1/3\,\mathrm{m}\cdot\mathrm{min}^{-1}$
X_*	0.1
T_{inlet}	$80°\mathrm{C}$
$x_{w,inlet}$	0.15
$x_{s,inlet}$	$1 - x_{w,inlet}$
$\dot{\mu}_{inlet}$	$10\,\mathrm{kg}\cdot\mathrm{min}^{-1}$
A_x	$\pi(0.5\,\mathrm{m})^2$
Φ_{inlet}	$\dot{\mu}_{inlet}/A_x$
$\overline{\rho}_{inlet}$	Φ_{inlet}/u_0
$\epsilon_{i,inlet}$	$x_{i,inlet}\overline{\rho}_{inlet}$

Sample Results

We solve the TEM in equilibrium with $\partial/\partial t \to 0$. (Numerics: ODE45)



Plot showing the behavior of the equilibrium solution

- X drops to equilibrium level over length of drier – good choice of parameters.
- T dips near inlet (evaporation / cooling) and then rises to a maximum at outlet (heating).
- Observed behaviour in industrial driers.

Optimal Control Theory

We know from the SMM how the Optimal Control theory works – apply it now to the Three-Equation Model (TEM).

1. Initial guess for $q(t)$. Solve the Forward Equations.
2. Solve the adjoint equation system:

$$-\frac{\partial}{\partial t}\begin{pmatrix}\psi_s\\ \psi_l\\ \psi_T\end{pmatrix} - u_0\frac{\partial}{\partial x}\begin{pmatrix}\psi_s\\ \psi_l\\ \psi_T\end{pmatrix} - \mathbb{J}^T\begin{pmatrix}\psi_s\\ \psi_l\\ \psi_T\end{pmatrix} = 0,$$

   where $\mathbb{J}$ is the Jacobian of the TEM at x and t (computationally intensive).
3. Terminal / outlet conditions:

$$\psi_s(\ell,t)=0, \psi_l(\ell,t)=0, \qquad \psi_T(\ell,t) = -(1/u_0)[T(\ell,t)-T_*].$$

4. Update $\dot{q}$ from step n to step $n+1$:

$$[\dot{q}(t)]^{n+1} = [\dot{q}(t)]^n + \lambda^n \int_0^\ell \psi_T^n(x,t)\eta^n(x,t)\mathrm{d}x,$$

   where $\eta(x,t) = [c_{p,l}\epsilon_l(x,t) + c_{p,s}\epsilon_s(x,t)]^{-1}$.

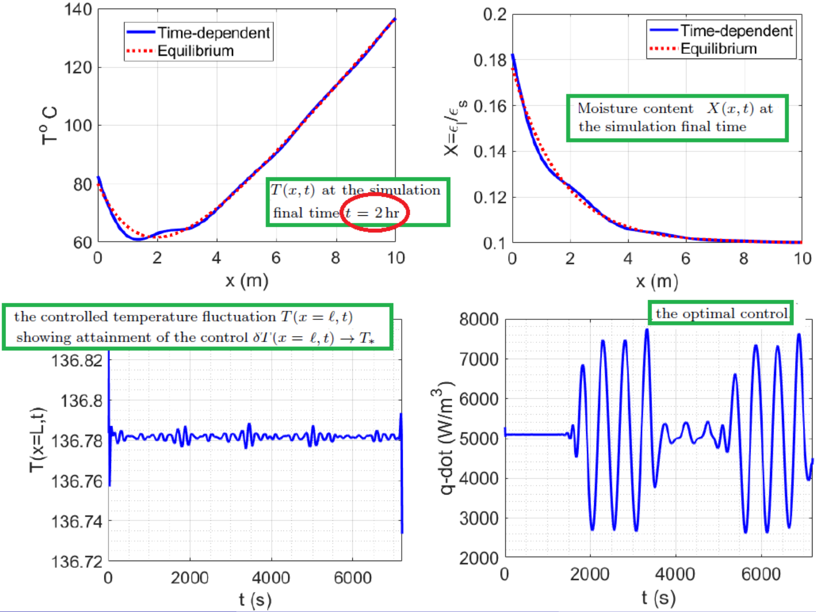
The inlet conditions are known for all time

The inlet conditions are known over the time horizon of the simulation ($=\tau$), and the optimal control is scheduled:

$$\epsilon_s(x=0,t) = \epsilon_{s,0},$$
$$\epsilon_l(x=0,t) = \epsilon_{l,eq}(x=0)\left[1+\delta\alpha\sin(\omega_F t)\right],$$
$$T(x=0,t) = T_{eq}(x=0)\left[1+\delta\alpha\sin(\omega_F t)\right],$$

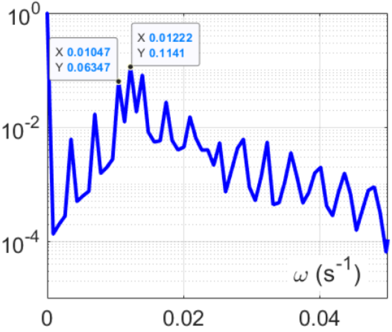
- 'eq' refers to the equilbrium solution (depends only on x),
- $\delta\alpha$ is a perturbation from equilbrium
- Monochromatic perturbation – $\omega_F = 2\pi/(8.5\,\mathrm{mins})$.

Results



Time-dependent
Equilibrium
T° C
x (m)
$T(x,t)$ at the simulation final time $t = 2\,\mathrm{hr}$
$X=\epsilon_l/\epsilon_s$
Moisture content $X(x,t)$ at the simulation final time
the controlled temperature fluctuation $T(x=\ell,t)$ showing attainment of the control $\delta T(x=\ell,t) \to T_*$
T(x=L,t)
t (s)
the optimal control
q-dot (W/m^3)

Beat Pattern

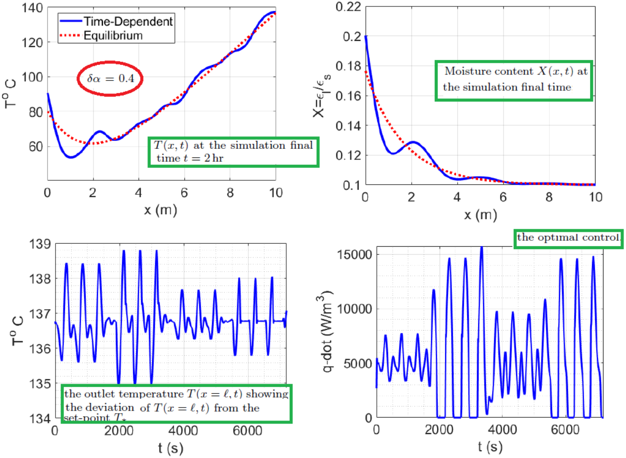


Normalized power spectrum of $\delta\dot{q}$ taken with respect to a simulation over 2 hours (nonlinear case). The maximum at $\omega \approx \omega_F$ is highlighted.

- Broad power spectrum because of 'ramp up' in forcing term (hyperbolic system).
- Secondary peak close to the primary peak at ω_F.
- Two neighbouring peaks in Fourier space – beat.

Nonlinear Case

- Large $\delta\alpha(= 0.4)$.
- Possibility of $\dot{q} < 0$.
- Constrain optimization by taking $\dot{q} = (1/2)\theta(t)^2$.
- Optimize over $\theta(t)$.
- Results qualitatively similar to before.



Complex inlet forcing

$$\begin{aligned} \epsilon_s(x=0,t) &= \epsilon_{s,0}, \\ \epsilon_s(x=0,t) &= \epsilon_{l,0}\left[1+\delta\alpha R_l(t)\right], \\ T(x=0,t) &= T_0\left[1+\delta\alpha R_T(t)\right], \end{aligned}$$

where

$$R_j(t) = \frac{1}{N_{modes}} \sum_{k=1}^{N_{modes}} \cos\left(2\pi \frac{kt}{T_{max}} + \chi_{j,k}\right),$$

Parameter	Value
P	$--$
ℓ	$10\,\mathrm{m}$
u_0	$1/3\,\mathrm{m}\cdot\mathrm{min}^{-1}$
X_*	0.1
T_{inlet}	$80°\mathrm{C}$
T_*	$120°\mathrm{C}$
$x_{w,inlet}$	0.15
$\dot{\mu}_{inlet}$	$10\,\mathrm{kg}\cdot\mathrm{min}^{-1}$
A_x	$\pi(0.5\,\mathrm{m})^2$
$\delta\alpha$	0.2
N_{modes}	10,2
τ	$4\,\mathrm{h}$

Parameter values for the adiabatic control

Adiabatic Control Algorithm

Algorithm 2 Algorithm for Adiabatic Control

1: **if** $t < \ell / u_0$ **then**
2: Keep $q(t)$ at the initial value $q(0)$;
3: **else**
4: **if** If T_{update} seconds have elapsed: **then**
5: Compute rolling averages $\overline{\epsilon_l(x = 0, t)}$ and $\overline{T(x = 0, t)}$ over preceding T_{avg} seconds;
6: Rolling averages used as inlet conditions for the **equilibrium model**;
7: Compute $\dot{q}_* =$ Const. such that $T(\ell) = T_*$ in the eqm model;
8: Update $\dot{q}(t)$ with $\dot{q}_*$.
9: **end if**
10: Repeat Steps 5–8 every T_{update} seconds.
11: **end if**

Adiabatic Control Algorithm

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9: **end if**
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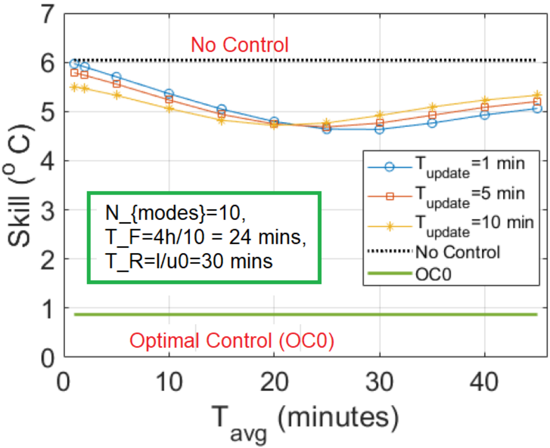
Optimal Control Theory allows us to quantify how effective the Adiabatic Control Algorithm is.

Adiabatic Control is poor for rapid inlet variations

- Skill $\propto J(T,q)$:

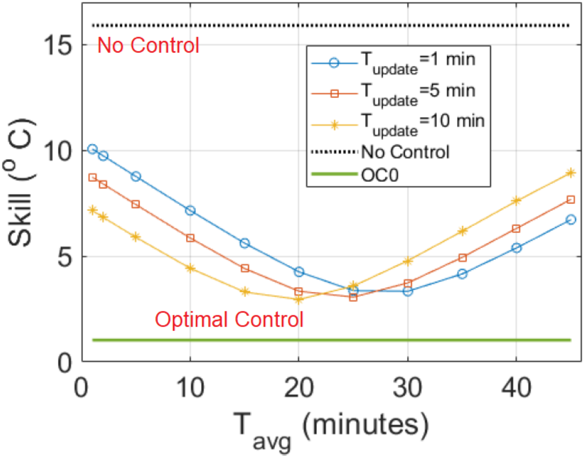
  Skill
  $$= \int_0^\tau [T(\ell,t) - T_*]^2 \mathrm{d}t.$$

- Low Skill score indicates good control.
- Skill depends on T_{avg} and T_{update}.
- Expect results to depend strongly on residence time $T_R = \ell/u_0$.



Score the control algorithm for the case of 'fast' inlet forcing, as a function of T_{avg}.

Adiabatic Control is good for slow inlet variations



Skill of the control algorithm for the case of 'slow' inlet forcing, as a function of T_{avg}.

N_{modes}=2, T_F=4h/2=120 mins, T_R=30 mins

Best skill score at T_{avg}\approx T_R

Conclusions

- Formulated a set of models of industrial drying – simple model $\rightarrow$ detailed model.
- Applied Matheamtics approach combines elements of different disciplines:
  - Functional Analysis
  - Control Theory
  - Physics (Drying Rate)
- Optimal Control of outlet temperature achieved via PDE-constrained Optimal Control Theory / Gradient Descent Method.
- Optimal Control is a 'best case' to aim for, other simplified controls can be introduced and performance quantified with respect to the best case.
- Adiabatic control is one such promising approach, others (e.g. Kalman Filter) can be investigated using our modelling framework.

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