Bounds on the spreading radius in droplet impact: the viscous case

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Introduction

- I will look at droplet impact on a smooth surface.
- **Impact, Spread, Retraction**
  In the land of splashes, what the scientist knows as Inertia and Surface Tension are the sculptors in liquids, and fashion from them delicate shapes none the less beautiful because they are too ephemeral for any eye but that of the high-speed camera [Yarin, Annu. Rev. Fluid Mech. (2006)]
- Highlights the importance of **parameters** in such studies; key parameters are the **Weber number**, $\mathrm{We} =$ Inertia/Surface Tension, and the **Reynolds number** – *sculptor has two tools*.

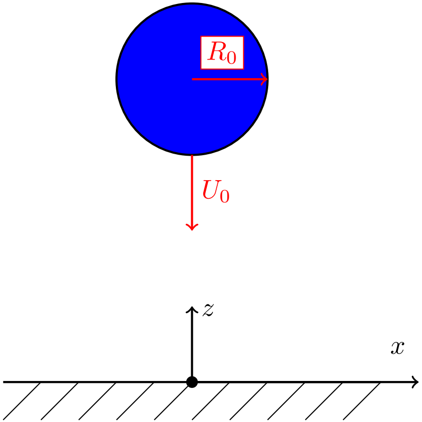
Fix Definitions

For the avoidance of doubt, we use the following definitions for the Reynolds and Weber numbers:

$$\mathrm{Re} = \frac{\rho U_0 R_0}{\mu},$$

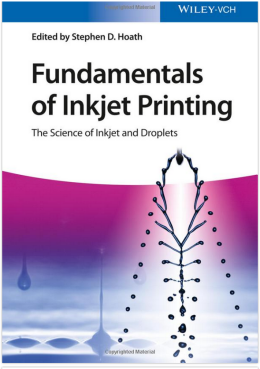
$$\mathrm{We} = \frac{\rho U_0^2 R_0}{\gamma},$$

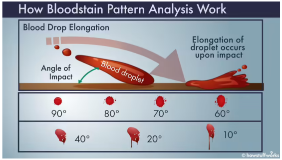
where ρ is the fluid density, μ the viscosity, and γ the surface tension.

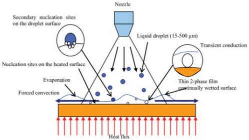


Motivation

- Industry (inket printing, cooling, bloodstain pattern analysis)



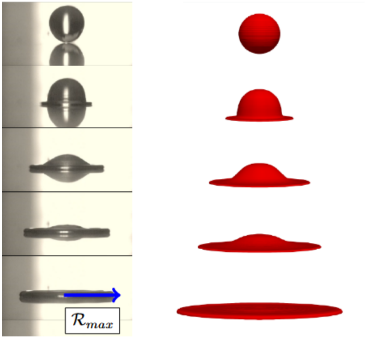




- **Scientific curiosity**...

Rim-Lamella Structure

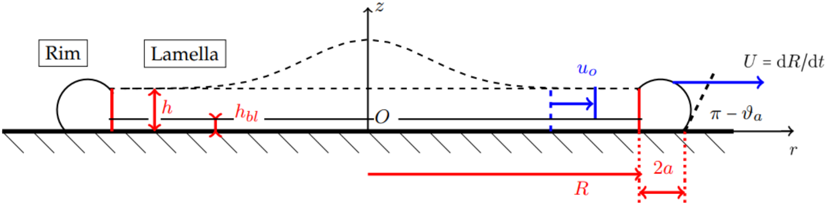
- Droplet spreading below **splash threshold** (no splash), $K \lesssim 3,000$, where $K = \mathrm{We}\sqrt{\mathrm{Re}}$
- At low We, droplet spreads out into a pancake structure – rim and lamella.
- Focus of this talk is on the rim-lamella structure.
- Of interest is the **maximum spreading radius** $\mathcal{R}_{max}$ and its dependence on We and Re.



Droplet impact study. Left: high-speed camera. Right: OpenFOAM simulations. Credit: Conor Quigley. Parameters: $\mathrm{Re} = 1700$ and $\mathrm{We} = 20$.

Rim-Lamella Models

- General model for describing dynamics of rim-lamella structure.
- Mass and momentum equations for the rim.
- Driven by fluxes from the lamella into the rim.
- Balanced by the tendency of surface tenstion to promote retraction.



- Key variables are rim position R, rim velocity U, rim volume V, and lamella height h.

Aim of present work

- We won't introduce any new models.
- Instead, we will **rigorously analyse** existing models.

Aim instead is to prove rigorously the scaling law

$$\frac{\mathcal{R}_{max}}{R_0} = k\mathrm{We}^{1/2}, \qquad \mathrm{Re} = \infty,$$

in the **inviscid case**, and the bounds:

$$k_1\mathrm{Re}^{1/5} - k_2(1-\cos\vartheta_a)^{1/2}\mathrm{Re}^{2/5}\mathrm{We}^{-1/2} \le \frac{\mathcal{R}_{max}}{R_0} \le k_1\mathrm{Re}^{1/5}, \qquad \mathrm{Re} < \infty.$$

in the **viscous** case.

Here, k, k_1, and k_2 are constant.

Plan of Talk

- In-depth description of Rim-Lamella Model in **inviscid case** with $\mathrm{Re} = \infty$;
- Key results.
- Sketch out extension to viscous case.

Rim-Lamella Modelling

After impact, a rim-lamella structure forms. Radially symmetric flow in the lamella. Mass and momentum balances:[4]

$$\begin{aligned}\frac{\partial}{\partial t}(rh) + \frac{\partial}{\partial r}(urh) &= 0,\\ \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial r} &= 0.\end{aligned}$$

- Valid for $t \geq \tau$ and $r \in (0, R)$.
- R marks the end of the lamella and the start of the rim.
- Exact solution:
$$u = \frac{r}{t + t_0}.$$
- In the viscous case (later on), this gives the **outer flow** far from the boundary layer (u_o).

[4] Yarin AL, Weiss DA. 1995 Impact of drops on solid surfaces: self-similar capillary waves, and splashing as a new type of kinematic discontinuity. Journal of Fluid Mechanics 283, 141–173.

Solution for h

Solution for h:

$$h = (t + t_0)^{-2} f\left(\frac{r}{t + t0}\right).$$

- The function f is not specified in this analysis.
- We use Roisman's ‘engineering approximation for the drop height’[5] because it fits the data:

$$\frac{h(r,t)}{R_0} = \frac{\eta}{(t+t_0)^2}\frac{R_0^2}{U_0^2}\mathrm{e}^{-(3\eta/4U_0^2)[r/(t+t_0)]^2}.$$

- Pre-factors for dimensional reasons. Parameter η is free. R_0 is the drop radius prior to impact.

[5] Roisman IV, Berberovic E, Tropea C. 2009 Inertia dominated drop collisions. I. On the universal flow in the lamella. Physics of Fluids 21, 052103

Mass and Momentum Balance in the Rim

Mass and momentum balance in the rim are described by the following ordinary differential equations, valid in the inviscid limit:

$$
\begin{aligned}
\frac{\mathrm{d}V}{\mathrm{d}t} &= 2\pi R\,(u_0 - U)\,h(R,t),\\
V\frac{\mathrm{d}U}{\mathrm{d}t} &= 2\pi R\Bigg[\underbrace{(u_0 - U)^2\,h(R,t)}_{=\text{Inertia}} - \underbrace{\frac{\sigma}{\rho}\,(1-\cos\vartheta_a)}_{=\text{Surface Tension}}\Bigg],\\
\frac{\mathrm{d}R}{\mathrm{d}t} &= U,
\end{aligned}
$$

where:

- V is the rim volume
- $u_0 = R/(t + t_0)$,
- U is the rim velocity
- ϑ_a is the advancing contact angle

Initial conditions:

$$R(\tau) = R_{init}, \qquad U(\tau) = U_{init}, \qquad V(\tau) = V_{init}.$$

Re-write

- Velocity defect:

$$\Delta = u_0 - U = \frac{R}{t+t_0} - U.$$

- Re-write momentum equation:

$$\frac{\mathrm{d}\Delta}{\mathrm{d}t} + \frac{\Delta}{t+t_0} = -\frac{2\pi Rh}{V}\left\{\Delta^2 - [c(t)]^2\right\}.$$

- Characteristic speed:

$$c(t) = \sqrt{\frac{\sigma(1-\cos\vartheta_a)}{\rho h(R,t)}} = \sqrt{\frac{\sigma(1-\cos\vartheta_a)}{\rho\eta R_0}(U_0/R_0)(t+t_0)\mathrm{e}^{(3\eta/8U_0^2)[R/(t+t_0)]^2}}.$$

- Reminiscent of the Taylor–Culick speed for the retraction of a liquid sheet of thickness h, $c = \sqrt{2\sigma/\rho h}$.

Gronwall's Inequality

Differential form [edit]

Let I denote an interval of the real line of the form $[a, \infty)$ or $[a, b]$ or $[a, b)$ with $a < b$. Let β and u be real-valued continuous functions defined on I. If u is differentiable in the interior I° of I (the interval I without the end points a and possibly b) and satisfies the differential inequality

$$u'(t) \leq \beta(t)\, u(t), \qquad t \in I^\circ,$$

then u is bounded by the solution of the corresponding differential *equation* $v'(t) = \beta(t)\, v(t)$:

$$u(t) \leq u(a) \exp\left(\int_a^t \beta(s)\, \mathrm{d}s \right)$$

for all $t \in I$.

Remark: There are no assumptions on the signs of the functions β and u.

- Used to put bounds on solutions of ODEs.
- E.g. Regularity of Navier–Stokes, Mixing Efficiency (advection-diffusion),... **Droplet Impact**

Key Result

Constraints on initial conditions:

- Advancing rim condition: $U_{init} > 0$;
- Deceleration condition: $0 < \Delta(\tau) \leq c(\tau)$;
- Rate of increase of $c(t)$ not too large:

$$\frac{3\eta}{2U_0^2}\left[\frac{R_{init}}{\tau + t_0} + \Delta(\tau)\right]^2 < 1.$$

If these conditions hold, then Gronwall's Inequality can be used to show:

$$0 < \Delta(\tau)\left(\frac{\tau + t_0}{t + t_0}\right) \leq \Delta \leq c,$$

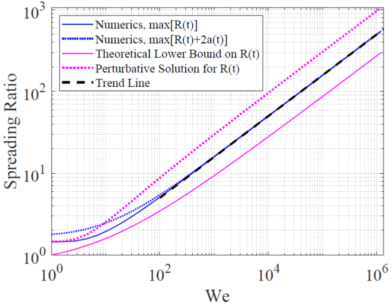
Further applications of Gronwall's Inequality yield:

$$\max[R(t)] \geq R_*, \qquad R_* = \frac{\tau + t_0}{4\widehat{c}(\tau)}\left[\widehat{c}(\tau) + \frac{R_{init}}{\tau + t_0}\right]^2,$$

where $\widehat{c}(\tau)$ is another grouping of parameters. **Lower Bound** on $\max[R(t)]$.

Sandwich Results

- Perturbation theory gives a **Upper Bound** on $\max[R(t)]$.
- Upper and Lower Bounds on $\max[R(t)]$.
- Dependent on initial conditions.
- With appropriate estimates on the initial conditions, both bounds possess $\mathrm{We}^{1/2}$ scaling at high Weber number.



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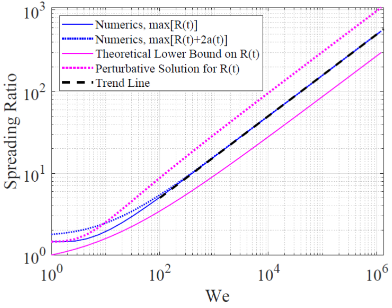


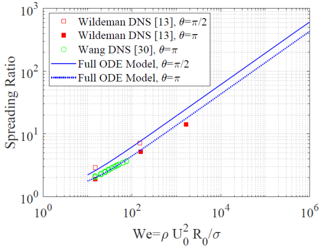
- By a ‘sandwich result’, we conclude that:

$$R_{max} = \mathrm{We}^{1/2} f(\mathrm{We}), \qquad \mathrm{We} \gg 1.$$

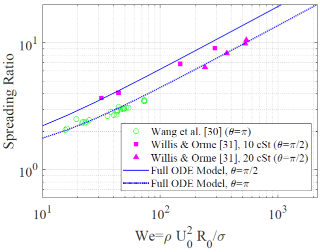
- Upper and Lower Bounds extend from $\max[R(t)]$ to $\mathcal{R}_{max}$ (geometric argument).
- Hence, $\mathcal{R}_{max} = O(\mathrm{We}^{1/2})$, for $\mathrm{We} \gg 1$.

Comparison with Experiments and DNS

Compare with results in the literature – Willis and Orme, Wang et al., and Wildeman et al.[6]



DNS: Wildeman et al. use a free-slip BC, equivalent to $Re = \infty$. Wang et al. use CA$= 180^\circ$.



Experiments: Willis and Orme use a head-on collision of two droplets... equivalent to CA$=90^\circ$. Wang et al. use an engineered superhydrophobic surface with CA$= 180^\circ$.

[6] Experiments in Fluids 34, 28–41 (2003); Energies 15, 8181 (2022); Journal of Fluid Mechanics 805, 636–655 (2016), respectively

Extension to Viscous Case

- We account for viscosity by considering the boundary layer in the schematic diagram.
- After lengthy derivations, the boundary-layer effect can be incorporated very simply into the rim-lamella model via a **depth-averaged velocity**

$$\overline{u} = u_o(R,t)\left(1 - \frac{h_{bl}}{h}\right), \qquad u_o(R,t) = \frac{R}{t+t_0},$$

  provided $h_{bl} < h$ (phase 1).
- Boundary-layer theory gives $h_{bl} \sim \sqrt{\nu r/u_o(r,t)}$. With $u_o(r,t) \sim r/(t+t_0)$, $h_{bl} \sim t^{1/2}$ (no space dependence).

Rim-Lamella model: Viscous Case

- Result is:

$$\begin{aligned}\frac{\mathrm{d}V}{\mathrm{d}t} &= 2\pi R h\left(\overline{u}-U\right),\\ V\frac{\mathrm{d}U}{\mathrm{d}t} &= 2\pi R h\rho\left(\overline{u}-U\right)^2-2\pi R\gamma\left(1-\cos\vartheta_a\right),\\ \frac{\mathrm{d}R}{\mathrm{d}t} &= U,\end{aligned}$$

  where $h\equiv h(R,t)$.

- After lengthy calculations (similar in spirt to the invisicd case), we arrive at the bounds:

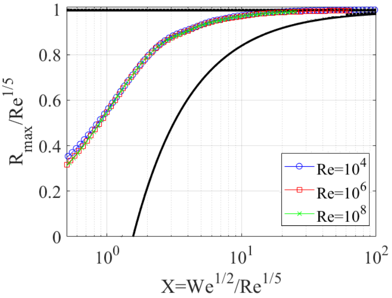
$$k_1\mathrm{Re}^{1/5}-k_2(1-\cos\vartheta_a)^{1/2}\mathrm{Re}^{2/5}\mathrm{We}^{-1/2}\leq\frac{\mathcal{R}_{max}}{R_0}\leq k_1\mathrm{Re}^{1/5},\qquad \mathrm{Re}<\infty.$$

Validation I

- Constant k_1 is **fitted** to agree with the Roisman correlation:

$$\frac{\mathcal{R}_{max}}{R_0} = 1.0\mathrm{Re}^{1/5} - 0.37\mathrm{Re}^{2/5}\mathrm{We}^{-1/2}.$$

- This fixes k_2 in our calculations.
- Solution of ODE remains with bounds (sanity test).



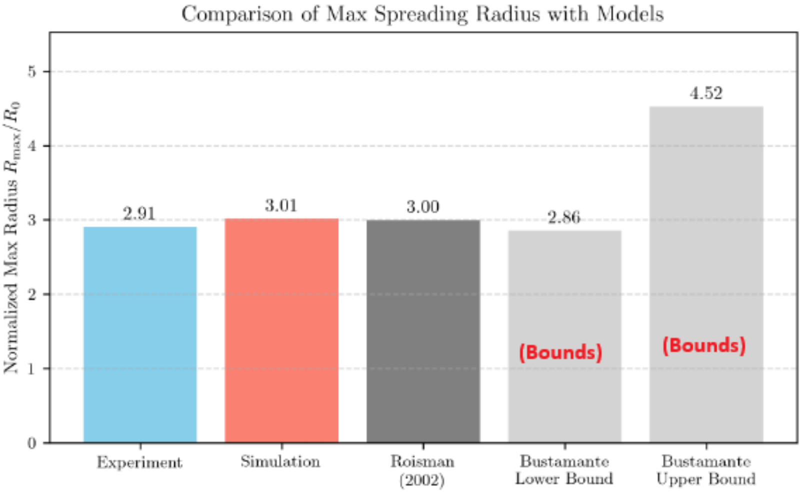
Colours: Full ODE model. Black lines: *a priori* bounds.

Validation II

Bounds also validated with respect to the **energy-budget analysis** of Wildeman et al. and with respect to **experiments**.

- Chronos 1.4 Camera
- Max: $1280 \times 1024@1069\,\mathrm{fps}$, Min: $320 \times 96@40,413\mathrm{fps}$.
- Perspex substrate.
- Results at $\mathrm{Re} = 1890$, $\mathrm{We} = 25$.

Validation III



Conclusions and Future Work

- Using Rim-Lamella models as a starting point, formulated *a priori* bounds on the spreading radius.
- Establishes rigorously the scaling behaviour of $\mathcal{R}_{max}$ with Re and We in the inviscid and viscous cases.
- Excellent agreement between theory numerical simulation, and experiments.
- Theory of **a priori bounds** has proved very fruitful – could find wider use in Fluid Mechanics: turbulence, mixing, spreading, **...**.