# Bounds on the spreading radius in droplet impact: the inviscid case

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THERMASMART
Thermal Management of Microprocessors

MARIE CURIE ACTIONS

# Introduction

- I will look at droplet impact on a smooth surface.
- **Impact, Spread, Retraction**
  *In the land of splashes, what the scientist knows as Inertia and Surface Tension are the sculptors in liquids, and fashion from them delicate shapes none the less beautiful because they are too ephemeral for any eye but that of the high-speed camera [Yarin, Annu. Rev. Fluid Mech. (2006)]*

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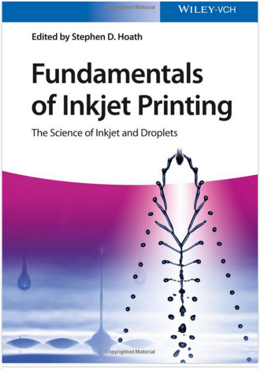
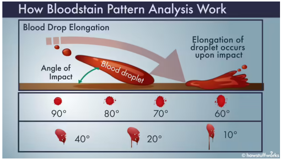
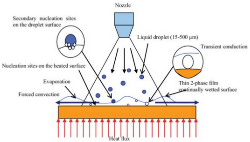
- Highlights the importance of **parameters** in such studies; key parameters are the **Weber number**, $\mathrm{We} =$ Inertia/Surface Tension, and the **Reynolds number** – *sculptor has two tools*.
- Droplet spreading below **splash threshold**, $K \lesssim 3,000$, where $K = \mathrm{We}\sqrt{\mathrm{Re}}$

# Motivation

- Industry (inket printing, cooling, bloodstain pattern analysis)



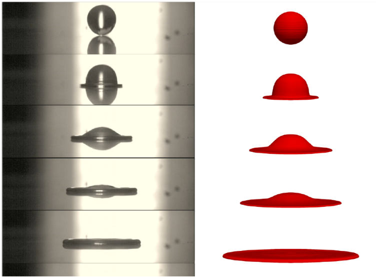




- **Scientific curiosity**...

# Why BIFD?

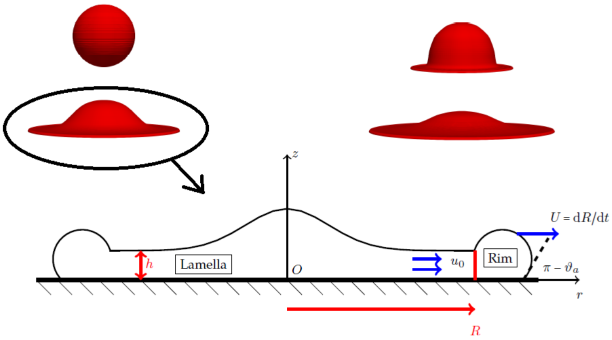
- At low We, droplet spreads out into a pancake structure – rim and lamella.
- At sufficiently high Weber number, an unstable wave develops on the rim, causing droplet splash (incl. famous **crown splash**)
- Understanding the rim and the lamella structure gives a **base state** for subsequent stability analysis.
- Focus of this talk is on the rim-lamella structure.



Droplet impact study. Left: high-speed camera. Right: OpenFOAM simulations. Credit: Conor Quigley. Parameters: $\mathrm{Re} = 1700$ and $\mathrm{We} = 20$.

# Aim of Present Work

To develop a better understanding of the Rim-Lamella dynamics in the inviscid limit.



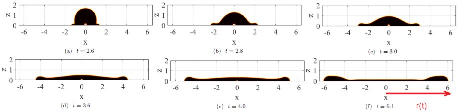

Top rows: OpenFOAM simulations. Credit: Conor Quigley. Parameters as before.
Bottom row: schematic diagram.

# Point of departure

- We won't introduce any new models.
- Instead, we will **rigorously analyse** existing models.
- Aim is to prove rigorously the scaling law

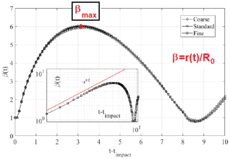
$$\mathcal{R}_{max} = k\mathrm{We}^{1/2}$$

valid in the inviscid limit $\mathrm{Re} = \infty$.




Want to find $r_{max}(\mathrm{We}, \mathrm{Re})$, or

$$\beta_{max} = r_{max}/R_0.$$



# Rim-Lamella Modelling

After impact, a rim-lamella structure forms. Radially symmetric flow in the lamella. Mass and momentum balances:[1]

$$\begin{aligned} \frac{\partial}{\partial t}(rh) + \frac{\partial}{\partial r}(urh) &= 0, \\ \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial r} &= 0. \end{aligned}$$

- Valid for $t \geq \tau$ and $r \in (0, R)$.
- $R$ marks the end of the lamella and the start of the rim.
- Exact solution:

$$u = \frac{r}{t + t_0}.$$

[1]Yarin AL,Weiss DA. 1995 Impact of drops on solid surfaces: self-similar capillary waves, and splashing as a new type of kinematic discontinuity. Journal of Fluid Mechanics 283, 141–173.

# Solution for $h$

Solution for $h$:

$$h = (t + t_0)^{-2} f \left( \frac{r}{t + t0} \right).$$

- The function $f$ is not specified in this analysis.
- We use Roisman's 'engineering approximation for the drop height'[2] because it fits the data:

$$\frac{h(r,t)}{R_0} = \frac{\eta}{(t + t_0)^2} \frac{R_0^2}{U_0^2} \mathrm{e}^{-(3\eta/4U_0^2)[r/(t+t_0)]^2}.$$

- Pre-factors for dimensional reasons. Parameter $\eta$ is free. $R_0$ is the drop radius prior to impact.

[2] Roisman IV, Berberovic E, Tropea C. 2009 Inertia dominated drop collisions. I. On the universal flow in the lamella. Physics of Fluids 21, 052103

# Mass and Momentum Balance in the Rim

Mass and momentum balance in the rim are described by the following ordinary differential equations, valid in the inviscid limit:

$$\begin{aligned}
\frac{\mathrm{d}V}{\mathrm{d}t} &= 2\pi R\,(u_0 - U)\,h(R,t),\\
V\frac{\mathrm{d}U}{\mathrm{d}t} &= 2\pi R\Bigg[\underbrace{(u_0 - U)^2\,h(R,t)}_{=\text{Inertia}} - \underbrace{\frac{\sigma}{\rho}\,(1-\cos\vartheta_a)}_{=\text{Surface Tension}}\Bigg],\\
\frac{\mathrm{d}R}{\mathrm{d}t} &= U,
\end{aligned}$$

where:

- $V$ is the rim volume
- $u_0 = R/(t+t_0)$,
- $U$ is the rim velocity
- $\vartheta_a$ is the advancing contact angle

Initial conditions:

$$R(\tau) = R_{init}, \qquad U(\tau) = U_{init}, \qquad V(\tau) = V_{init}.$$

# Re-write

- Velocity defect:

$$\Delta = u_0 - U = \frac{R}{t+t_0} - U.$$

- Re-write momentum equation:

$$\frac{\mathrm{d}\Delta}{\mathrm{d}t} + \frac{\Delta}{t+t_0} = -\frac{2\pi R h}{V}\big\{\Delta^2 - [c(t)]^2\big\}.$$

- Characteristic speed:

$$c(t) = \sqrt{\frac{\sigma(1-\cos\vartheta_a)}{\rho h(R,t)}} = \sqrt{\frac{\sigma(1-\cos\vartheta_a)}{\rho\eta R_0}}(U_0/R_0)(t+t_0)\mathrm{e}^{(3\eta/8U_0^2)[R/(t+t_0)]^2}.$$

- Reminiscent of the Taylor–Culick speed for the retraction of a liquid sheet of thickness $h$, $c = \sqrt{2\sigma/\rho h}$.

# Gronwall's Inequality

## Differential form [ edit ]

Let $I$ denote an interval of the real line of the form $[a, \infty)$ or $[a, b]$ or $[a, b)$ with $a < b$. Let $\beta$ and $u$ be real-valued continuous functions defined on $I$. If $u$ is differentiable in the interior $I^\circ$ of $I$ (the interval $I$ without the end points $a$ and possibly $b$) and satisfies the differential inequality

$$u'(t) \leq \beta(t)\, u(t), \qquad t \in I^\circ,$$

then $u$ is bounded by the solution of the corresponding differential *equation* $v'(t) = \beta(t)\, v(t)$:

$$u(t) \leq u(a) \exp\left( \int_a^t \beta(s)\, \mathrm{d}s \right)$$

for all $t \in I$.

**Remark:** There are no assumptions on the signs of the functions $\beta$ and $u$.

- Used to put bounds on solutions of ODEs.
- E.g. Regularity of Navier–Stokes, Mixing Efficiency (advection-diffusion),... **Droplet Impact**

# Key Result

Constraints on initial conditions:

- Advancing rim condition: $U_{init} > 0$;
- Deceleration condition: $0 < \Delta(\tau) \leq c(\tau)$;
- Rate of increase of $c(t)$ not too large:

$$\frac{3\eta}{2U_0^2}\left[\frac{R_{init}}{\tau + t_0} + \Delta(\tau)\right]^2 < 1.$$

If these conditions hold, then Gronwall's Inequality can be used to show:

$$0 < \Delta(\tau)\left(\frac{\tau + t_0}{t + t_0}\right) \leq \Delta \leq c,$$

Further applications of Gronwall's Inequality yield:

$$\max[R(t)] \geq R_*, \qquad R_* = \frac{\tau + t_0}{4\widehat{c}(\tau)}\left[\widehat{c}(\tau) + \frac{R_{init}}{\tau + t_0}\right]^2,$$

where $\widehat{c}(\tau)$ is another grouping of parameters. **Lower Bound** on $\max[R(t)]$.

# Perturbation Theory I

Three-equation Rim–Lamella model can be reduced to as single equation:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left\{(t+t_0)^2\frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{R}{t+t_0}-\frac{(\tau+t_0)^2}{3R_{init}^2}\left(\frac{R}{t+t_0}\right)^3\right]\right\}$$
$$=-\frac{2c(\tau)^2}{R_{init}^2}(t+t_0)^2\frac{R}{t+t_0}, \qquad t>\tau.$$

Close to a family of integrable systems found in Astrophysics (the Emden-Fowler Equation)) – can be solved by a simple perturbation method.

# Perturbation Theory II

New variable $L(t)$:

$$\frac{L}{t+t_0} := \frac{R}{t+t_0} - \epsilon \frac{(\tau+t_0)^2}{3R_{init}^2}\left(\frac{R}{t+t_0}\right)^3 .$$

Previous collapsed equation becomes:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[(t+t_0)^2 \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{L}{t+t_0}\right)\right] = -\frac{2c(\tau)^2}{R_{init}^2}(t+t_0)^2 \frac{R}{t+t_0} ,$$

or, developing the derivatives and rearranging,

$$\frac{\mathrm{d}^2 L}{\mathrm{d}t^2} + \Omega^2 R = 0 , \qquad \Omega := \sqrt{2}\, \frac{c(\tau)}{R_{init}} .$$

Perturbation solution:

$$L(t) = \sum_{k=0}^{\infty} \epsilon^k L_k(t) = L_0(t) + \epsilon L_1(t) + \epsilon^2 L_2(t) + \ldots .$$

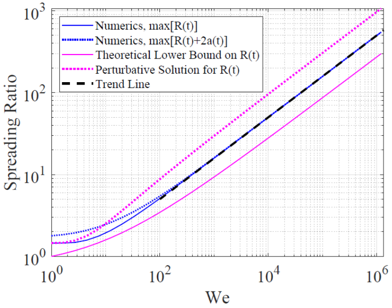
Invert cubic and find $R(t)$. Analysis gives **upper bound** on $\max[R(t)]$.

# Sandwich Results

- Upper and lower bounds on $\max[R(t)]$.
- Dependent on initial conditions.
- With appropriate estimates on the initial conditions, both bounds possess $\mathrm{We}^{1/2}$ scaling at high Weber number.
- By a 'sandwich result', we conclude that:

$$R_{max} = \mathrm{We}^{1/2} f(\mathrm{We}), \qquad \mathrm{We} \gg 1,$$

where $f(\mathrm{We})$ is a bounded function, $|f(\mathrm{We})| \leq M$.

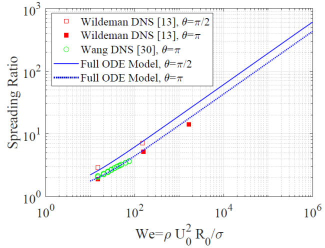



Hence, $R_{max} = O(\mathrm{We}^{1/2})$, for $\mathrm{We} \gg 1$.

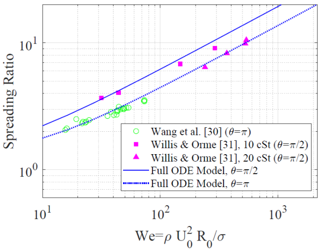
Upper and lower bounds extend from $\max[R(t)]$ to $\mathcal{R}_{max}$ (geometric argument).

# Comparison with Experiments and DNS

Compare with results in the literature – Willis and Orme, Wang et al., and Wildeman et al.[3]




DNS: Wildeman et al. use a free-slip BC, equivalent to $Re = \infty$. Wang et al. use CA$= 180^\circ$.




Experiments: Willis and Orme use a head-on collision of two droplets... equivalent to CA$=90^\circ$. Wang et al. use an engineered superhydrophobic surface with CA$= 180^\circ$.

[3] Experiments in Fluids 34, 28–41 (2003); Energies 15, 8181 (2022); Journal of Fluid Mechanics 805, 636–655 (2016), respectively

# Conclusions and Future Work

- Established rigorously the behaviour $\mathcal{R}_{max} \propto \mathrm{We}^{1/2}$ for droplet spreading at large Weber number and infinite Reynolds number.
- Showcased novel application of **bounds** in Fluids – adds to work in Turbulence / NS regularity, Fluid Mixing, etc.
- Next step is to carry out a similar exercise for the case of finite Reynolds number, where the appearance of a boundary layer complicates the rim-lamella model.

# Acknowledgments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