First: Recap the Tuesday online lecture.

Quadratic Model Problem (QMP):

$$f(\underline{x}) = c + \langle \underline{a}, \underline{x} \rangle + \frac{1}{2}\langle \underline{x}, B\underline{x} \rangle ,$$

where B is a symmetric positive-definite matrix.

Steepest Descent:

$$\underline{x}_{k+1} = \underline{x}_k + \alpha_k p_k$$
$$= \underline{x}_k - \alpha_k \nabla f_k$$

$\nabla f(\underline{x}_k)$

The α_k is obtained from the 1D subproblem:

$$\alpha_k = \arg\min_{\alpha > 0} f(\underline{x}_k - \alpha \nabla f_k)$$

Exact sl$^{\underline{n}}$ (QMP):

$$\alpha_k = \frac{\langle \nabla f_k , \nabla f_k \rangle}{\langle \nabla f_k , B \nabla f_k \rangle}$$

Minimizer (QMP):

$$B \underline{x}_* = -\underline{a} \quad \because \quad \nabla f(\underline{x}_*) = 0$$

$$\Rightarrow \underline{x}_* = -B^{-1}\underline{a}$$

B-norm:

$$\|\underline{v}\|_B^2 = \langle \underline{v}, B\underline{v} \rangle \quad \forall\ \underline{v} \in \mathbb{R}^n$$

Distance to the minimizer:

$$\|\underline{x}_k - \underline{x}_*\|_B$$

Key result in Tuesday lecture:

$$\boxed{\|\underline{x}_{k+1} - \underline{x}_*\|_B \leq \left(1 - \frac{\lambda_{min}}{\lambda_{max}}\right)^{1/2} \|\underline{x}_k - \underline{x}_*\|_B}$$

OR :

$$\|\underline{x}_{k+1} - \underline{x}_*\|_B \leq \left(1 - \frac{1}{\kappa(B)}\right)^{1/2} \|\underline{x}_k - \underline{x}_*\|_B$$

If $\kappa(B) >> 1$, then

$$1 - \frac{1}{\kappa(B)} \lesssim 1$$

Then,

$$\frac{\|\underline{x}_{k+1} - \underline{x}_*\|_B}{\|\underline{x}_k - \underline{x}_*\|_B} \lesssim 1 \Rightarrow \text{Slow convergence.}$$

Remark : When $\lambda_{min} = \lambda_{max}$, then

$$\|x_{k+1} - x_*\|_B \leq 0 \cdot \|x_k - x_*\|_B$$

Then, the steepest-descent method converges in one iteration.

When $\lambda_{min} = \lambda_{max}$, $B = \lambda \mathbb{I}$.

Theorem : When $B = \lambda \mathbb{I}$, the SD method converges in at most one step. $\square$

Moving on, we will extend this analysis to Newton and Quasi-Newton methods :

Theorem 6.3 : Suppose that f is twice differentiable and the Hessian Matrix $B(\underline{x})$ is Lipschitz in a neighbourhood of the minimiser x_* at which sufficient conditions for optimality hold ($\nabla f(x_*) = 0$, $B(x_*)$ is positive-definite). Suppose that the starting-point $\underline{x}_0$ is sufficiently close to the minimizer $\underline{x}_*$ and consider the iterates

$$\underline{x}_{k+1} = \underline{x}_k + \underline{p}_k^N .$$

Then:

1. The iterates converge to $\underline{x}_*$ as $k \to \infty$.
2. The rate of convergence is quadratic:

$$\| \underline{x}_{k+1} - \underline{x}_* \|_2 \leq C \| \underline{x}_k - \underline{x}_* \|_2^2 , \text{ some } C .$$

Remark: By $B(\underline{x})$ Lipschitz in a neighbourhood $\mathcal{N}$ of $\underline{x}_*$, we mean that there exists a positive constant K such that:

$$\| B(\underline{y}_2) - B(\underline{y}_1) \|_2 \leq K \| \underline{y}_2 - \underline{y}_1 \|_2$$

for all $\underline{y}_1$ and $\underline{y}_2$ in the neighbourhood $\mathcal{N}$.

Remark: As you go through the different steps in the proof, we get down to:

$$\frac{\| \underline{x}_1 - \underline{x}_* \|_2}{\| \underline{x}_0 - \underline{x}_* \|_2} \leq \underbrace{C \| \underline{x}_0 - \underline{x}_* \|_2}_{\leq 1/2} , \text{ for}$$

some constant C. "Sufficiently close" means the RHS should be $\leq 1/2$.

One last remark: A similar result holds for quasi-Newton methods such as BFGS:

$$\|\underline{x}_{k+1} - \underline{x}_*\|_2 \leq C \|\underline{x}_k - \underline{x}_*\|_2^{1+\epsilon}, \quad \epsilon > 0.$$

i.e. super-linear convergence.

Exercises #2

Question 1 Newton Method, with

given

$$\|\underline{x}_{k+1} - \underline{x}_*\|_2 \leq C \|\underline{x}_k - \underline{x}_*\|_2^2 \quad \text{some } C, \quad (1)$$

whenever $\|\underline{x}_k - \underline{x}_*\|_2 < \delta$.

Choose a starting-value $\underline{x}_0$ such that:

$$\|\underline{x}_0 - \underline{x}_*\|_2 < \delta \quad \text{AND} \quad \|\underline{x}_0 - \underline{x}_*\|_2 < \frac{1}{2C}.$$

i.e. $\|\underline{x}_0 - \underline{x}_*\|_2 < \min\left(\delta, \frac{1}{2C}\right)$.

Aim: We want to show:

$$\|\underline{x}_k - \underline{x}_*\|_2 \leq \frac{1}{2^{2^k - 1}} \|\underline{x}_0 - \underline{x}_*\|_2$$

a) Take Eq$^{\underline{n}}$ (1) with $k=0$:

$$\|\underline{x}_1 - \underline{x}_*\|_2 \leq C \underbrace{\|\underline{x}_0 - \underline{x}_*\|_2}_{< \frac{1}{2C}} \|\underline{x}_0 - \underline{x}_*\|_2$$

$$< C \cdot \frac{1}{2C} \|\underline{x}_0 - \underline{x}_*\|_2$$

$$\Rightarrow \|\underline{x}_1 - \underline{x}_*\|_2 < \frac{1}{2} \|\underline{x}_0 - \underline{x}_*\|_2 < \frac{1}{2} \min\left(\delta, \frac{1}{2C}\right)$$

Apply Eq$^{\underline{n}}$ (1) with $k = 1$:

$$\|\underline{x}_2 - \underline{x}_*\|_2 \leq C \underbrace{\|\underline{x}_1 - \underline{x}_*\|_2} \|\underline{x}_1 - \underline{x}_*\|_2$$

$$< \frac{1}{2} \underbrace{C\|\underline{x}_0 - \underline{x}_*\|_2}_{< \frac{1}{2C}} \cdot \frac{1}{2} \|\underline{x}_0 - \underline{x}_*\|_2$$

$$< \frac{1}{2^3} \|\underline{x}_0 - \underline{x}_*\|_2$$

Apply Eq$^{\underline{n}}$ (1) with $k = 2$:

$$\|\underline{x}_3 - \underline{x}_*\|_2 \leq C \underbrace{\|\underline{x}_2 - \underline{x}_*\|_2} \|\underline{x}_2 - \underline{x}_*\|_2$$

$$< \frac{1}{2^3} \underbrace{C\|\underline{x}_0 - \underline{x}_*\|_2}_{< C \cdot \frac{1}{2C}} \frac{1}{2^3} \|\underline{x}_0 - \underline{x}_*\|_2$$

$$< \frac{1}{2^3} \cdot \frac{1}{2^3} \cdot \frac{1}{2} \|\underline{x}_0 - \underline{x}_*\|_2$$

$$= \frac{1}{2^7} \|\underline{x}_0 - \underline{x}_*\|_2$$

Part b): $\|\underline{x}_k - \underline{x}_0\|_2 \leq \frac{1}{2^{p_k}} \|\underline{x}_0 - \underline{x}_*\|_2$

$k = 1$, $p_k = 1$

$k = 2$, $p_k = 3$

$k = 3$, $p_k = 7$

Guess:
$$\begin{cases} p_k = 2p_{k-1} + 1, & k = 2, 3, \ldots \\ p_k = 1 & k = 1 \end{cases} \qquad (2)$$

Part c)

$$p_k = B \cdot \lambda^k + A$$

Sub in to (2) :

$$B \cdot \lambda^k + A = 2(B\lambda^{k-1} + A) + 1$$

$$\Rightarrow \quad B \cdot \lambda^k + A = 2B\lambda^{k-1} + 2A + 1$$

Equate: $\begin{cases} \cancel{B}\lambda^k = 2\cancel{B}\lambda^{k-1} \Rightarrow 1 = 2 \cdot \lambda^{-1} \Rightarrow \lambda = 2 \\ A = 2A + 1 \Rightarrow 0 = A + 1 \Rightarrow A = -1 \end{cases}$

$$p_k = B \cdot 2^k - 1$$

But $p_1 = 1$

$$\Rightarrow \quad B \cdot 2 - 1 = 1$$

$$\Rightarrow \quad B = 1.$$

Hence: $\quad p_k = 2^k - 1$

d) $$\|\underline{x}_k - \underline{x}_*\|_2 \leq \frac{1}{2^{p_k}} \|\underline{x}_0 - \underline{x}_*\|_2$$

$$= \frac{1}{2^{2^k - 1}} \|\underline{x}_0 - \underline{x}_*\|_2$$

Hence:

$$\lim_{k \to \infty} \|\underline{x}_k - \underline{x}_*\|_2 = 0 \qquad \blacksquare$$

<u>Question 2</u> : Show that if $0 < c_2 < c_1 < 1$

then there may be no step lengths satisfying then there may be no step lengths satisfying the SWCs.

Hint : $\phi(\alpha) = a + b\alpha + c\alpha^2$

where $b < 0$, $c > 0$

<u>Solution</u> : $\phi(0) = a$, $\phi'(0) = b$, $b<0$.

SWC1 : $\phi(\alpha) \le \phi(0) + c_1 \alpha \phi'(0)$.

$\Rightarrow \cancel{a} + b\alpha + c\alpha^2 \le \cancel{a} + c_1 \alpha b$

Solve for α: $\alpha = 0$, OR

$$b + c\alpha \le c_1 b$$

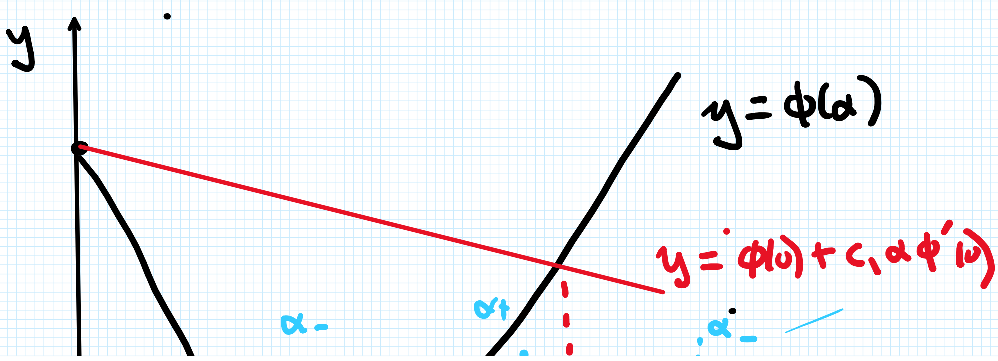
$$\Rightarrow c\alpha \le -b + c_1 b$$

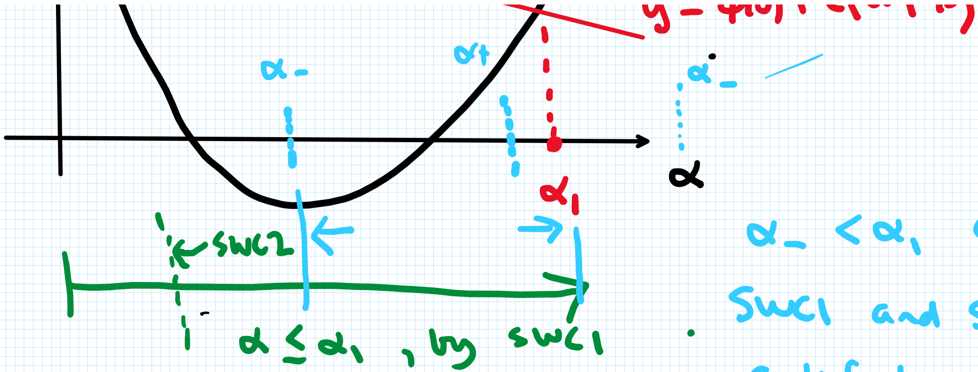
$$\Rightarrow c\alpha \le -b(1-c_1)$$

$$\Rightarrow c\alpha \le |b|(1-c_1)$$

$$\Rightarrow \alpha \le \frac{|b|\overbrace{(1-c_1)}^{\text{pos}}}{c}$$

Take: $\alpha_1 = \dfrac{|b|(1-c_1)}{c}$





$\alpha_- < \alpha_1$ OK

SWC1 and SWC2 satisfied simultaneously.

$\alpha_- > \alpha_1$ BAD

SWC1 and SW2 not satisfied simultan.....

SWC2 : We require

$$|\phi'(\alpha)| \leq c_2 |\phi'(0)|$$

$$\Rightarrow |b + 2c\alpha| \leq c_2 |b|$$

$$b^2 + 4bc\alpha + 4c^2\alpha^2 \leq c_2^2 b^2$$

$$\Rightarrow 4c^2\alpha^2 + 4bc\alpha + b^2(1-c_2^2) \leq 0.$$

Critical points, where equality holds :

$$4c^2\alpha^2 + 4bc\alpha + b^2(1-c_2^2) = 0 .$$

$$\alpha = \frac{-4bc \pm \sqrt{16b^2c^2 - 16b^2c^2(1-c_2^2)}}{8c^2} \qquad \sqrt{b^2} = |b|$$

$$= \frac{-4bc \pm 4|b|c\,c_2}{8c^2}$$

$$= \frac{4|b|\cancel{c}\cdot(1 \pm c_2)}{8\cdot c\cdot \cancel{c}}$$

$$\Rightarrow \alpha_{\pm} = \frac{|b|}{2c}(1 \pm c_2)$$

Referring to the figure, we attempt to make

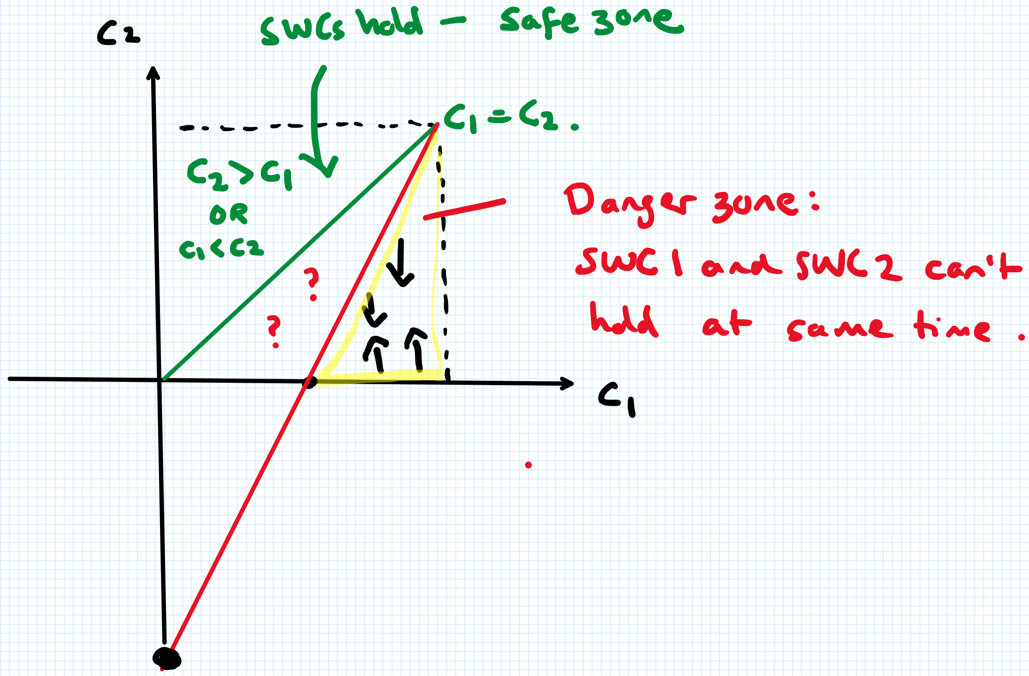
$\alpha_- > \alpha_1$.

$$\frac{|b|}{2\cancel{c}}(1-c_2) > \frac{|\cancel{b}|(1-c_1)}{\cancel{c}}$$

$$\Rightarrow \quad 1-c_2 > 2-2c_1$$

$$\Rightarrow \quad \boxed{2c_1 > c_2+1} \qquad \text{BAD.}$$

Plot in c_1–c_2 space:



Danger zone: $\quad 2c_1 > c_2+1$

$\qquad 2x > y+1$

Margin: $\quad 2x = y+1$

$$\left\{\begin{array}{l} x=0,\ y=-1. \\ y=0,\ x=1/2 \\ x=1,\ y=1 \end{array}\right.$$

Hence, to stay out of danger, we should take

$$\boxed{c_1 < c_2}$$

On the other hand, if $c_1 > c_2$, we cannot rule out being in the danger zone in the figure, where SWCs 1 and 2 can't hold simultaneously. ∎

We look at another question, which will help us to understand Q5.

Q4, ACM40990 Midterm Exam, 2025.

Consider the Steepest-Descent method with exact line search applied to the quadratic cost function

$$f(\boldsymbol{x}) = c + \langle \boldsymbol{a}, \boldsymbol{x} \rangle + \tfrac{1}{2}\langle \boldsymbol{x}, B\boldsymbol{x} \rangle, \qquad (2)$$

QMP

where $c \in \mathbb{R}$, $\boldsymbol{a} \in \mathbb{R}^n$ and $B \in \mathbb{R}^{n\times n}$ are constant. Take as given that B is symmetric positive-definite.

Question 4(b)

(b) Consider again the cost function in Equation (2). Take $n = 2$ and

$$B = \begin{pmatrix} 2\times 10^8 & 10^4 \\ 10^4 & 1 \end{pmatrix}.$$

Comment on the convergence of the Steepest-Descent algorithm applied to this problem. **[5 marks]**

Solution: We calculate the condition number of B.

Since B is symmetric,

$$\kappa(B) = \|B\|_2 \|B^{-1}\|_2 = \frac{\max_i |\lambda_i|}{\min_i |\lambda_i|}$$

So we compute the eigenvalues of B:

$$\begin{vmatrix} 2\times10^8-\lambda & 10^4 \\ 10^4 & 1-\lambda \end{vmatrix} = 0$$

$$(2\times10^8-\lambda)(1-\lambda) - 10^8 = 0$$

Let $\rho = 10^8$.

$$(1-\lambda)(2\rho-\lambda) - \rho = 0$$

$$\Rightarrow (\lambda-2\rho)(\lambda-1) - \rho = 0$$

$$\Rightarrow \lambda^2 - \lambda - 2\rho\lambda + \rho = 0.$$

$$\Rightarrow \lambda^2 - \lambda(1+2\rho) + \rho = 0.$$

$$\Rightarrow \lambda = \frac{(1+2\rho) \pm \sqrt{(1+2\rho)^2 - 4\rho}}{2}$$

$$\Rightarrow \lambda = \frac{(1+2\rho) \pm \sqrt{1+\cancel{4\rho}+4\rho^2-\cancel{4\rho}}}{2}$$

Since $\rho = 10^8$ is large:

$$\lambda = \begin{cases} \dfrac{(1+2\rho)+\sqrt{1+4\rho^2}}{2} \simeq \dfrac{1+2\rho+2\rho}{2} \simeq 2\rho \\[2ex] \dfrac{1+2\rho-\sqrt{1+4\rho^2}}{2} \simeq \dfrac{1+\cancel{2\rho}-\cancel{2\rho}}{2} = 1/2 \end{cases}$$

Or, more accurately,

$$\lambda = \begin{cases} 2\rho + O(\rho^0) \\ \frac{1}{2} + O(\rho^{-1}) \end{cases}$$

$$\left\{ \quad \frac{1}{2} + O(\rho^{-1}) \right)$$

$$\max_{i=1,2} |\lambda_i| = 2\rho + O(\rho^0)$$

$$\min_{i=1,2} |\lambda_i| = 1/2 + O(\rho^{-1})$$

$$\kappa(B) = \frac{2\rho + O(\rho^0)}{1/2 + O(\rho^{-1})}$$

$$= 4\rho + O(\rho^0).$$

But $\rho = 10^8$.

$$\Rightarrow \kappa(B) = O(10^8).$$

Remark: It suffices to estimate the order of magnitude of the condition number.

$$\|\underline{x}_{k+1} - \underline{x}_*\|_B \leq \underbrace{\left(1 - \frac{1}{\kappa(B)}\right)}_{\text{prefactor}}^{1/2} \|\underline{x}_k - \underline{x}_*\|_B$$

Prefactor close to 1, convergence of SD algorithm is very slow.

<u>Question 5</u> : Coding question

- SD is very slow

– Newton is fast.

Check $\kappa(B)$, $\kappa(B) \gg 1$, so the SD is slow.

One last remark about condition numbers:

For a symmetric matrix,

$$\kappa(B) = \frac{\max_i |\lambda_i|}{\min_i |\lambda_i|}$$

If the denominator is small, then $\kappa(B)$ is large. Then one of the eigenvalues is close to zero. Then, there exists a perturbation of small size:

$$B + \overbrace{\epsilon}^{\text{small}} B_1$$

such that $B + \epsilon B_1$ is a singular matrix

If $\epsilon = O(10^{-16})$ then the matrix is effectively singular, in double precision.

<u>Moral</u>: Be very careful about inverting (numerically) a matrix with a large condition number.