Chapter 10

Stokes's and Gauss's Theorems

Overview

In ordinary calculus, recall the rule of integration by parts:

$$\int_a^b u\,\mathrm{d}v = (uv)\,|_a^b - \int_a^b v\,\mathrm{d}u.$$

That is, a difficult integral $u\,\mathrm{d}v$ can be split up into an easier integral $v\,\mathrm{d}u$ and a 'boundary term' $u(b)v(b) - u(a)v(a)$. In this section we do something similar for vector integrals.

10.1 Gauss's Theorem (or the Divergence Theorem)

Theorem 10.1 *Let V be a region in space bounded by a closed surface S, and let $\boldsymbol{v}(\boldsymbol{x})$ be a vector field with continuous derivatives. Then*

$$\int_V \nabla\cdot\boldsymbol{v}\,\mathrm{d}V = \int_S \boldsymbol{v}\cdot\mathrm{d}\boldsymbol{S},$$

where $\mathrm{d}\boldsymbol{S}$ is outward-pointing surface-area element associated with the surface S.

Proof: First, consider a parallelepiped of sides of length Δx, Δy, and Δz, with one vertex positioned at (x, y, z) (Fig. 10.1). As in previous exercises, label the faces Fxp, Fxm, Fyp, Fym, Fzp, and Fzm. We compute

$$\sum_{\text{all faces}} \boldsymbol{v}\cdot\Delta\boldsymbol{S},$$

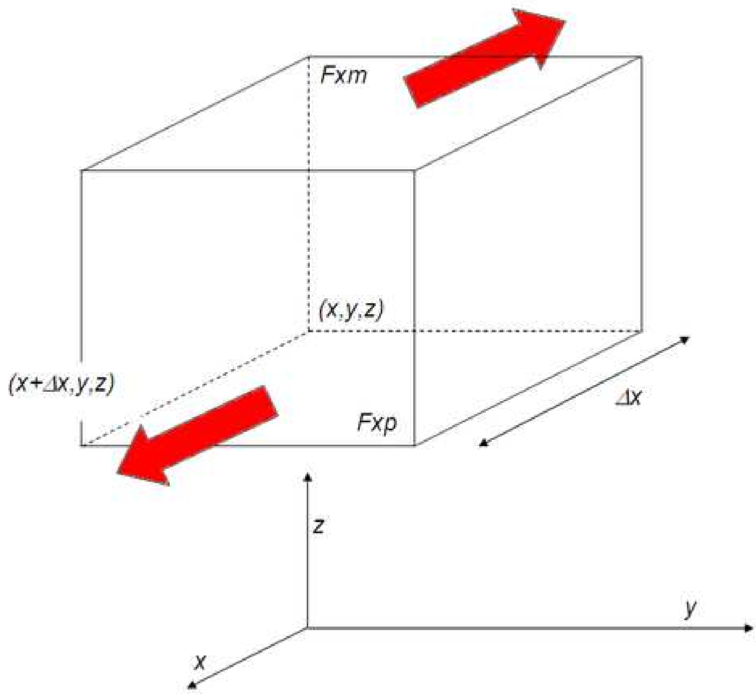

Figure 10.1: Area integration over a parallelepiped, as applied to Gauss's theorem.

where $\Delta \boldsymbol{S}$ is the area element on each face. For example, in the x-direction, we have a positive contribution from Fxp and a negative one from Fxm, to give

$$-v_1(x,y,z)\Delta y\Delta z + v_1(x+\Delta x,y,z)\Delta y\Delta z.$$

We immediately write down the other contributions: From Fyp and Fym, we have

$$-v_2(x,y,z)\Delta x\Delta z + v_2(x,y+\Delta y,z)\Delta x\Delta z,$$

and from Fzp and Fzm, we have

$$-v_3(x,y,z)\Delta x\Delta y + v_2(x,y,z+\Delta z)\Delta x\Delta y.$$

Summing over all six contributions (i.e. over all six faces), we have

$$\sum_{\text{all faces}} \boldsymbol{v}\cdot\Delta\boldsymbol{S} =$$
$$v_1(x+\Delta x,y,z)\Delta y\Delta z - v_1(x,y,z)\Delta y\Delta z + v_2(x,y+\Delta y,z)\Delta x\Delta z - v_2(x,y,z)\Delta x\Delta z +$$
$$v_3(x,y,z+\Delta z)\Delta x\Delta y - v_3(x,y,z)\Delta x\Delta y.$$

We apply Taylor's theorem to these increments, and omit terms that are $O(\Delta x^2, \Delta y^2, \Delta z^2)$. This becomes rigorous in the limit when the parallelepiped volume go to zero. In this way, we obtain

$$\sum_{\text{all faces}} \boldsymbol{v} \cdot \mathrm{d}\boldsymbol{S} = \nabla \cdot \boldsymbol{v}\, \mathrm{d}V.$$

For the second and final step, consider an arbitrary shape of volume V in three dimensions. We break this volume up into many infinitesimally small parallelepipeds. By the previous result, we have

$$\sum_{\text{all parallelepipeds}} \nabla \cdot \boldsymbol{v}\, \mathrm{d}V = \sum_{\text{all parallelepipeds}} \left(\sum_{\text{all faces}} \boldsymbol{v} \cdot \mathrm{d}\boldsymbol{S} \right). \tag{10.1}$$

Consider, however, two neighbouring parallelepipeds (Fig. 10.2). Call them A and B These will share a common face, F, with normal vector $\hat{\boldsymbol{n}}$ and area $\mathrm{d}S$. Parallelepiped A gives a contribution $\hat{\boldsymbol{n}} \cdot \boldsymbol{v}(F) dS$, say, to the sum (10.1), while parallelepiped B must give a contribution $-\hat{\boldsymbol{n}} \cdot \boldsymbol{v}(F)\mathrm{d}S$. The only place where such a cancellation cannot occur is on exterior faces. Thus,

$$\sum_{\text{all parallelepipeds}} \nabla \cdot \boldsymbol{v}\, \mathrm{d}V = \sum_{\text{all exterior faces}} \boldsymbol{v} \cdot \mathrm{d}\boldsymbol{S}.$$

But the parallelepiped volumes are infinitesimally small, so this sum converts into an integral:

$$\int_V \nabla \cdot \boldsymbol{v}\, \mathrm{d}V = \int_S \boldsymbol{v} \cdot \mathrm{d}\boldsymbol{S}.$$

This completes the proof.

10.1.1 Green's theorem

A frequently used corollary of Gauss's theorem is a relation called **Green's theorem**. If ϕ and ψ are two scalar fields, then we have the identities

$$\begin{aligned} \nabla \cdot (\phi \nabla \psi) &= \phi \nabla \cdot \nabla \psi + \nabla \phi \cdot \nabla \psi, \\ \nabla \cdot (\psi \nabla \phi) &= \psi \nabla \cdot \nabla \phi + \nabla \psi \cdot \nabla \phi. \end{aligned}$$

Subtracting these equations gives

$$\begin{aligned} \nabla \cdot (\phi \nabla \psi - \psi \nabla \phi) &= \phi \nabla \cdot \nabla \psi - \psi \nabla \cdot \nabla \phi, \\ &= \phi \nabla^2 \psi - \psi \nabla^2 \phi. \end{aligned}$$

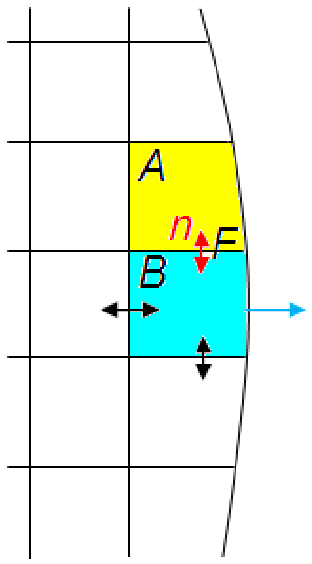

Figure 10.2: Cancellations in Gauss's theorem.

We integrate over a volume V whose boundary is a closed set S. Applying Gauss's theorem gives

$$\begin{aligned} \int_V \left(\phi\nabla^2\psi - \psi\nabla^2\phi\right) \mathrm{d}V &= \int_V \left[\nabla\cdot(\phi\nabla\psi - \psi\nabla\phi)\right] \mathrm{d}V, \\ &= \int_S (\phi\nabla\psi - \psi\nabla\phi)\cdot \mathrm{d}\boldsymbol{S}. \end{aligned}$$

Thus, we have Green's theorem:

$$\int_V \left(\phi\nabla^2\psi - \psi\nabla^2\phi\right) \mathrm{d}V = \int_S (\phi\nabla\psi - \psi\nabla\phi)\cdot \mathrm{d}\boldsymbol{S},$$

where V is a region of $\mathbb{R}^3$ whose boundary is the closed set S.

10.1.2 Other forms of Gauss's theorem

Although the form $\int_V \nabla\cdot\boldsymbol{v}\mathrm{d}V = \int_S \boldsymbol{v}\cdot\mathrm{d}\boldsymbol{S}$ is the most common statement of Gauss's theorem, there are other forms. For example, let

$$\boldsymbol{v}(\boldsymbol{x}) = v(\boldsymbol{x})\boldsymbol{a},$$

where $\boldsymbol{a}$ is a *constant vector*. We have

$$\int_V \nabla\cdot\boldsymbol{v}\,\mathrm{d}V = \int_V \nabla\cdot\boldsymbol{v}\,\mathrm{d}V = \boldsymbol{a}\cdot\int_V (\nabla v)\mathrm{d}V.$$

However, applying Gauss's theorem gives

$$\int_V \nabla \cdot \boldsymbol{v}\,\mathrm{d}V = \int_S v\boldsymbol{a} \cdot \mathrm{d}\boldsymbol{S} = \boldsymbol{a} \cdot \int_S v\,\mathrm{d}\boldsymbol{S}.$$

Equating both sides,

$$\boldsymbol{a} \cdot \int_V \nabla v\,\mathrm{d}V = \boldsymbol{a} \cdot \int_S v\,\mathrm{d}\boldsymbol{S},$$

or

$$\boldsymbol{a} \cdot \left[\int_V \nabla v\,\mathrm{d}V - \int_S v\,\mathrm{d}\boldsymbol{S}\right] = 0.$$

Since this holds for arbitrary vector fields of the form $\boldsymbol{v} = v(\boldsymbol{x})\boldsymbol{a}$, it must be true that $[\cdots] = 0$, or

$$\boxed{\int_V \nabla v\,\mathrm{d}V = \int_S v\,\mathrm{d}\boldsymbol{S}.}$$

Similarly, letting $\boldsymbol{v}(\boldsymbol{x}) = \boldsymbol{a} \times \boldsymbol{u}(\boldsymbol{x})$, where $\boldsymbol{a}$ is a constant vector, gives

$$\boxed{\int_V \nabla \times \boldsymbol{u}\,\mathrm{d}V = \int_S \mathrm{d}\boldsymbol{S} \times \boldsymbol{u}.}$$

Worked examples

1. Evaluate by using Gauss's theorem $\int_S \boldsymbol{v} \cdot \mathrm{d}\boldsymbol{S}$, where

$$\boldsymbol{v} = 8xz\hat{\boldsymbol{x}} + 2y^2\hat{\boldsymbol{y}} + 3yz\hat{\boldsymbol{z}}$$

and S is the surface of the cube in the positive octant, one of whose vertices lies at $(0,0,0)$.

We compute:

$$\begin{aligned}
\int_S \boldsymbol{v} \cdot \mathrm{d}\boldsymbol{S} &= \int_V \mathrm{d}V\,\nabla \cdot \boldsymbol{v}, \\
&= \int_0^1 \mathrm{d}x \int_0^1 \mathrm{d}y \int_0^1 \mathrm{d}z\ (8z + 4y + 3y), \\
&= 1 \cdot 1 \cdot \int_0^1 8z\,\mathrm{d}z + 1 \cdot 1 \cdot \int_0^1 7y\,\mathrm{d}y, \\
&= 4 + \tfrac{7}{2} = \tfrac{15}{2}.
\end{aligned}$$

2. A fluid is confined in a container of volume V with closed boundary S. The velocity of the fluid is $\boldsymbol{v}(\boldsymbol{x},t)$. The velocity satisfies the so-called no-throughflow condition

$$\boldsymbol{v}\cdot\hat{\boldsymbol{n}} = 0, \text{ on } S,$$

where $\hat{\boldsymbol{n}}$ is the outward-pointing normal to the surface. Now suppose that a pollutant is introduced to the fluid, of concentration $C(\boldsymbol{x},t)$. The pollutant must satisfy the equation

$$\frac{\partial C}{\partial t} + \nabla\cdot(\boldsymbol{v}C) = 0.$$

Prove that the total amount of pollutant,

$$P(t) = \int_V C(\boldsymbol{x},t)\,\mathrm{d}V,$$

stays the same over time (hence P is in fact independent of time).

Proof: We have

$$\begin{aligned}\frac{dP}{dt} &= \frac{d}{dt}\int_V C(\boldsymbol{x},t)\,\mathrm{d}V,\\ &= \int_V \frac{\partial C(\boldsymbol{x},t)}{\partial t}\,\mathrm{d}V,\\ &= -\int_V \nabla\cdot(\boldsymbol{v}C)\,\mathrm{d}V,\\ &= -\int_S C(\boldsymbol{x}\in S,t)\boldsymbol{v}(\boldsymbol{x}\in S,t)\cdot\mathrm{d}\boldsymbol{S}.\end{aligned}$$

But

$$\hat{\boldsymbol{n}}\cdot\boldsymbol{v}|_{\boldsymbol{x}\in S} = 0,$$

hence

$$\frac{dP}{dt} = 0,$$

and the amount of pollutant P is constant ('conserved').

10.2 Stokes's Theorem

Theorem 10.2 *Let S be an open, two-sided surface bounded by a closed, non-intersecting*

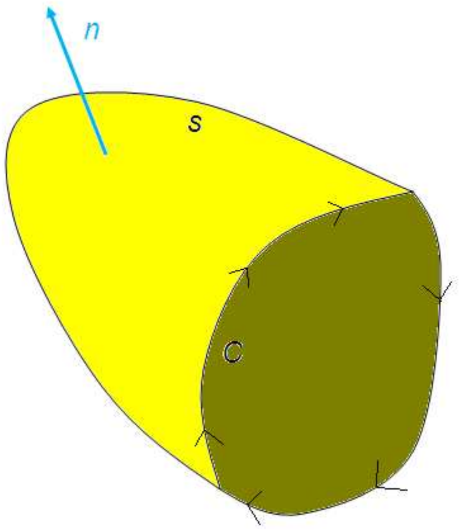

Figure 10.3: Stokes theorem: S is a surface; C is its boundary. The boundary can be given a definite orientation so the curve is called **two-sided**.

curve C, and let $\boldsymbol{v}(\boldsymbol{x})$ be a vector field with continuous derivatives. Then,

$$\oint_C \boldsymbol{v} \cdot \mathrm{d}\boldsymbol{x} = \int_S (\nabla \times \boldsymbol{v}) \cdot \mathrm{d}\boldsymbol{S},$$

where C is treated in the positive direction: an observer walking along the boundary of S, with his head pointing in the direction of the positive normal to S, has the surface on his left.

For the $S - C$ curve to which the theorem refers, see Fig. 10.3.

Proof: First, consider a rectangle in the x-y plane of sides of length Δx and Δy, with one vertex positioned at (x, y) (Fig. 10.4). Label the edges Exp, Exm, Eyp, and Eym. We compute

$$\sum_{\text{all edges}} \boldsymbol{v} \cdot \Delta\boldsymbol{x},$$

where $\Delta\boldsymbol{x}$ is the line element on each edge, and we compute in an anticlockwise sense. For example, in the x-direction, along Exp we have $\mathrm{d}\boldsymbol{x} = \hat{\boldsymbol{x}}\mathrm{d}x$ and along Exm we have $\mathrm{d}\boldsymbol{x} = -\hat{\boldsymbol{x}}\mathrm{d}x$. Adding up these contributions to $\boldsymbol{v} \cdot \Delta\boldsymbol{x}$ gives

$$[v_1(x, y, z)\Delta x - v_1(x, y + \Delta y, z)]\,\Delta x.$$

Similarly, the contributions along Eyp and Eym give

$$[v_2(x + \Delta x, y) - v_2(x, y)]\,\Delta y.$$

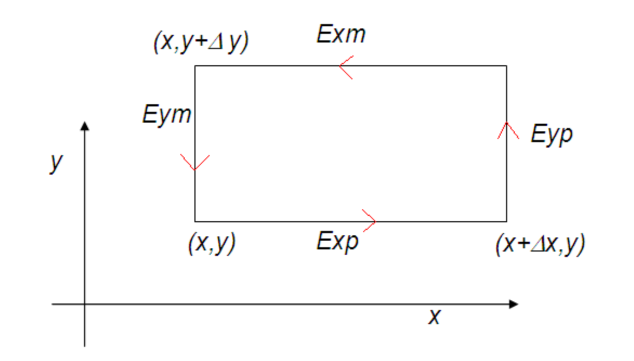

Figure 10.4: Line integration over a rectangle.

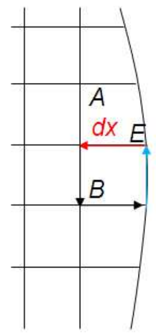

Figure 10.5: Cancellations in Stokes's theorem.

Summing over these four contributions (i.e. summing over the four edges), we have

$$\sum_{\text{all edges}} \boldsymbol{v} \cdot \Delta \boldsymbol{x} = [v_1(x,y) - v_1(x,y+\Delta y)]\,\Delta x + [v_2(x+\Delta x,y) - v_2(x,y)]\,\Delta y$$

We apply Taylor's theorem to these increments and omit terms that are $O(\Delta x^2, \Delta y^2)$. This procedure is rigorous in the limit as the parallelogram area goes to zero. We obtain

$$\begin{aligned}\sum_{\text{all edges}} \boldsymbol{v} \cdot \Delta \boldsymbol{x} &= [v_1(x,y) - v_1(x,y+\Delta y)]\,\Delta x + [v_2(x+\Delta x,y) - v_2(x,y)]\,\Delta y \\ &= \left(\frac{\partial v_2}{\partial x} - \frac{\partial v_1}{\partial y}\right)_{(x,y)} \Delta x \Delta y.\end{aligned}$$

However, $\mathrm{d}\boldsymbol{S} = \Delta x \Delta y$ pointing out of the page, hence

$$\sum_{\text{all edges}} \boldsymbol{v} \cdot \mathrm{d}\boldsymbol{x} = (\nabla \times \boldsymbol{v}) \cdot \mathrm{d}\boldsymbol{S}.$$

For the second and final step, consider a surface S with boundary C. We break this surface up into many infinitesimally small parallelograms. By the previous result, we have

$$\sum_{\text{all parallelograms}} (\nabla \times \boldsymbol{v}) \cdot \mathrm{d}\boldsymbol{S} = \sum_{\text{all parallelograms}} \left(\sum_{\text{all edges}} \boldsymbol{v} \cdot \mathrm{d}\boldsymbol{x} \right). \tag{10.2}$$

Consider, however, two neighbouring parallelograms (Fig. 10.5). Call them A and B These will share a common edge, E, with line element $\mathrm{d}\boldsymbol{x}$. Parallelogram A gives a contribution a, say, to the sum (10.1), while parallelepiped B must give a contribution $-a$. The only place where such a cancellation cannot occur is on exterior edges. Thus,

$$\sum_{\text{all parallelograms}} (\nabla \times \boldsymbol{v}) \cdot \mathrm{d}\boldsymbol{S} = \sum_{\text{all exterior edges}} \boldsymbol{v} \cdot \mathrm{d}\boldsymbol{x}.$$

But the parallelogram areas are infinitesimally small, so this sum converts into an integral:

$$\int_S (\nabla \times \boldsymbol{v}) \cdot \mathrm{d}\boldsymbol{S} = \oint_C \boldsymbol{v} \cdot \mathrm{d}\boldsymbol{x}.$$

This completes the proof.

Example: Given a vector $\boldsymbol{v} = -\hat{\boldsymbol{x}}y + \hat{\boldsymbol{y}}x$, using Stokes's theorem, show that the integral around a continuous closed curve in the xy plane

$$\tfrac{1}{2} \oint \boldsymbol{v} \cdot \mathrm{d}\boldsymbol{x} = \tfrac{1}{2} \oint (x\,\mathrm{d}y - y\,\mathrm{d}x) = S,$$

the area enclosed by the curve.

Proof:

$$\begin{aligned}
\tfrac{1}{2} \oint_C \boldsymbol{v} \cdot \mathrm{d}\boldsymbol{x} &= \tfrac{1}{2} \int_S [\nabla \times (-\hat{\boldsymbol{x}}y + \hat{\boldsymbol{y}}x)] \cdot \mathrm{d}\boldsymbol{S}, \\
&= \tfrac{1}{2} \int_S (2\hat{\boldsymbol{z}}) \cdot \mathrm{d}\boldsymbol{S}, \\
&= \tfrac{1}{2} \int_S (2\hat{\boldsymbol{z}}) \cdot (\mathrm{d}x\,\mathrm{d}y\,\hat{\boldsymbol{z}}), \\
&= \int_S \mathrm{d}x\,\mathrm{d}y = S.
\end{aligned}$$

Green's theorem in the plane

The last example hints at the following result: let S be a patch of area entirely contained in the xy plane, with boundary C, and let $\boldsymbol{v} = (v_1(x,y), v_2(x,y), 0)$ be a smooth vector field. Then,

$$\begin{aligned}
\int_S (\nabla \times \boldsymbol{v}) \cdot \mathrm{d}\boldsymbol{S} &= \int_S (\nabla \times \boldsymbol{v}) \cdot (\mathrm{d}x\,\mathrm{d}y\,\hat{\boldsymbol{z}}), \\
&= \int_S \left(\frac{\partial v_2}{\partial x} - \frac{\partial v_1}{\partial y} \right) \mathrm{d}x\,\mathrm{d}y.
\end{aligned}$$

But by Stokes's theorem,

$$\begin{aligned}
\int_S (\nabla \times \boldsymbol{v}) \cdot \mathrm{d}\boldsymbol{S} &= \int_C \boldsymbol{v} \cdot \mathrm{d}\boldsymbol{x}, \\
&= \int_C (v_1 \mathrm{d}x + v_2 \mathrm{d}y).
\end{aligned}$$

Putting these equations together, we have Green's theorem in the plane:

$$\int_S \left(\frac{\partial v_2}{\partial x} - \frac{\partial v_1}{\partial y} \right) \mathrm{d}x \, \mathrm{d}y = \int_C (v_1 \mathrm{d}x + v_2 \mathrm{d}y) \, .$$

10.3 Potential theory

A vector field $\boldsymbol{v}$ is irrotational if and only if

- $\nabla \times \boldsymbol{v} = 0$ if and only if
- $\boldsymbol{v} = -\nabla \mathcal{U}$ if and only if
- The line integral $\int_C \boldsymbol{v} \cdot \mathrm{d}\boldsymbol{x}$ depends only on the initial and final points of the path C and is independent of the details of the path between these terminal points.

Proving that $\boldsymbol{v} = -\nabla \mathcal{U} \implies \nabla \times \boldsymbol{v} = 0$ was trivial and we have done this already. Until now, we have been unable to prove the converse, namely that $\nabla \times \boldsymbol{v} \implies \boldsymbol{v} = -\nabla \mathcal{U}$. Let us do so now.

Consider an open subset $\Omega \in \mathbb{R}^3$ that is **simply connected**, i.e. contains no 'holes'. Let us take an arbitrary closed, smooth curve C in Ω. Because Ω is simply connected, it is possible to find a surface S that lies entirely in Ω, such that (S, C) have the properties mentioned in Stokes's theorem. Suppose now that $\nabla \times \boldsymbol{v} = 0$ for **all points** $\boldsymbol{x} \in \Omega$. Now, by Stokes's theorem,

$$\begin{aligned} 0 &= \int_S (\nabla \times \boldsymbol{v}) \cdot \mathrm{d}\boldsymbol{S}, \\ &= \oint_C \boldsymbol{v} \cdot \mathrm{d}\boldsymbol{x}. \end{aligned}$$

This last result is true for **all closed, piecewise smooth contours** in the domain Ω. The only way for this relationship to be satisfied for **all contours** is if $\boldsymbol{v} = -\nabla \mathcal{U}$, for some function $\mathcal{U}(\boldsymbol{x})$, since then,

$$\begin{aligned} \oint_C \boldsymbol{v} \cdot \mathrm{d}\boldsymbol{x} &= -\oint_C (\nabla \mathcal{U}) \cdot \mathrm{d}\boldsymbol{x}, \\ &= -\left[\mathcal{U}(a) - \mathcal{U}(a) \right], \\ &= 0, \end{aligned}$$

for some reference point a on the contour C. Thus, we have proved that a vector field $\boldsymbol{v}$ is irrotational **if and only if** $\boldsymbol{v} = -\nabla \mathcal{U}$.

Note: Simple-connectedness will not be an issue in this module, as we usually work with vector fields defined on the whole of $\mathbb{R}^3$. On the other hand, it is not hard to find a domain Ω that is not simply connected. For example, consider a portion of the xy plane with a hole (Fig. 10.6). The closed

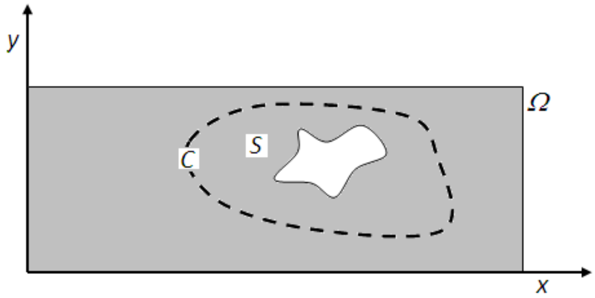

Figure 10.6: The set Ω is not simply connected.

curve C surrounds a region S; however, S is not contained entirely in Ω. We have knowledge of $\nabla \times \boldsymbol{v}$ only in Ω; we are unable to say anything about $\nabla \times \boldsymbol{v}$ in certain parts of the region S, and are therefore unable to apply the arguments of Stokes's theorem to this particular (S, C) pair.

A more precise definition of simple-connectedness than the vague condition that 'the set should contain no holes' is the following: for any two closed paths $C_0 : [0,1] \to \Omega$, $C_1 : [0,1] \to \Omega$ based at $\boldsymbol{x}_0$, i.e.

$$\boldsymbol{x}_{C_0}(0) = \boldsymbol{x}_{C_1}(0) = \boldsymbol{x}_0,$$

there exists a continuous map

$$H : [0,1] \times [0,1] \to \Omega,$$

such that

$$\begin{aligned} H(t,0) &= \boldsymbol{x}_{C_0}(t), \qquad 0 \le t \le 1, \\ H(t,1) &= \boldsymbol{x}_{C_1}(t), \qquad 0 \le t \le 1, \\ H(0,s) &= H(1,s) = \boldsymbol{x}_0, \qquad 0 \le s \le 0. \end{aligned}$$

Such a map is called a **homotopy** and C_0 and C_1 are called homotopy equivalent. One can think of this map as a 'continuous deformation of one loop into another'. Because a point is, trivially, a loop, in a simply-connected set, a loop can be continuously deformed into a point. Note in the example Fig. 10.6, the loop C cannot be continuously deformed into a point without leaving the set Ω. This is a more relational - or topological way - of describing the 'hole' in the set in Fig. 10.6.

Worked examples

1. In thermodynamics, the energy of a system of gas particles is expressed in differential form:

$$A(x,y)\mathrm{d}x + B(x,y)\mathrm{d}y,$$

where

- A is the temperature;
- B is minus the pressure;
- x has the interpretation of entropy;
- y has the interpretation of container volume.

The temperature and the pressure are known to satisfy the following relation:

$$\frac{\partial A}{\partial y} = \frac{\partial B}{\partial x}.$$

Prove that for any closed path C in xy-space (i.e. in entropy/volume-space),

$$\oint_C \left[A(x,y)\mathrm{d}x + B(x,y)\mathrm{d}y\right] = 0.$$

Proof: We may regard

$$\boldsymbol{v}(x,y) = (A(x,y), B(x,y))$$

as a vector field, and we may take

$$\mathrm{d}\boldsymbol{S} = \mathrm{d}x\,\mathrm{d}y\hat{\boldsymbol{z}}$$

as an area element, pointing out of the xy-plane. Now let S be the patch of area in xy space enclosed by the curve C. We have

$$\begin{aligned}
\int_S (\nabla\times\boldsymbol{v})\cdot\mathrm{d}\boldsymbol{S} &= \int_S \left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y}\right)\mathrm{d}x\,\mathrm{d}y,\\
&= \int_S \left[\frac{\partial B}{\partial x} - \frac{\partial A}{\partial y}\right]\mathrm{d}x\,\mathrm{d}y,\\
&= \int_S \left(\frac{\partial A}{\partial y} - \frac{\partial A}{\partial y}\right)\mathrm{d}x\,\mathrm{d}y,\\
&= 0.
\end{aligned}$$

But by Stokes's theorem,

$$
\begin{aligned}
0 &= \int_S (\nabla \times \boldsymbol{v}) \cdot \mathrm{d}\boldsymbol{S}, \\
&= \int_C \boldsymbol{v} \cdot \mathrm{d}\boldsymbol{x}, \\
&= \int_C \left[A\mathrm{d}x + B\mathrm{d}y\right],
\end{aligned}
$$

as required. Because $A(x,y)\mathrm{d}x + B(x,y)\mathrm{d}y$ integrates to zero when the integral is a closed contour, there exists a potential $E(x,y)$, such that

$$\mathrm{d}E = A(x,y)\mathrm{d}x + B(x,y)\mathrm{d}y.$$

The function E is called the **thermodynamic energy**. The integral of $\mathrm{d}E$ around a closed path is identically zero, and **the energy is path-independent**.

In general, the **differential form**

$$A(x,y)\mathrm{d}x + B(x,y)\mathrm{d}y$$

is **exact** if and only if

- There is a function $\phi(x,y)$, such that

$$A(x,y)\mathrm{d}x + B(x,y)\mathrm{d}y = \frac{\partial \phi}{\partial x}\mathrm{d}x + \frac{\partial \phi}{\partial y}\mathrm{d}y := \mathrm{d}\phi,$$

  if and only if

- The following relation holds:

$$\frac{\partial A(x,y)}{\partial y} = \frac{\partial B(x,y)}{\partial x}$$

2. In mechanics, particles experience a **force field** $\boldsymbol{F}(\boldsymbol{x})$. The force is called conservative if a potential function exists:

$$\boldsymbol{F} = -\nabla \mathcal{U}.$$

Thus, a force is conservative if and only if $\nabla \times \boldsymbol{F} = 0$.

3. Show that the three-dimensional gravitational force

$$\boldsymbol{F} = -\frac{\alpha \boldsymbol{r}}{|\boldsymbol{r}|^3}$$

is a conservative force, where α is a positive constant. We compute $\nabla \times \boldsymbol{F}$ and apply the chain rule:

$$\begin{aligned}\nabla \times \boldsymbol{F} &= -\alpha \left[\frac{1}{r^3}\nabla \times \boldsymbol{r} + \boldsymbol{r} \times \nabla\left(r^{-3}\right)\right], \\ &= -\alpha \left[\frac{1}{r^3}\nabla \times \boldsymbol{r} - 3r^{-4}\boldsymbol{r} \times \nabla r\right]\end{aligned}$$

Now

$$\nabla \times \boldsymbol{r} = \begin{vmatrix} \hat{\boldsymbol{x}} & \hat{\boldsymbol{y}} & \hat{\boldsymbol{z}} \\ \partial_x & \partial_y & \partial_z \\ x & y & z \end{vmatrix} = \hat{\boldsymbol{x}}\left(\partial_y z - \partial_z y\right) - \hat{\boldsymbol{y}}\left(\partial_x z - \partial_z x\right) + \hat{\boldsymbol{z}}\left(\partial_x y - \partial_y x\right) = 0.$$

Formally,

$$\boldsymbol{r} \times \nabla = \begin{vmatrix} \hat{\boldsymbol{x}} & \hat{\boldsymbol{y}} & \hat{\boldsymbol{z}} \\ x & y & z \\ \partial_x & \partial_y & \partial_z \end{vmatrix} = \hat{\boldsymbol{x}}\left(y\partial_z - z\partial_y\right) + \text{Cyclic permutations.}$$

Hence,

$$\begin{aligned}\boldsymbol{r} \times \nabla r &= \hat{\boldsymbol{x}}\left(y\partial_z - z\partial_y\right)\sqrt{x^2+y^2+z^2}^{1/2} + \text{Cyclic permutations,} \\ &= \hat{\boldsymbol{x}}\left[y\tfrac{1}{2}2z\left(x^2+y^2+z^2\right)^{-1/2} - z\tfrac{1}{2}2y\left(x^2+y^2+z^2\right)^{-1/2}\right] + \text{Cyclic permutations,} \\ &= 0.\end{aligned}$$

Thus, both contributions to $\nabla \times \boldsymbol{F}$ are zero, so $\nabla \times \boldsymbol{F} = 0$, and gravity is conservative. See if you can show that

$$\mathcal{U} = -\frac{\alpha}{r}$$

is a suitable potential, $\boldsymbol{F} = -\nabla\left(-\alpha r^{-1}\right)$.

4. Show that the force

$$\boldsymbol{F} = \alpha(x^2\hat{\boldsymbol{x}} + y\hat{\boldsymbol{y}})$$

is a conservative force and construct its potential.

We have

$$\nabla \times \boldsymbol{F} = \alpha \begin{vmatrix} \hat{\boldsymbol{x}} & \hat{\boldsymbol{y}} & \hat{\boldsymbol{z}} \\ \partial_x & \partial_y & \partial_z \\ x^2 & y & 0 \end{vmatrix} = \alpha \hat{\boldsymbol{z}} \left(\partial_x y - \partial_y x^2 \right) = 0.$$

Next, we take

$$F_x = \alpha x^2 = -\partial_x \mathcal{U}.$$

Ordinary integration gives

$$\mathcal{U}(x, y) = -\tfrac{1}{3}\alpha x^3 + f(y),$$

where $f(y)$ is a function to be determined. But we also have

$$F_y = \alpha y = -\partial_y \mathcal{U},$$

which gives

$$\mathcal{U}(x, y) = -\tfrac{1}{2}\alpha y^2 + g(x).$$

Putting these results together, we have

$$\mathcal{U}(x, y) = -\alpha \left(\tfrac{1}{3}x^3 + \tfrac{1}{2}y^2 \right) + \text{Const.},$$

and the constant is immaterial because only *gradients* of the potential are important.

5. Recall that the vorticity $\boldsymbol{\omega}(\boldsymbol{x})$ measures the amount of swirl in a fluid velocity field $\boldsymbol{v}(\boldsymbol{x})$, $\boldsymbol{\omega} = \nabla \times \boldsymbol{v}$. Show that all irrotational flows

$$\boldsymbol{\omega} = 0,$$

are potential flows,

$$\boldsymbol{v} = \nabla \phi.$$

Show that the potential for an incompressible irrotational flow satisfies Laplace's equation:

$$\nabla \cdot \boldsymbol{v} = 0 \text{ and } \boldsymbol{\omega} = 0 \implies \nabla^2 \phi = 0.$$

The study of the equation $\nabla^2 \phi = 0$ is called **harmonic analysis**.

If the flow is irrotational, then $\nabla \times \boldsymbol{v} = 0$, which implies, by Stokes's theorem,

$$\boldsymbol{v} = \nabla \phi,$$