Some new mathematical insights into the rim-lamella model for droplet spreading

Lennon Ó Náraigh[1*]

[1]School of Mathematics and Statistics, University College Dublin, Belfield, Dublin 4, Ireland

October 2025

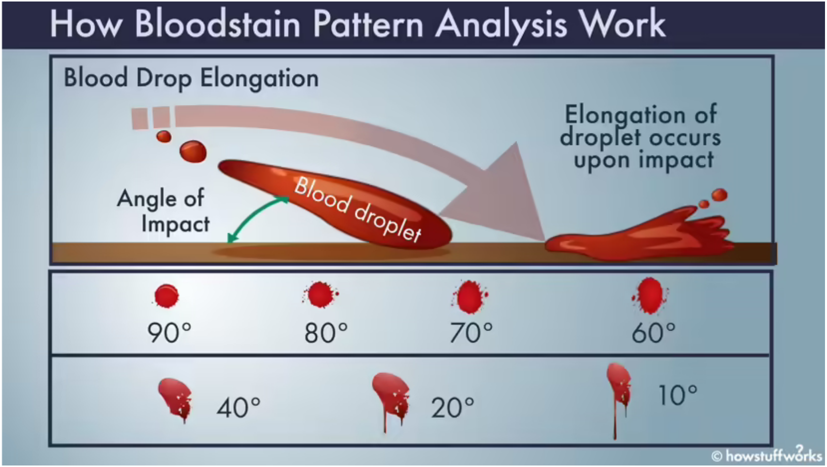
*All Collaborators: slide 14

Introduction

- I will look at droplet impact on a smooth surface.
- **Impact, Spread, Retraction**
  In the land of splashes, what the scientist knows as Inertia and Surface Tension are the sculptors in liquids, and fashion from them delicate shapes none the less beautiful because they are too ephemeral for any eye but that of the high-speed camera [Yarin, Annu. Rev. Fluid Mech. (2006)]
- Highlights the importance of **parameters** in such studies; key parameters are the **Weber number**, $\mathrm{We} =$ Inertia/Surface Tension, and the **Reynolds number** – *sculptor has two tools*.

Motivation

One particular application in Bloodstain Pattern analysis:



Bloodstain Pattern Analysis – Brief Tutorial

Six parameters needed to trace back to the original source of the bloodstain:

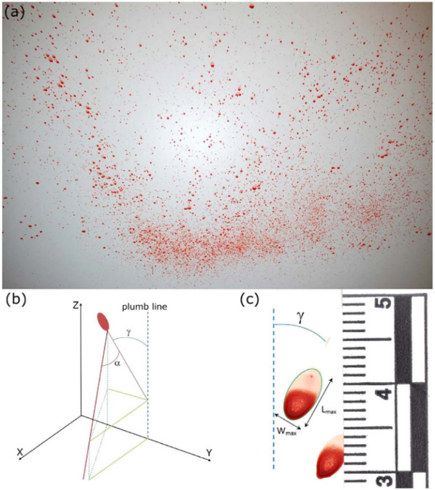
- x, y, z
- α, γ,
- **Impact speed** U_0

Parameters x, y, z, and γ can be measured directly.

Parameter α can be inferred from $\sin\alpha = W_{max}/L_{max}$.

This leaves the impact speed.

Source in footnote.[2]



[2] Laan, N., de Bruin, K.G., Slenter, D., Wilhelm, J., Jermy, M. and Bonn, D., 2015. Bloodstain pattern analysis: implementation of a fluid dynamic model for position determination of victims. Scientific reports, 5(1), p.11461.

Bloodstain Pattern Analysis – Brief Tutorial

- Impact speed can be estimated from a correlation:

$$\frac{W_{max}}{D_0} = a_0 \mathrm{Re}^{1/5} \frac{P^{1/2}}{a_1 + P^{1/2}(\sin\alpha)^{4/5}}, \qquad P = \mathrm{We}\,\mathrm{Re}^{-2/5}.$$

- a_0 and a_1 are fitting parameters.
- W_{max} can be measured, D_0 can be inferred through the volume of a bloodstain.
- $\mathrm{We} \propto U_0^2$ and $\mathrm{Re} \propto U_0$ – equation can be solved to yield U_0.

Spreading radius

- Normal impact, $\alpha = \pi/2$:

$$\frac{D_{max}}{D_0} = a_0 \mathrm{Re}^{1/5} \frac{P^{1/2}}{a_1 + P^{1/2}}$$

- A classical problem in droplet impact – D_{max}/D_0 is the **spreading ratio**.
- For large P (binomial approximation) we obtain:

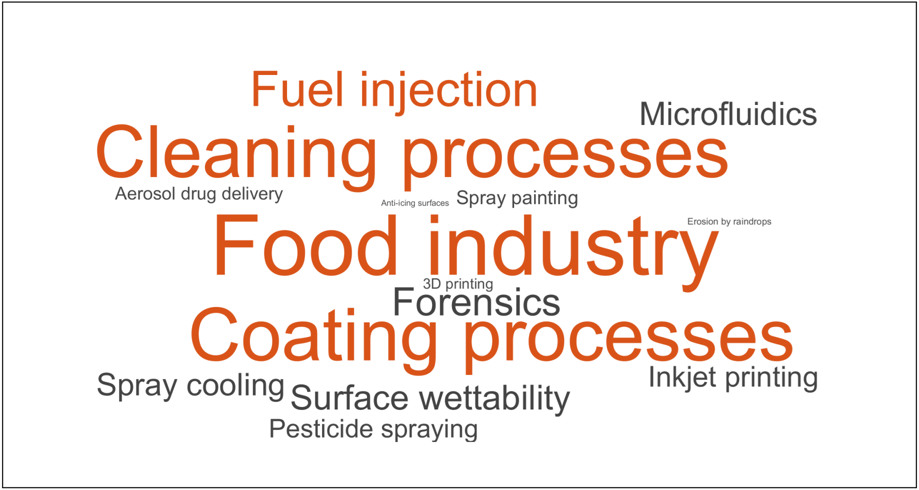
$$\frac{D_{max}}{D_0} = a_0 \mathrm{Re}^{1/5} - a_0 \cdot a_1 \, \mathrm{We}^{-1/2} \mathrm{Re}^{2/5},$$

which is the semi-empirical correlation obtained by Roisman[3].

[3] Roisman, I.V., 2009. Inertia dominated drop collisions. II. An analytical solution of the Navier–Stokes equations for a spreading viscous film. Physics of Fluids, 21(5).

Other applications

Wordcloud, weighted by Google Scholar hits on 22/09:



Important to emphazie that **scientific curiosity** is a main motivation here.

Fix Ideas...

The main question in this talk is: **Where does $P = \mathrm{We}^{1/2}\mathrm{Re}^{-1/5}$ come from?**

Droplet impact can be categorized as involving[4]:

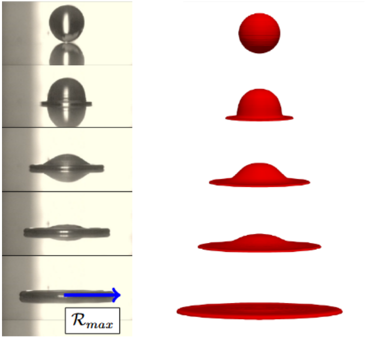
- Splash (prompt / corona)
- Rebound
- Deposition

We focus on **deposition**, which occurs below the splash threshold.

[4] Josserand, C. and Thoroddsen, S.T., 2016. Drop impact on a solid surface. Annual review of fluid mechanics, 48(1), pp.365-391.

Splash Threshold

- Droplet spreading below **splash threshold** (no splash), $K \lesssim 3,000$, where $K = \mathrm{We}\sqrt{\mathrm{Re}}$
- At low We, droplet spreads out into a pancake structure – rim and lamella.
- Of interest is the **maximum spreading radius** $\mathcal{R}_{max}$ and its dependence on We and Re.



Droplet impact study. Left: high-speed camera. Right: OpenFOAM simulations. Credit: Conor Quigley. Parameters: $\mathrm{Re} = 1700$ and $\mathrm{We} = 20$.

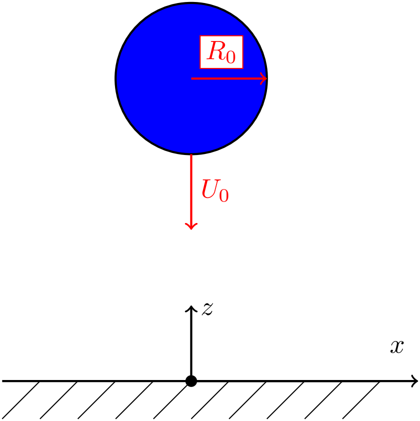
For the avoidance of doubt...

We use the following definitions for the Reynolds and Weber numbers:

$$\mathrm{Re} = \frac{\rho U_0 R_0}{\mu},$$

$$\mathrm{We} = \frac{\rho U_0^2 R_0}{\gamma},$$

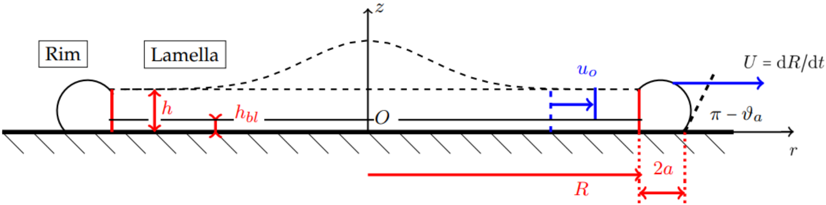
where ρ is the fluid density, μ the viscosity, and γ the surface tension.



Rim-Lamella Models

The answer to the research question comes through the study of **rim-lamella models**.

- General model for describing dynamics of rim-lamella structure.
- Mass and momentum equations for the rim.
- Driven by fluxes from the lamella into the rim.
- Balanced by the tendency of surface tension to promote retraction.



- Key variables are rim position R, rim velocity U, rim volume V, and lamella height h.

Aim of present work

- We won't introduce any new models.
- Instead, we will **rigorously analyse** existing models.

Aim is to prove rigorously the scaling law

$$\frac{\mathcal{R}_{max}}{R_0} = k\mathrm{We}^{1/2}, \qquad \mathrm{Re} = \infty,$$

in the **inviscid case**, and the bounds:

$$k_1\mathrm{Re}^{1/5} - k_2(1-\cos\vartheta_a)^{1/2}\mathrm{Re}^{2/5}\mathrm{We}^{-1/2} \leq \frac{\mathcal{R}_{max}}{R_0} \leq k_1\mathrm{Re}^{1/5}, \qquad \mathrm{Re} < \infty.$$

in the **viscous** case.

Here, k, k_1, and k_2 are constant.

Plan of Talk

- In-depth description of Rim-Lamella Model in **inviscid case** with $\mathrm{Re} = \infty$.
  - Present key results.
- Sketch out extension to viscous case.
- Sketch out future work... droplet impact on to a porous medium.

Collaborators

Faculty	Alidad Amirfazli (York University Toronto) Miguel Bustamante (UCD) *Conceptualization, Analysis*
PhD Student	Yating Hu (York University Toronto) *Analysis*
Master's Student	Juan Mairal (UCD) *CFD*
Final-Year Project Students	Conor Quigley (UCD), Joseph Anderson (UCD), Patrick Murray (UCD) *High-speed camera work, CFD*
Summer Research Student	Nicola Young (UCD – now MSc Applied Mathematics at Imperial) *Analysis, CFD*

Rim-Lamella Modelling

After impact, a rim-lamella structure forms. Radially symmetric flow in the lamella. Mass and momentum balances:[5]

$$\begin{aligned} \frac{\partial}{\partial t}(rh) + \frac{\partial}{\partial r}(urh) &= 0, \\ \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial r} &= 0. \end{aligned}$$

- Valid for $t \geq \tau$ and $r \in (0, R)$.
- R marks the end of the lamella and the start of the rim.
- Exact solution:

$$u = \frac{r}{t + t_0}.$$

[5] Yarin AL,Weiss DA. 1995 Impact of drops on solid surfaces: self-similar capillary waves, and splashing as a new type of kinematic discontinuity. Journal of Fluid Mechanics 283, 141–173.

Solution for h

Solution for h:

$$h = (t + t_0)^{-2} f\left(\frac{r}{t + t_0}\right).$$

- The function f is not specified in this analysis (method of characteristics).
- We use Roisman's 'engineering approximation for the drop height'[6] because it fits the data:

$$\frac{h(r,t)}{R_0} = \frac{\eta}{(t + t_0)^2}\frac{R_0^2}{U_0^2}\mathrm{e}^{-(3\eta/4U_0^2)[r/(t+t_0)]^2}.$$

- Pre-factors for dimensional reasons. Parameter η is free. R_0 is the drop radius prior to impact.

[6] Roisman IV, Berberovic E, Tropea C. 2009 Inertia dominated drop collisions. I. On the universal flow in the lamella. Physics of Fluids 21, 052103

Mass and Momentum Balance in the Rim

Mass and momentum balance in the rim are described by the following ordinary differential equations, valid in the inviscid limit:

$$\begin{aligned}
\frac{\mathrm{d}V}{\mathrm{d}t} &= 2\pi R\,(u_0 - U)\,h(R,t), \\
V\,\frac{\mathrm{d}U}{\mathrm{d}t} &= 2\pi R\Bigg[\underbrace{(u_0 - U)^2\,h(R,t)}_{=\text{Inertia}} - \underbrace{\frac{\gamma}{\rho}\,(1-\cos\vartheta_a)}_{=\text{Surface Tension}}\Bigg], \\
\frac{\mathrm{d}R}{\mathrm{d}t} &= U,
\end{aligned}$$

where:

- V is the rim volume
- $u_0 = R/(t+t_0)$,
- U is the rim velocity
- ϑ_a is the advancing contact angle

Initial conditions:

$$R(\tau) = R_{init}, \qquad U(\tau) = U_{init}, \qquad V(\tau) = V_{init}.$$

Re-write

- Velocity defect:

$$\Delta = u_0 - U = \frac{R}{t + t_0} - U.$$

- Re-write momentum equation:

$$\frac{\mathrm{d}\Delta}{\mathrm{d}t} + \frac{\Delta}{t + t_0} = -\frac{2\pi R h}{V}\left\{\Delta^2 - [c(t)]^2\right\}.$$

- Characteristic speed:

$$c(t) = \sqrt{\frac{\gamma(1 - \cos\vartheta_a)}{\rho h(R,t)}} = \sqrt{\frac{\gamma(1 - \cos\vartheta_a)}{\rho\eta R_0}}(U_0/R_0)(t+t_0)\mathrm{e}^{(3\eta/8U_0^2)[R/(t+t_0)]^2}.$$

- Reminiscent of the Taylor–Culick speed for the retraction of a liquid sheet of thickness h, $c = \sqrt{2\gamma/\rho h}$.

Gronwall's Inequality

Differential form [edit]

Let I denote an interval of the real line of the form $[a, \infty)$ or $[a, b]$ or $[a, b)$ with $a < b$. Let β and u be real-valued continuous functions defined on I. If u is differentiable in the interior I° of I (the interval I without the end points a and possibly b) and satisfies the differential inequality

$$u'(t) \leq \beta(t)\, u(t), \qquad t \in I^\circ,$$

then u is bounded by the solution of the corresponding differential *equation* $v'(t) = \beta(t)\, v(t)$:

$$u(t) \leq u(a) \exp\left(\int_a^t \beta(s)\, \mathrm{d}s\right)$$

for all $t \in I$.

Remark: There are no assumptions on the signs of the functions β and u.

- Used to put bounds on solutions of differential equations.
- E.g. Regularity of Navier–Stokes, Mixing Efficiency (advection-diffusion),... **Droplet Impact**

Key Result

Constraints on initial conditions:

- Advancing rim condition: $U_{init} > 0$;
- Deceleration condition: $0 < \Delta(\tau) \leq c(\tau)$;
- Rate of increase of $c(t)$ not too large:

$$\frac{3\eta}{2U_0^2}\left[\frac{R_{init}}{\tau + t_0} + \Delta(\tau)\right]^2 < 1.$$

If these conditions hold, then Gronwall's Inequality can be used to show:

$$0 < \Delta(\tau)\left(\frac{\tau + t_0}{t + t_0}\right) \leq \Delta \leq c,$$

Further applications of Gronwall's Inequality yield:

$$\max_{t\in[\tau,\infty)}[R(t)] \geq R_*, \qquad R_* = \frac{\tau + t_0}{4\widehat{c}(\tau)}\left[\widehat{c}(\tau) + \frac{R_{init}}{\tau + t_0}\right]^2,$$

where $\widehat{c}(\tau)$ is another grouping of parameters. **Lower Bound** on $\max[R(t)]$.

Perturbation Theory I

Three-equation Rim–Lamella model can be reduced to as single equation:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left\{(t+t_0)^2\frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{R}{t+t_0}-\frac{(\tau+t_0)^2}{3R_{init}^2}\left(\frac{R}{t+t_0}\right)^3\right]\right\} = -\frac{2c(\tau)^2}{R_{init}^2}(t+t_0)^2\frac{R}{t+t_0}, \qquad t>\tau.$$

Close to a family of integrable systems found in Astrophysics (the Emden-Fowler Equation)) – can be solved by a simple perturbation method.

Perturbation Theory II

New variable $L(t)$:

$$\frac{L}{t+t_0} := \frac{R}{t+t_0} - \epsilon \frac{(\tau+t_0)^2}{3R_{init}^2}\left(\frac{R}{t+t_0}\right)^3 .$$

Previous collapsed equation becomes:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[(t+t_0)^2 \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{L}{t+t_0}\right)\right] = -\frac{2c(\tau)^2}{R_{init}^2}(t+t_0)^2 \frac{R}{t+t_0} ,$$

or, developing the derivatives and rearranging,

$$\frac{\mathrm{d}^2 L}{\mathrm{d}t^2} + \Omega^2 R = 0 , \qquad \Omega := \sqrt{2}\,\frac{c(\tau)}{R_{init}} .$$

Perturbation solution:

$$L(t) = \sum_{k=0}^{\infty} \epsilon^k L_k(t) = L_0(t) + \epsilon L_1(t) + \epsilon^2 L_2(t) + \dots .$$

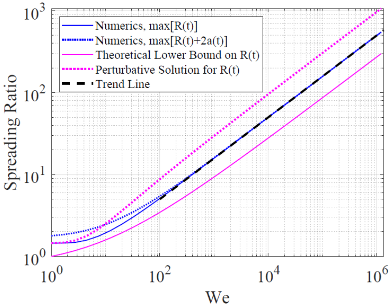
Invert cubic and find $R(t)$. Analysis gives **upper bound** on $\max[R(t)]$.

Sandwich Results

- Upper and lower bounds on $\max[R(t)]$.
- Dependent on initial conditions.
- With appropriate estimates on the initial conditions, both bounds possess $\mathrm{We}^{1/2}$ scaling at high Weber number.
- By a 'sandwich result', we conclude that:

$$R_{max} = \mathrm{We}^{1/2} f(\mathrm{We}), \qquad \mathrm{We} \gg 1,$$

where $f(\mathrm{We})$ is a bounded function, $|f(\mathrm{We})| \leq M$.

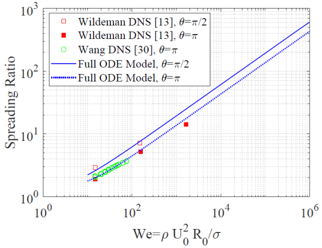


Hence, $R_{max} = O(\mathrm{We}^{1/2})$, for $\mathrm{We} \gg 1$.

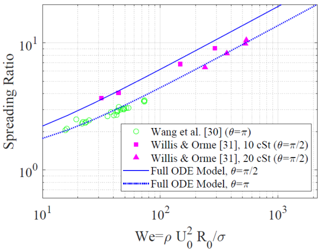
Upper and lower bounds extend from $\max[R(t)]$ to $\mathcal{R}_{max}$ (geometric argument).

Comparison with Experiments and DNS

Compare with results in the literature – Willis and Orme, Wang et al., and Wildeman et al.[7]



DNS: Wildeman et al. use a free-slip BC, equivalent to $Re = \infty$. Wang et al. use CA$= 180^\circ$.



Experiments: Willis and Orme use a head-on collision of two droplets... equivalent to CA$=90^\circ$. Wang et al. use an engineered superhydrophobic surface with CA$= 180^\circ$.

[7]Experiments in Fluids 34, 28–41 (2003); Energies 15, 8181 (2022); Journal of Fluid Mechanics 805, 636–655 (2016), respectively

Rim-Lamella model: Viscous Case

Previous description is modified by the presence of a viscous **boundary layer**.

- BL theory gives $h_{bl} \sim \sqrt{\nu r / u_o}$
- u_o is the **outer flow**, $u_o = r/(t + t_0)$
- Combination gives $h_{bl} \sim \sqrt{\nu(t + t_0)}$, independent of r.
- Introduce extra degrees of freedom to get $h_{bl} = \xi \sqrt{\nu(t + t_1)}$.

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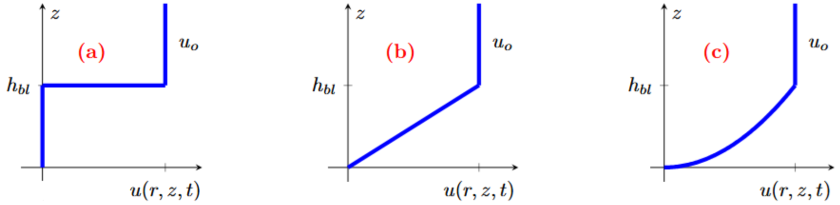
Flow transitions rapidly to zero across BL:



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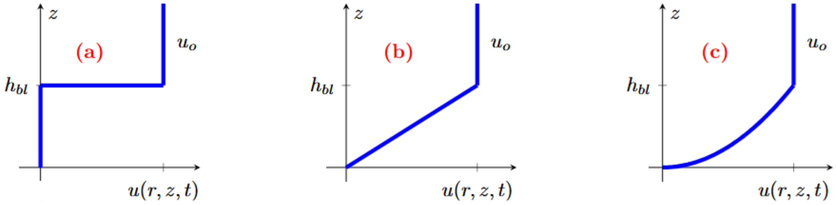

Option (c) is the Kármán's Momentum Integral Theory, but Option (a) is more analytically tractable for our purposes.

Kinematic Condition

We use Option 1 – kinematic condition (h-equation) becomes:

$$\frac{\partial h}{\partial t} + u_o(r,t)\frac{\partial h}{\partial r} = -\frac{2}{t+t_0}(h - h_{bl}),$$

valid for $t > \tau$, $h < h_{bl}$.

By the method of characteristics:

$$h(r,t) = \frac{1}{(t+t_0)^2} g\left(\frac{r}{t+t_0}\right) + h_{PI}(t).$$

Particular integral:

$$h_{PI}(t) = \tfrac{4}{15}\left[h_{bl}(t)\left(\frac{3t+5t_0-2t_1}{t+t_0}\right) - h_{bl}(\tau)\left(\frac{3\tau+5t_0-2t_1}{\tau+t_0}\right)\right].$$

Systematic Derivations – Mass Conservation

Systematic derivation of rim-lamella model. E.g. for mass conservation,

$$V_{tot} = V + 2\pi \int_0^R rh(r,t)\mathrm{d}r = \text{Const.}$$

We take $\mathrm{d}V_{tot}/\mathrm{d}t = 0$ and differentiate under the integral, using $\dot{R} = U$ to get:

$$\frac{\mathrm{d}V}{\mathrm{d}t} = 2\pi R \left[u_o \left(h - h_{bl}\right) - Uh\right],$$

where u_o and h are evaluated at $r = R$.

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where u_o and h are evaluated at $r = R$.

Equivalently,

$$\frac{\mathrm{d}V}{\mathrm{d}t} = 2\pi R h\left(\overline{u} - U\right),$$

where

$$\overline{u} = u_o(R,t)\left(1 - \frac{h_{bl}}{h}\right)$$

is the depth-averaged velocity $\overline{u} = h^{-1}\int_0^h u(R,z,t)\mathrm{d}z$.

Rim-Lamella model: Final Results

A similar approach for the momentum equation. Final results after simplifications:

$$\frac{\mathrm{d}V}{\mathrm{d}t} = 2\pi Rh\left(\overline{u} - U\right), \qquad \frac{\mathrm{d}R}{\mathrm{d}t} = U$$

$$V\frac{\mathrm{d}U}{\mathrm{d}t} = 2\pi Rh\rho\left(\overline{u} - U\right)^2 - 2\pi R\gamma\left(1 - \cos\vartheta_a\right),$$

where $h \equiv h(R,t)$.

Initial conditions are key:

- Zero initial mass in the rim: $V(\tau) = 0$.
- Finite acceleration at $t = \tau$:

$$\left(\overline{u} - U\right)^2 h - \frac{\gamma\left(1 - \cos\vartheta_{ap}\right)}{\rho} = 0.$$

hence:

$$U(t = \tau) = \overline{u} - \sqrt{\frac{\gamma\left(1 - \cos\vartheta_{ap}\right)}{\rho h}}$$

Geometric conditions in initial lamella shape [7]: $h_{init} = R_0/2$, $R_{init} = R_0\sqrt{8/3}$.

[7] Eggers, J., Fontelos, M.A., Josserand, C. and Zaleski, S., 2010. Drop dynamics after impact on a solid wall: theory and simulations. Physics of Fluids, 22(6).

Rim-Lamella model: Bounds

After lengthy calculations (similar in spirit to the invisicd case), we arrive at the bounds:

$$k_1 \mathrm{Re}^{1/5} - k_2(1-\cos\vartheta_a)^{1/2}\mathrm{Re}^{2/5}\mathrm{We}^{-1/2} \leq \frac{\mathcal{R}_{max}}{R_0} \leq k_1\mathrm{Re}^{1/5}, \qquad \mathrm{Re} < \infty.$$

- The time t_* where $h(t) = h_{bl}(t_*)$ is key to the calculations. Correspondingly, $h_* = h(t_*)$.
- Constants k_1 and k_2 arise from fitting $h_*/R_0 = k_h\mathrm{Re}^{-2/5}$ and $t_*/T = k_t\mathrm{Re}^{1/5}$ to the h-equation, these can be **quantified**.
- Connecting everything:

$$k_1 = R_{init} h_{init}^{1/2} k_h^{-1/2}, \qquad k_2 = k_h^{-1/2} k_t.$$

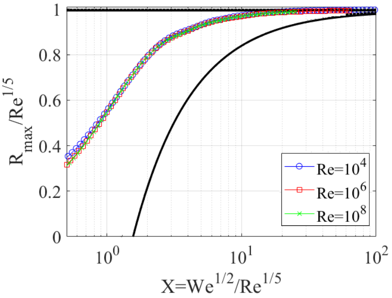
- Further constants (ξ, t_1, t_0) can also be quantified to yield a closed-form model.

Fitting constants and Validation I

- Constant k_1 is **fitted** to agree with the Roisman correlation:

$$\frac{\mathcal{R}_{max}}{R_0} = 1.0\mathrm{Re}^{1/5} - 0.37\mathrm{Re}^{2/5}\mathrm{We}^{-1/2}.$$

- This fixes k_2 in our calculations.
- Solution of ODE remains with bounds (sanity test).



Colours: Full ODE model. Black lines: *a priori* bounds.

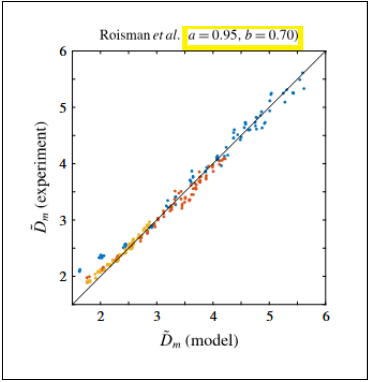
Validation II

Bounds validated with respect to the **Roisman's correlation** [8]

$$\mathcal{R}_{max}/R_0 = a\,\mathrm{Re}^{1/5} - b\,\mathrm{Re}^{2/5}\mathrm{We}^{-1/2}.$$

- k_1 identified with a;
- $k_2\sqrt{1-\cos\vartheta_a}$ identified with b.
- Value: $\vartheta_a = \pi/2$.

	a	b
Lower Bound	1.00	0
Roisman	1.00	-0.37
Upper Bound	1.00	- 1.55



- Roisman is empirical (see on right);
- As $b_{Roisman}$ is sandwiched between the bounds...

bounds are consistent with experimental data

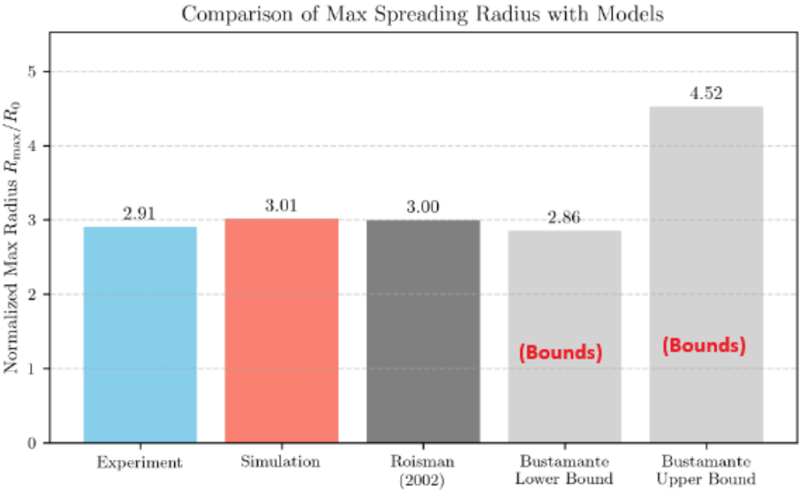
[8] Wildeman, S., Visser, C.W., Sun, C. and Lohse, D., 2016. On the spreading of impacting drops. Journal of fluid mechanics, 805, pp.636-655.

Nice illustration – FYP

Bounds also validated with respect to **experiments**.

- Chronos 1.4 Camera
- Max: $1280 \times 1024@1069\,\mathrm{fps}$, Min: $320 \times 96@40,413\mathrm{fps}$.
- Perspex substrate.
- Results at $\mathrm{Re} = 1890$, $\mathrm{We} = 25$.

Nice illustration – FYP (again)



Phase 2 of motion

- Something glossed over – because shape of model boundary-layer profile, all fluid motion 'stops' when $h(t) = h_{bl}(t_*)$, at $t = t_*$.
- Thereafter, RL equations simplify and describe **retraction** – *Phase 2* of the model.

$$
\begin{aligned}
\frac{\mathrm{d}V}{\mathrm{d}t} &= -2\pi R h_* U, \qquad \frac{\mathrm{d}R}{\mathrm{d}t} = U \\
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$$
\begin{aligned}
\frac{\mathrm{d}V}{\mathrm{d}t} &= -2\pi R h_* U, \qquad \frac{\mathrm{d}R}{\mathrm{d}t} = U \\
V\frac{\mathrm{d}U}{\mathrm{d}t} &= 2\pi R h \rho U^2 - 2\pi R \gamma \left(1 - \cos\vartheta_a\right).
\end{aligned}
$$

Use mass conservation, $V = V_{tot} - \pi R^2 h_*$ to re-write as two-equation system:

$$
\left(V_{tot} - \pi R^2 h_*\right)\frac{\mathrm{d}U}{\mathrm{d}t} = 2\pi R h_* \left(U^2 - c_*^2\right), \qquad \frac{\mathrm{d}R}{\mathrm{d}t} = U,
$$

where c_* is the TC speed evaluated at h_*.

Hamiltonian System

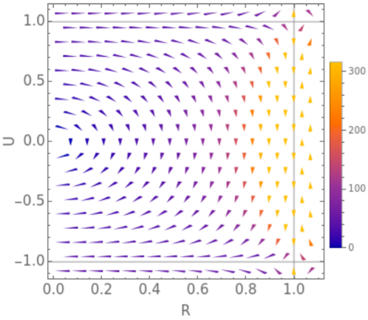
Using a Jacobi last multiplier, the equations of motion can re-expressed as a Hamiltonian system:

$$\frac{\mathrm{d}R}{\mathrm{d}t} = \frac{U}{c_*^2 - U^2} =: \frac{\partial H}{\partial U},$$

$$\rho \frac{\mathrm{d}U}{\mathrm{d}t} = -\frac{2\pi R h_*}{V_{tot} - \pi R^2 h_*} =: -\frac{\partial H}{\partial R},$$

where:

$$H(R, U) = -\tfrac{1}{2} \log\left(c_*^2 - U^2\right) - \log\left(V_{tot} - \pi R^2 h_*\right).$$



Invariant domain with $c_*^2 - U^2 > 0$ and $0 < V_{tot} - \pi R^2 h_*$ (stop model when $R = 0$).

Integrals

Using conservation of energy, model reduced further to a single equation:

$$\left(\frac{\mathrm{d}R}{\mathrm{d}t}\right) = c_*^2 - \frac{\beta^2}{\left(V_{tot} - \pi R^2 h_*\right)^2}, \qquad \beta := \mathrm{e}^{-H}.$$

Time T_4 for one-quarter period is the time from onset of Phase 2 until full retraction (mathematical):

$$T_4 = \sqrt{\frac{V_{tot}}{\pi h_* c_*^2}} \underbrace{\frac{(\chi+1)E\left(\frac{1-\chi}{1+\chi}\right) - \chi K\left(\frac{1-\chi}{1+\chi}\right)}{\sqrt{\chi+1}}}_{\text{Weak dependence: @0: 1, @1: 1.11}}, \qquad \chi := \frac{\beta}{c_* V_{tot}}.$$

R is maximum ($=R_{\mathsf{ph}-2,\mathsf{max}}$) at start of Phase 2:

$$R_{\mathsf{ph}-2,\mathsf{max}} = \sqrt{\frac{V_{tot}}{\pi h_*}}\sqrt{1-\chi} \le \sqrt{\frac{V_{tot}}{\pi h_*}},$$

giving

$$T_4 \sim \mathrm{We}^{1/2}, \qquad R_{\mathsf{ph}-2,\mathsf{max}} \sim \mathrm{Re}^{1/5}.$$

Summary of where we are at...

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  - Used Gronwall's Inequality to place bounds on the maximum spreading radius. ☑

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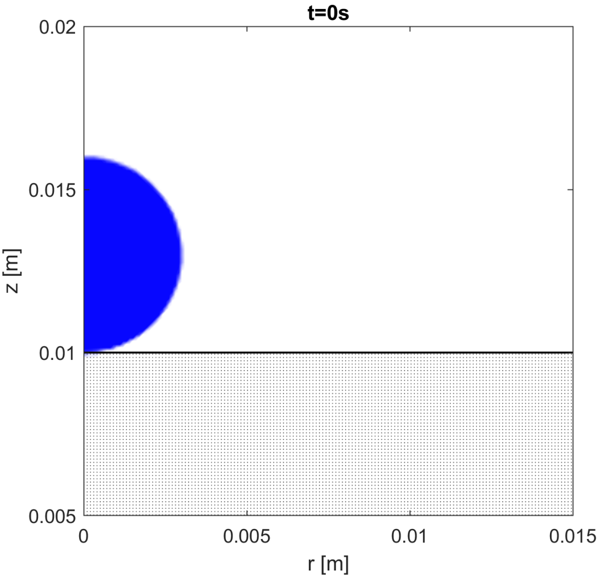
  - Used Gronwall's Inequality to place bounds on the maximum spreading radius. ☑
- Re-capitulated the Rim-Lamella model in 3D axisymmetry in the viscous case. ☑
  - Not a whole load of new things here... a minor-reformulation of the model due to Eggers *et al.*.
  - Point of departure is again to use Gronwall's Inequality to place bounds on the maximum spreading radius.
  - Identified Hamiltonian system in *Phase 2* of the motion ☑

Future work – Droplet Impact on to porous medium (Coupled NS / Darcy)



Acknowledgments

- Funding from ThermaSMART, 2017-2023 (Marie Sklodowska–Curie grant agreement No. 778104).
- Research Ireland (Dublin) and the Department of Agriculture, Food and Marine (Dublin) under grant 21/RC/10303_P2 (VistaMilk Phase 2, 2024-).
- School of Mathematics and Statistics Summer Research Projects (2025).