Understanding turbulence with mathematical models: from 2x2 matrices to 1,000 CPU simulations

Lennon Ó Náraigh

School of Mathematics and Statistics and CASL, University College Dublin

7th April 2016
What is turbulence?

Turbulence in a fluid is characterized by chaotic motion on many lengthscales.

Mathematically and computationally, a very tough problem:
- No analytical solutions
What is turbulence?

Turbulence in a fluid is characterized by chaotic motion on many lengthscales.

Mathematically and computationally, a very tough problem:
- No analytical solutions
- Numerics require resolution of a vast array of timescales and lengthscales
What is turbulence?

Turbulence in a fluid is characterized by chaotic motion on many lengthscales.

Mathematically and computationally, a very tough problem:

- No analytical solutions
- Numerics require resolution of a vast array of timescales and lengthscales
- Dimensionality is important
How can it be characterized?

Turbulence depends on the dimension – 2D turbulence very different from 3D turbulence. Take 3D turbulence as a paradigm. The fundamental element is the eddy – a coherent patch of fluid motion on a particular scale.
How can it be characterized?

Turbulence depends on the dimension – 2D turbulence very different from 3D turbulence. Take 3D turbulence as a paradigm. The fundamental element is the **eddy** – a coherent patch of fluid motion on a particular scale.

Typically, the turbulence is forced at the large scales. The eddies interact and transfer energy to smaller eddies, and so on.

Best summarized in poetry:

> Big whorls have little whorls
> Which feed on their velocity,
> And little whorls have lesser whorls
> And so on to viscosity.

(L.F. Richardson)
How can it be characterized?

Turbulence depends on the dimension – 2D turbulence very different from 3D turbulence. Take 3D turbulence as a paradigm. The fundamental element is the eddy – a coherent patch of fluid motion on a particular scale.

Typically, the turbulence is forced at the large scales. The eddies interact and transfer energy to smaller eddies, and so on.

**Best summarized in poetry:**

*Big whorls have little whorls*

*Which feed on their velocity,*

*And little whorls have lesser whorls*

*And so on to viscosity.*

(L.F. Richardson)
How to characterize turbulence

- Turbulence can be characterized by statistical quantities, e.g. an average velocity field

\[ U(x, t) = \langle u(x, t) \rangle, \]

averaged over an ensemble of experiments.
How to characterize turbulence

- Turbulence can be characterized by statistical quantities, e.g. an average velocity field

\[ U(x, t) = \langle u(x, t) \rangle, \]

averaged over an ensemble of experiments.

- Then, introduce fluctuating velocity fields,

\[ u'(x, t) = \langle u(x, t) \rangle - u(x, t) \]
How to characterize turbulence

- Turbulence can be characterized by statistical quantities, e.g. an average velocity field
  \[ U(x, t) = \langle u(x, t) \rangle, \]
  averaged over an ensemble of experiments.
- Then, introduce fluctuating velocity fields,
  \[ u'(x, t) = \langle u(x, t) \rangle - u(x, t) \]
- ... and two-point correlations \( \langle u'_i u'_j \rangle \).
How to characterize turbulence

- Turbulence can be characterized by statistical quantities, e.g. an average velocity field
  \[ U(x, t) = \langle u(x, t) \rangle, \]
  averaged over an ensemble of experiments.
- Then, introduce fluctuating velocity fields,
  \[ u'(x, t) = \langle u(x, t) \rangle - u(x, t) \]
- ... and two-point correlations \( \langle u'_i u'_j \rangle \).

Crucial information about the spatial structure is then supplied by the following correlation function:

\[ R_{ij}(x, t) = \langle u'_i(x + y, t) u'_j(y, t) \rangle, \]

and its Fourier transform \( E_{ij}(k) \).
Turbulence is generic

Kolmogorov similarity hypotheses:

- On a sufficiently small scale, all turbulence is homogeneous and isotropic
- and can be characterized by universal functional forms...

meaning that

\[ E_{ij}(k) = f(\eta|k|) \text{ for all } i, j \]

where \( \eta \) is a generic lengthscale called the **Kolmogorov scale**.
Turbulence is generic

Kolmogorov similarity hypotheses:

- On a sufficiently small scale, all turbulence is homogeneous and isotropic
- and can be characterized by universal functional forms...

meaning that

$$E_{ij}(k) = f(\eta|k|)$$

for all $i, j$

where $\eta$ is a generic lengthscale called the **Kolmogorov scale**.
How do we develop an understanding of turbulence?

We aim to answer two key questions:

- Where does turbulence come from?
- Once turbulence is established, can we provide evidence for the turbulence phenomenology using numerical simulations?

To answer these questions, we need a key assumption:

All relevant fluid motions – including turbulence – can be completely characterized by the Navier–Stokes equations:

\[
\rho \left( \frac{\partial u}{\partial t} + u \cdot \nabla u \right) = -\nabla p + \mu \nabla^2 u + f, \quad \nabla \cdot u = 0,
\]

where \( f \) includes the effects of large-scale forcing.
How do we develop an understanding of turbulence?

We aim to answer two key questions:

- Where does turbulence come from?
- Once turbulence is established, can we provide evidence for the turbulence phenomenology using numerical simulations?

To answer these questions, we need a key assumption:

All relevant fluid motions – including turbulence – can be completely characterized by the Navier–Stokes equations:

\[
\rho \left( \frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} \right) = -\nabla p + \mu \nabla^2 \mathbf{u} + \mathbf{f}, \quad \nabla \cdot \mathbf{u} = 0,
\]

where \( \mathbf{f} \) includes the effects of large-scale forcing.
Where does turbulence come from?

We will look at the specific example of **channel flow**.

We look at the passage to turbulence via linear instability.
Where does turbulence come from?

We will look at the specific example of channel flow.

We look at the passage to turbulence via linear instability.

For unidirectional steady flow with constant forcing (pressure drop) the Navier–Stokes equations have an analytical solution:

\[ u(z) = \frac{H^2}{2\mu} \left| \frac{dP}{dL} \right| \left[ \frac{z}{H} - \left( \frac{z}{H} \right)^2 \right]. \]
Where does turbulence come from?

We will look at the specific example of channel flow.

![Diagram of channel flow]

We look at the passage to turbulence via linear instability.

For unidirectional steady flow with constant forcing (pressure drop) the Navier–Stokes equations have an analytical solution:

\[
\begin{align*}
    u(z) &= \frac{H^2}{2\mu} \left| \frac{dP}{dL} \right| \left[ \frac{z}{H} - \left( \frac{z}{H} \right)^2 \right].
\end{align*}
\]

This suggests a nondimensionalization based on the friction velocity \( V = \sqrt{(H/2\rho)|dP/dL|} \), with

\[
\begin{align*}
    \tilde{u}(\tilde{z}) &= Re_* \tilde{z}(1 - \tilde{z}), \\
    Re_* &= \frac{\rho V H}{\mu}.
\end{align*}
\]
Linear stability analysis I

- Introduce a tiny sinusoidal perturbation (wavenumber $\alpha$) around the base flow.
- Produces pressure and velocity fluctuations that satisfy linearized equations of motion.
- Linearized equations of motion solved via eigenvalue analysis for complex eigenvalue $\lambda = -i\omega$ (Orr–Sommerfeld equation)
Linear stability analysis I

- Introduce a tiny sinusoidal perturbation (wavenumber $\alpha$) around the base flow.
- Produces pressure and velocity fluctuations that satisfy linearized equations of motion.
- Linearized equations of motion solved via eigenvalue analysis for complex eigenvalue $\lambda = -i\omega$ (Orr–Sommerfeld equation)

Instability predicted beyond $Re^* = 214.9$
Introduce a tiny sinusoidal perturbation (wavenumber $\alpha$) around the base flow.

- Produces pressure and velocity fluctuations that satisfy linearized equations of motion.
- Linearized equations of motion solved via eigenvalue analysis for complex eigenvalue $\lambda = -i \omega$ (Orr–Sommerfeld equation)

Instability predicted beyond $Re^* = 214.9$
Linear stability analysis II

- Theory shows base flow is unstable beyond $Re_* = Re_{*c} \approx 214.9$.
- Passage to turbulence is clear – crank up the Reynolds number, wait for the flow to go unstable, hope that the instabilities generate a complicated chaotic motion on all scales.
Linear stability analysis II

- Theory shows base flow is unstable beyond $Re_* = Re_{*c} \approx 214.9$.
- Passage to turbulence is clear – crank up the Reynolds number, wait for the flow to go unstable, hope that the instabilities generate a complicated chaotic motion on all scales.
- Seems like the end of the story, but there is a problem – Transition to turbulence is observed below $Re_{*c}$ – **subcritical transition to turbulence**.
Theory shows base flow is unstable beyond $Re_\ast = Re_{\ast c} \approx 214.9$.

Passage to turbulence is clear – crank up the Reynolds number, wait for the flow to go unstable, hope that the instabilities generate a complicated chaotic motion on all scales.

Seems like the end of the story, but there is a problem – Transition to turbulence is observed below $Re_{\ast c}$ – subcritical transition to turbulence.

Subcritical transition in channel flow can be understood using two theoretical tools – transient growth and coherent states. We can understand these using $2 \times 2$ matrices.
Consider simple two-dimensional autonomous dynamical system:

\[ \dot{x} = F(x). \]

- Fixed points \( x_0 \) satisfy \( F(x_0) = 0 \) (base state!)
- Fixed points are classified by their stability: form Jacobian matrix

\[
J = \begin{pmatrix}
\frac{\partial F_1}{\partial x} & \frac{\partial F_1}{\partial y} \\
\frac{\partial F_2}{\partial x} & \frac{\partial F_2}{\partial y}
\end{pmatrix}_{x_0}
\]

and compute eigenvalues \( \lambda = \text{spec}(J) \).
- If \( \Re(\lambda) > 0 \) for some eigenvalue, then system is unstable, otherwise it is stable or neutral.
Forcing can be introduced by looking at
\[ \dot{x} = \mu I x + F(x), \quad \mu \in \mathbb{R}^+. \]

Fixed-point analysis as before, compute eigenvalues of Jacobian.
Forcing can be introduced by looking at
\[ \dot{x} = \mu I x + F(x), \quad \mu \in \mathbb{R}^+. \]

- Fixed-point analysis as before, compute eigenvalues of Jacobian.
- Generally, \( \Re(\lambda) < 0 \) for \( \mu < \mu_c \) and \( \Re(\lambda) > 0 \) for \( \mu > \mu_c \) indicating a transition from stability to instability at the critical value \( \mu_c \).
Forcing

- Forcing can be introduced by looking at

\[ \dot{x} = \mu I x + F(x), \quad \mu \in \mathbb{R}^+. \]

- Fixed-point analysis as before, compute eigenvalues of Jacobian.

- Generally, \( \Re(\lambda) < 0 \) for \( \mu < \mu_c \) and \( \Re(\lambda) > 0 \) for \( \mu > \mu_c \) indicating a transition from stability to instability at the critical value \( \mu_c \).

- But this is not the end of the story!
Transient growth

- Transient growth can occur in linear systems where the Jacobian is **non-normal**:

\[ J J^\dagger - J^\dagger J \neq 0. \]

- There are situations where all eigenvalues are linearly stable but solutions of \( \dot{\mathbf{u}} = J\mathbf{u} \) grow rapidly before the eigenvalue theory eventually kicks in and forces

\[ \|\mathbf{u}\| \to 0 \text{ as } t \to \infty. \]

- Growth is measured by amplification factor

\[ G(t) = \sup_{\mathbf{u}_0 \text{ s.t. } \|\mathbf{u}_0\| = 1} \|e^{Jt}\mathbf{u}_0\| \]
Transient growth

- Transient growth can occur in linear systems where the Jacobian is non-normal:
  \[ JJ^\dagger - J^\dagger J \neq 0. \]

- There are situations where all eigenvalues are linearly stable but solutions of \( \dot{u} = Ju \) grow rapidly before the eigenvalue theory eventually kicks in and forces \( \|u\| \to 0 \) as \( t \to \infty \).

- Growth is measured by amplification factor
  \[ G(t) = \sup_{\|u_0\|=1} \|e^{Jt}u_0\| \]

We look at a simple concrete example (motivated by physics) that will make this much less mysterious.
Two-level system – linear theory I

Two-level system:

\[ i \frac{\partial u}{\partial t} = \mathcal{H}u + i (\mu_0 I + \mathcal{G}) u, \quad u \in \mathbb{C}^2, \]

where

\[ \mathcal{H} = \begin{pmatrix} E_0 & A \\ A & E_0 \end{pmatrix}, \quad \mathcal{G} = \text{diag}(-g_1, -g_2). \]

Note that \([\mathcal{H}, \mathcal{G}] \neq 0\) implies that the operator

\[ \mathcal{L} = \mathcal{H} + i (\mu_0 I + \mathcal{G}) \]

is non-normal, with \([\mathcal{L}, \mathcal{L}^\dagger] \propto g_2 - g_1\).
Eigenvalues: let \( u(t) = u_0 e^{-i\omega t} \), to give

\[
\Omega_r = E_0 \pm \sqrt{4A^2 - (g_1 - g_2)^2}, \quad \Omega_i = \mu_0 - \frac{1}{2}(g_1 + g_2),
\]

\[
4A^2 > (g_1 - g_2)^2, \quad \text{Case 1},
\]

\[
\Omega_r = E_0, \quad \Omega_i = \mu_0 - \frac{1}{2}(g_1 + g_2) \pm \sqrt{(g_1 - g_2)^2 - 4A^2}, \quad \text{Case 2}.
\]

We work in **Case 2** (crossover is called the diabolic point).
Two-level system – linear theory II

Eigenvalues: let \( u(t) = u_0 e^{-i\omega t} \), to give

\[
\Omega_r = E_0 \pm \sqrt{4A^2 - (g_1 - g_2)^2}, \quad \Omega_i = \mu_0 - \frac{1}{2}(g_1 + g_2),
\]

\[
4A^2 > (g_1 - g_2)^2, \quad \text{Case 1},
\]

\[
\Omega_r = E_0, \quad \Omega_i = \mu_0 - \frac{1}{2}(g_1 + g_2) \pm \sqrt{(g_1 - g_2)^2 - 4A^2}, \quad \text{Case 2}.
\]

We work in Case 2 (crossover is called the diabolic point).

We use

\[
\frac{1}{2} \frac{d}{dt} \|u\|_2^2 \leq [\mu_0 - \min(g_1, g_2)] \|u\|_2^2
\]

to identify subcritical parameter values for the forcing \( \mu_0 \) where transient growth is possible:

\[
\min(g_1, g_2) < \mu_0 < \frac{1}{2}(g_1 + g_2) - \sqrt{(g_1 - g_2)^2 - 4A^2}.
\]
Transient growth by itself won’t induce a subcritical transition because eventually the disturbance will die out.

The idea is that the transient growth will excite a nonlinear solution:

- Transient growth excites nonlinear solution,
- Nonlinear solution has a tendency to decay over time (damping) but this is counteracted by further transient growth.
- Nonlinear solution is therefore a quasi-steady structure (coherent state)
- Nonlinear solution can itself be unstable to secondary instability leading to a cascade whereby more and more nonlinear solutions of increasing complexity are excited.

We therefore look to add some nonlinear terms to the two-level system to see what might happen...
Introduction of nonlinear terms II

- Nonlinear two-level system:

\[ i \frac{\partial u}{\partial t} = \mathcal{L}u + a \begin{pmatrix} |u_1|^2 & 0 \\ 0 & |u_2|^2 \end{pmatrix} u. \]
Introduction of nonlinear terms II

- Nonlinear two-level system:

\[ i \frac{\partial u}{\partial t} = \mathcal{L}u + a \begin{pmatrix} |u_1|^2 & 0 \\ 0 & |u_2|^2 \end{pmatrix} u. \]

- We search for a self-sustained oscillatory solution:

\[ u = \text{Re}^{i\Omega t} u_0, \quad \|u_0\|_2^2 = 1, \quad \Omega \in \mathbb{R}. \]
Introduction of nonlinear terms II

- Nonlinear two-level system:

\[ i \frac{\partial u}{\partial t} = \mathcal{L}u + a \begin{pmatrix} |u_1|^2 & 0 \\ 0 & |u_2|^2 \end{pmatrix} u. \]

- We search for a **self-sustained oscillatory solution**:

\[ u = \text{Re}^{i\Omega t} u_0, \quad \|u_0\|_2^2 = 1, \quad \Omega \in \mathbb{R}. \]

- Such a solution can be found for \( g_2 < \mu_0 < g_2 \): we have \( \Omega = E_0 + aR^2 \), where \( R \) has the special value

\[ R^2 = \frac{g_1 - g_2}{a} \sqrt{\frac{1}{X^2} - 1}, \quad X^2 = -\frac{(\mu_0 - g_1)(\mu_0 - g_2)}{A^2}. \]
Introduction of nonlinear terms II

- Nonlinear two-level system:

\[
\imath \frac{\partial u}{\partial t} = \mathcal{L}u + a \begin{pmatrix}
|u_1|^2 & 0 \\
0 & |u_2|^2
\end{pmatrix} u.
\]

- We search for a **self-sustained oscillatory solution**:

\[
u = \text{Re}^{\imath \Omega t} u_0, \quad \|u_0\|_2^2 = 1, \quad \Omega \in \mathbb{R}.
\]

- Such a solution can be found for \( g_2 < \mu_0 < g_2 \): we have \( \Omega = E_0 + aR^2 \), where \( R \) has the special value

\[
R^2 = \frac{g_1 - g_2}{a} \sqrt{\frac{1}{X^2} - 1}, \quad X^2 = -\frac{(\mu_0 - g_1)(\mu_0 - g_2)}{A^2}.
\]

- Recall, the linearized problem was non-normal for \( g_1 \neq g_2 \). The same condition implies the existence of the non-trivial nonlinear solution!
Nonlinear two-level system:

\[ i \frac{\partial u}{\partial t} = \mathcal{L}u + a \begin{pmatrix} |u_1|^2 & 0 \\ 0 & |u_2|^2 \end{pmatrix} u. \]

We search for a self-sustained oscillatory solution:

\[ u = \text{Re}^{i\Omega t} u_0, \quad \|u_0\|_2^2 = 1, \quad \Omega \in \mathbb{R}. \]

Such a solution can be found for \( g_2 < \mu_0 < g_2 \): we have \( \Omega = E_0 + aR^2 \), where \( R \) has the special value

\[ R^2 = \frac{g_1 - g_2}{a} \sqrt{\frac{1}{X^2} - 1}, \quad X^2 = -\frac{(\mu_0 - g_1)(\mu_0 - g_2)}{A^2}. \]

Recall, the linearized problem was non-normal for \( g_1 \neq g_2 \). The same condition implies the existence of the non-trivial nonlinear solution!

Floquet analysis reveals that the nonlinear oscillation is always unstable to a secondary instability (exact result).
Numerical solution

Numerical simulation with 8th-order accurate Runge-Kutta scheme, with initial condition \( u(t = 0) = \left( \frac{A_0}{\sqrt{2}} \right) (i, 1)^T \).

**FIG. 2.** Solutions of (a) the non-Hermitian linear Schrödinger equation; (b) the non-Hermitian nonlinear Schrödinger equation. The initial data are the parameters are the same in (a) and (b).
Transient growth is found in Orr–Sommerfeld equation

Figure 6.2: Validation of our code for the maximum transient growth rate compared to known benchmark case in the literature (data from Reference [SH01]). The small discrepancies between the two datasets are due to errors in scanning and digitizing the data from the reference text.
Transient growth is much more important for 3D modes than for 2D modes.

Figure 6.3: Time evolution of the optimal transient growth rate as a function of the wavenumbers \( \alpha \) (streamwise) and \( \beta \) (spanwise). Between \( t = 0.1 \) and \( t = 10 \) the optimal disturbance moves from being spanwise-dominated to streamwise-dominated.

Figure 6.4: Eigenvalue of most-dangerous mode of the Orr–Sommerfeld–Squire equations, with \( Re = 5000 \). The most-dangerous mode according to eigenvalue analysis (valid as \( t \to \infty \)) is a streamwise one.
Transient growth is found in Orr–Sommerfeld equation

Figure 6.2: Validation of our code for the maximum transient growth rate compared to known benchmark case in the literature (data from Reference [SH01]). The small discrepancies between the two datasets are due to errors in scanning and digitizing the data from the reference text.
Coherent structures are found in channel flows as unstable travelling waves

**Exact coherent structures in channel flow**

\[ \text{(Left:} \pm 60\% \max [\omega_x(x,y,z)], \text{ right:} \pm 40\% \max [\omega_x(x,y,z)]) \]

**Figure 5.** Lower branch at \( Re = 415 \) (\( Re \approx 58 \)). Level curves of streamwise velocity \( u \) at \( y = 0 \) overlayed with isosurfaces of streamwise vorticity.

[Waleffe, JFM, 2001]
The second question

Once turbulence is established, can we provide evidence for the turbulence phenomenology using numerical simulations?

Answer is yes, but direct numerical simulation is extremely costly: all energy-containing scales down to the Kolmogorov microscale $\eta$ need to be resolved. Back-of-the-envelope scaling calculations indicate that $\eta \sim Re^{-3/4}$, where $Re$ is the Reynolds number based on the large-scale forcing and domain size. Hence, $\Delta x \sim Re^{-3/4}$ in a numerical simulation, requiring $N_T \sim \Delta x^{-3} \sim Re^{9/4}$.

For flow around an aeroplane, we might have $Re = 2 \times 10^7$, leading to a requirement of $N_T \sim 10^{17}$ gridpoints.
The second question

Once turbulence is established, can we provide evidence for the turbulence phenomenology using numerical simulations?

Answer is yes, but **direct numerical simulation** is extremely costly:

- All energy-containing scales down to the Kolmogorov microscale $\eta$ need to be resolved.
- Back-of-the-envelope scaling calculations indicate that
  \[ \eta \sim Re^{-3/4} \]
  where $Re$ is the Reynolds number based on the large-scale forcing and domain size.
- Hence, $\Delta x \sim Re^{-3/4}$ in a numerical simulation, requiring $N_T$ gridpoints, where
  \[ N_T \sim \Delta x^{-3} \sim Re^{9/4}. \]
The second question

Once turbulence is established, can we provide evidence for the turbulence phenomenology using numerical simulations?

Answer is yes, but **direct numerical simulation** is extremely costly:

- All energy-containing scales down to the Kolmogorov microscale $\eta$ need to be resolved.
- Back-of-the-envelope scaling calculations indicate that $$\eta \sim Re^{-3/4}$$

where $Re$ is the Reynolds number based on the large-scale forcing and domain size.

- Hence, $\Delta x \sim Re^{-3/4}$ in a numerical simulation, requiring $N_T$ gridpoints, where $$N_T \sim \Delta x^{-3} \sim Re^{9/4}.$$  

- For flow around an aeroplane, we might have $Re = 2 \times 10^7$, leading to a requirement of $$N_T \sim 10^{17}$$ gridpoints.
Turbulence modelling

- DNS are restricted to low-to-intermediate Reynolds numbers.
- Also, for doing parameter studies, running hundreds of DNS may be infeasible – even at the low-Reynolds number end.
- Requirement for turbulence models more sophisticated than two-level systems (!) but less computationally intensive than high-res DNS.
- The best tradeoff so far (used commonly by engineers, scientists, and designers) is **large-eddy simulation**.
Large-eddy simulation (LES)

Filter the velocity field:

\[ \mathbf{u}(\mathbf{x}, t) = \overline{\mathbf{u}}(\mathbf{x}, t) + \mathbf{u}'(\mathbf{x}, t) \]

\[ \overline{\mathbf{u}}(\mathbf{x}, t) = \int_{\Omega} G(\mathbf{r}) \mathbf{u}(\mathbf{x} - \mathbf{r}, t) \, d\mathbf{r} \]

Apply the filtering process to the Navier-Stokes equations:

\[ \frac{\partial \overline{u}_i}{\partial t} + \frac{u_j}{\partial t} \frac{\partial \overline{u}_i}{\partial x_j} = -\frac{1}{\rho} \frac{\partial p}{\partial x_i} + \frac{1}{\rho} \frac{\partial}{\partial x_j} \left( \mu \left( \frac{\partial \overline{u}_i}{\partial x_j} + \frac{\partial \overline{u}_j}{\partial x_i} \right) \right) - \frac{\partial \tau_{ij}}{\partial x_j}, \quad \frac{\partial \overline{u}_i}{\partial x_i} = 0 \]
Large-eddy simulation (LES)

Filter the velocity field:

\[ u(x, t) = \bar{u}(x, t) + u'(x, t) \]

\[ \bar{u}(x, t) = \int_{\Omega} G(r) u(x - r, t) \, dr \]

Apply the filtering process to the Navier-Stokes equations:

\[
\frac{\partial \bar{u}_i}{\partial t} + u_j \frac{\partial \bar{u}_i}{\partial x_j} = -\frac{1}{\rho} \frac{\partial p}{\partial x_i} + \frac{1}{\rho} \frac{\partial}{\partial x_j} \left( \mu \left( \frac{\partial \bar{u}_i}{\partial x_j} + \frac{\partial \bar{u}_j}{\partial x_i} \right) \right) - \frac{\partial \tau_{ij}}{\partial x_j}, \quad \frac{\partial \bar{u}_i}{\partial x_i} = 0
\]

**Closure problem** – additional stresses \( \tau_{ij} \) to be modelled
Closure problem – Smagorinsky model

Smagorinsky (1963) proposed:

\[ \tau_{ij} = -\nu_t \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) = -2\nu_t s_{ij} \]

\[ \frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j} = -\frac{1}{\rho} \frac{\partial p}{\partial x_i} + \frac{1}{\rho} \frac{\partial}{\partial x_j} \left( (\mu + \nu_t) \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) \right) \]

\[ \nu_t = \left( C_S \Delta \phi_w (z) \right)^2 |\mathbf{s}|, \quad |\mathbf{s}| = \sqrt{2 \left( s_{ij} \right) \left( s_{ij} \right)^T} \]

\[ \Rightarrow \text{Implement eddy viscosity, initialise system and run the simulation!} \]
Computational framework

Even a large eddy-simulation can be challenging to implement numerically – In-house implementation uses TPLS computational framework developed by yours truly:

- Finite-volume projection method to solve single-phase and two-phase incompressible Navier–Stokes equations.
- Fully MPI-parallelized code runs on 10s–1000s of CPU cores.
- Research-level version deployed to solve problems in two-phase flows where interface is captured using levelset formulation.
- Single-phase version available for demonstration purposes, and incorporates the Smagorinsky-LES model.
Computational framework

Even a large eddy-simulation can be challenging to implement numerically – In-house implementation uses TPLS computational framework developed by yours truly:

- Finite-volume projection method to solve **single-phase and two-phase** incompressible Navier–Stokes equations.
- Fully MPI-parallelized code runs on 10s–1000s of CPU cores.

Research-level version deployed to solve problems in two-phase flows where interface is captured using levelset formulation.

Single-phase version available for demonstration purposes, and incorporates the Smagorinsky-LES model.
Computational framework

Even a large eddy-simulation can be challenging to implement numerically – In-house implementation uses TPLS computational framework developed by yours truly:

- Finite-volume projection method to solve **single-phase and two-phase** incompressible Navier–Stokes equations.
- Fully MPI-parallelized code runs on 10s–1000s of CPU cores.
- Research-level version deployed to solve problems in two-phase flows where interface is captured using levelset formulation.
Computational framework

Even a large eddy-simulation can be challenging to implement numerically – In-house implementation uses TPLS computational framework developed by yours truly:

- Finite-volume projection method to solve **single-phase and two-phase** incompressible Navier–Stokes equations.
- Fully MPI-parallelized code runs on 10s–1000s of CPU cores.
- Research-level version deployed to solve problems in two-phase flows where interface is captured using levelset formulation.

Single-phase version available for demonstration purposes, and incorporates the Smagorinsky-LES model.
Aside – why write your own code?

https://www.youtube.com/watch?v=gzSMkKeF9nQ
LES Results – instantaneous snapshots I
LES Results – instantaneous snapshots II

Top – streamwise
Middle – spanwise
Bottom – wall-normal
All taken in a particular $xz$ plane
LES Results – turbulent statistics I

- Spacetime average of physical quantities is steady – forcing and dissipation in balance on average.

Prediction (S-E): \( U_C \approx 19.56 \)
LES: \( U_C \approx 19.54 \)
LES Results – turbulent statistics I

- Spacetime average of physical quantities is steady – forcing and dissipation in balance on average.
- Introduce spacetime average variables (Reynolds averaging), in particular average streamwise velocity $U_0(z)$, averaged over space, time, and symmetry plane.

![Evolution of the centreline velocity](image)

Prediction (S-E): $U_C \approx 19.56$
LES: $U_C \approx 19.54$
LES Results – turbulent statistics I

- Spacetime average of physical quantities is steady – forcing and dissipation in balance on average.
- Introduce spacetime average variables (Reynolds averaging), in particular average streamwise velocity $U_0(z)$, averaged over space, time, and symmetry plane.

Prediction (S-E): $U_C \approx 19.56$
LES: $U_C \approx 19.54$

<table>
<thead>
<tr>
<th></th>
<th>$U_{\text{mean}}$</th>
<th>$U_C / U_{\text{mean}}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Kim et al. (1986)</td>
<td>15.63</td>
<td>1.16</td>
</tr>
<tr>
<td>LES</td>
<td>16.75</td>
<td>1.167</td>
</tr>
</tbody>
</table>
LES Results – law of the wall

- Reynolds averaging also gives a balance equation for the mean velocity:

\[
\frac{d}{dz} \left( \tau_R + \mu \frac{dU_0}{dz} \right) - \frac{dP}{dL} = 0.
\]

- \( \tau_R \) is the **Reynolds stress** – similar to the residual stress of the LES technique.

- Not known *a priori* – needs to be modelled (closure problem again!!)
LES Results – law of the wall

- Reynolds averaging also gives a balance equation for the mean velocity:

\[
\frac{d}{dz} \left( \tau_R + \mu \frac{dU_0}{dz} \right) - \frac{dP}{dL} = 0.
\]

- \( \tau_R \) is the Reynolds stress – similar to the residual stress of the LES technique.

- Not known \textit{a priori} – needs to be modelled (closure problem again!!)

- Eddy viscosity:

\[
\tau_R = \mu_T \frac{dU_0}{dz}
\]
LES Results – law of the wall

- Reynolds averaging also gives a balance equation for the mean velocity:

\[
\frac{d}{dz} \left( \tau_R + \mu \frac{dU_0}{dz} \right) - \frac{dP}{dL} = 0.
\]

- \(\tau_R\) is the Reynolds stress – similar to the residual stress of the LES technique.

- Not known \textit{a priori} – needs to be modelled (closure problem again!!)

- Eddy viscosity:

\[
\tau_R = \mu_T \frac{dU_0}{dz}
\]

- Somewhere between the wall and the centreline,

\[
\mu_T \sim \kappa z^2 \left| \frac{dU_0}{dz} \right|
\]

leading to law of the wall

\[
\frac{U_0}{u_*} = \frac{1}{\kappa} \log z + \text{Const}.
\]
LES Results – law of the wall

- Reynolds averaging also gives a balance equation for the mean velocity:
  \[ \frac{d}{dz} \left( \tau_R + \mu \frac{dU_0}{dz} \right) - \frac{dP}{dL} = 0. \]
- \( \tau_R \) is the Reynolds stress – similar to the residual stress of the LES technique.
- Not known \textit{a priori} – needs to be modelled (closure problem again!!)
- Eddy viscosity:
  \[ \tau_R = \mu_T \frac{dU_0}{dz} \]

Somewhere between the wall and the centreline,
\[ \mu_T \sim \kappa z^2 \left| \frac{dU_0}{dz} \right| \]
leading to law of the wall
\[ \frac{U_0}{u_*} = \frac{1}{\kappa} \log z + \text{Const}. \]
Conclusions

- Fluid turbulence can be understood in the context of instability.
- But it is more than linear instability – subcritical transition to turbulence is common.
- Can be understood by combining transient growth with coherent structures.
- ... which can in turn be understood with a two-level dynamical system.
Conclusions

- Fluid turbulence can be understood in the context of instability.
- But it is more than linear instability – subcritical transition to turbulence is common.
- Can be understood by combining transient growth with coherent structures.
- ... which can in turn be understood with a two-level dynamical system.
- For fully-developed turbulence, DNS can be expensive, but LES provides lots of understanding at a lower computational cost.
- TPLS computational framework has been used to illustrate some standard results in channel turbulence.
Conclusions

- Fluid turbulence can be understood in the context of instability.
- But it is more than linear instability – subcritical transition to turbulence is common.
- Can be understood by combining transient growth with coherent structures.
- ... which can in turn be understood with a two-level dynamical system.
- For fully-developed turbulence, DNS can be expensive, but LES provides lots of understanding at a lower computational cost.
- TPLS computational framework has been used to illustrate some standard results in channel turbulence.
- But TPLS is much more than this...
TPLS has been used in a wide variety of applications and is available as open-source software:

http://sourceforge.net/projects/tpls/

Evaporation of non-spherical droplets

Phase separation in binary liquids

Waves in two-layer flows

Countercurrent gas-liquid flows