Optimization Algorithms (ACM 41030)

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Exercises #6

6. Formulate the dual problem for the following OPs:

   (a) Minimize:
   $$\min_{\boldsymbol{x}\in\mathbb{R}^n} \langle \boldsymbol{c}, \boldsymbol{x}\rangle, \text{ subject to } A\boldsymbol{x} - \boldsymbol{b} \geq 0.$$
   Here, $\boldsymbol{c} \in \mathbb{R}^n$ is a constant vector, $\boldsymbol{b} \in \mathbb{R}^m$ is a constant vector, and $A \in \mathbb{R}^{m\times n}$ is a contant matrix.

   (b) Minimize:
   $$\min_{\boldsymbol{x}\in\mathbb{R}^n} \tfrac{1}{2}\langle \boldsymbol{x}, G\boldsymbol{x}\rangle, \text{ subject to } A\boldsymbol{x} - \boldsymbol{b} \geq 0.$$
   Here, A and b are as before, and $G \in \mathbb{R}^{n\times n}$ is a constant symmetric positive-definite matrix.

6a) Linear cost function $f(\underline{x}) = \langle \underline{c}, \underline{x}\rangle$

m-dimensional

$$\mathcal{L}(\underline{x}, \underline{\lambda}) = \langle \underline{c}, \underline{x}\rangle - \sum_{i=1}^{m} \lambda_i \,(A\underline{x} - \underline{b})_i$$

$$= \langle \underline{c}, \underline{x}\rangle_{\mathbb{R}^n} - \langle \underline{\lambda}, A\underline{x} - \underline{b}\rangle_{\mathbb{R}^m}$$

$$= \langle \underline{c}, \underline{x}\rangle_{\mathbb{R}^n} - \langle A^T\underline{\lambda}, \underline{x}\rangle_{\mathbb{R}^n} + \langle \underline{\lambda}, \underline{b}\rangle_{\mathbb{R}^m}$$

Step 1: $\quad q(\underline{\lambda}) = \inf_{\underline{x}\in\mathbb{R}^n} \mathcal{L}(\underline{x}, \underline{\lambda})$

So we minimize $\mathcal{L}(\underline{x}, \underline{\lambda})$ w.r.t. $\underline{x}$:

$$\nabla_x \mathcal{L}(\underline{x}, \underline{\lambda}) = 0 .$$

$$\nabla_x \mathcal{L} = \underline{c} - A^T\underline{\lambda} + 0$$

Set $\nabla_x \mathcal{L} = 0$ to obtain $q(\underline{\lambda})$. I.e.

$$\underline{c} = A^T\underline{\lambda} .$$

Sub back into $\mathcal{L}(\underline{x}, \underline{\lambda})$:

$$\mathcal{L}(x, \lambda)\Big| = \Big[\langle c, x\rangle - \langle \lambda, Ax\rangle + \langle \underline{\lambda}, \underline{b}\rangle\Big]$$

$$\mathcal{L}(\underline{x},\underline{\lambda})\Big|_{\underline{c}=A^T\underline{\lambda}} = \Big[\langle \underline{c},\underline{x}\rangle - \langle \underline{\lambda}, A\underline{x}\rangle + \langle \underline{\lambda},\underline{b}\rangle\Big]_{\underline{c}=A^T\underline{\lambda}}$$

$$= \cancel{\langle A^T\underline{\lambda},\underline{x}\rangle} - \cancel{\langle A^T\underline{\lambda},\underline{x}\rangle} + \langle \underline{\lambda},\underline{b}\rangle$$

$$= \langle \underline{\lambda},\underline{b}\rangle$$

$$= \langle \underline{\lambda},\underline{b}\rangle_{\mathbb{R}^m}$$

$$\therefore \quad q(\underline{\lambda}) = \langle \underline{\lambda},\underline{b}\rangle$$

Dual problem:

$$\boxed{\max_{\underline{\lambda}\in\mathbb{R}^m} \langle \underline{\lambda},\underline{b}\rangle \quad \text{subject to: } \begin{cases} \underline{\lambda} \geq 0 \\ A^T\underline{\lambda} = \underline{c} \end{cases}}$$

6b) Quadratic primal problem:

$$f(\underline{x}) = \frac{1}{2}\langle \underline{x}, G\underline{x}\rangle$$

Constraint: $A\underline{x} - \underline{b} \geq 0$, m constraints

Given: $G \in \mathbb{R}^{n\times n}$ is symmetric P.D.

Dual formulation:

$$\mathcal{L}(\underline{x},\underline{\lambda}) = \frac{1}{2}\langle \underline{x}, G\underline{x}\rangle_{\mathbb{R}^n} - \langle \underline{\lambda}, A\underline{x}-\underline{b}\rangle_{\mathbb{R}^m}$$

$$= \frac{1}{2}\langle \underline{x}, G\underline{x}\rangle - \langle A^T\underline{\lambda},\underline{x}\rangle_{\mathbb{R}^n} + \langle \underline{\lambda},\underline{b}\rangle_{\mathbb{R}^m}$$

$$= \frac{1}{2}\langle \underline{x}, G\underline{x}\rangle - \langle A^T\underline{\lambda}, \underline{x}\rangle_{\mathbb{R}^n} + \langle \underline{\lambda}, \underline{b}\rangle_{\mathbb{R}^m}$$

<u>Minimize over $\underline{x}$:</u>

$$\nabla_x \mathcal{L} = G\underline{x} - A^T\underline{\lambda}$$

$$\nabla_x \mathcal{L} = 0 \Rightarrow G\underline{x} = A^T\underline{\lambda} \Rightarrow \boxed{\underline{x} = G^{-1}A^T\underline{\lambda}}$$

$$\mathcal{L}(\underline{x} = G^{-1}A^T\underline{\lambda}, \underline{\lambda}) = \left[\frac{1}{2}\langle \underline{x}, G\underline{x}\rangle - \langle \underline{\lambda}, A\underline{x}\rangle + \langle \underline{\lambda}, \underline{b}\rangle\right]_{\underline{x} = G^{-1}A^T\underline{\lambda}}$$

$$= \frac{1}{2}\langle G^{-1}A^T\underline{\lambda}, \cancel{G}\cancel{G^{-1}}A^T\underline{\lambda}\rangle - \langle \underline{\lambda}, A\, G^{-1}A^T\underline{\lambda}\rangle + \langle \underline{\lambda}, \underline{b}\rangle$$

$$= \frac{1}{2}\langle A^T\underline{\lambda}, G^{-1}A^T\underline{\lambda}\rangle - \langle A^T\underline{\lambda}, G^{-1}A^T\underline{\lambda}\rangle + \langle \underline{\lambda}, \underline{b}\rangle$$

$$= -\frac{1}{2}\langle A^T\underline{\lambda}, \underset{\sim}{G^{-1}} A^T\underline{\lambda}\rangle_{\mathbb{R}^n} + \langle \underline{\lambda}, \underline{b}\rangle_{\mathbb{R}^m}$$

$$= q(\underline{\lambda})$$

<u>Dual problem :</u>

$$\boxed{\begin{array}{l} \text{Maximize} \\ \quad q(\underline{\lambda}) = -\frac{1}{2}\langle A^T\underline{\lambda}, G^{-1}A^T\underline{\lambda}\rangle + \langle \underline{\lambda}, \underline{b}\rangle \\ \text{Subject to} \\ \quad \underline{\lambda} \geq 0 \end{array}}$$

Back to Q.3

3. Consider the half-space defined by:

$$H_\alpha = \{\boldsymbol{x} \in \mathbb{R}^n | \boldsymbol{a} \cdot \boldsymbol{x} + \alpha \geq 0\},$$

where $\boldsymbol{a} \in \mathbb{R}^n$ is a constant non-zero vector and $\alpha \in \mathbb{R}$ is a constant scalar. Formulate and solve the OP for finding the point $\boldsymbol{x} \in H_\alpha$ with the smallest Euclidean norm.

$$f(\underline{x}) = \frac{1}{2}\langle \underline{x}, \underline{x} \rangle \equiv \frac{1}{2}\underline{x}\cdot\underline{x}$$

Single constraint: $\underline{a}\cdot\underline{x} + \alpha \geq 0$.

$$\mathcal{L}(\underline{x}, \lambda) = \frac{1}{2}\underline{x}\cdot\underline{x} - \lambda(\underline{a}\cdot\underline{x} + \alpha)$$

Step 1: Minimize wrt $\underline{x}$:

$$\nabla_x \mathcal{L} = \underline{x} - \lambda\underline{a}$$

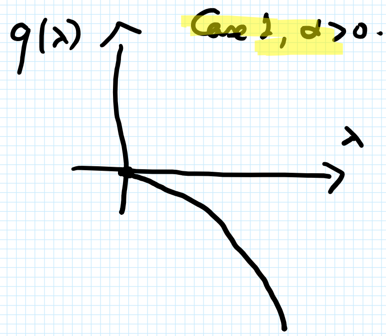
$$\nabla_x \mathcal{L} = 0 \Rightarrow \underline{x} = \lambda\underline{a}.$$

$$\mathcal{L}(\underline{x} = \lambda\underline{a}, \lambda) = \frac{1}{2}(\lambda\underline{a})\cdot(\lambda\underline{a}) - \lambda\underline{a}\cdot(\lambda\underline{a}) - \lambda\alpha$$

$$= \frac{1}{2}\lambda^2\underline{a}\cdot\underline{a} - \lambda^2\underline{a}\cdot\underline{a} - \lambda\alpha$$

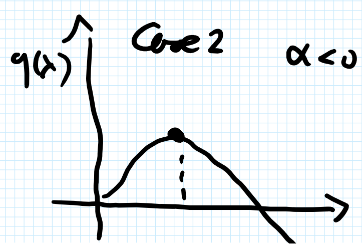
$$= -\frac{1}{2}\lambda^2\underline{a}\cdot\underline{a} - \lambda\alpha$$

$$= q(\lambda)$$



Dual problem:

$$\boxed{\begin{aligned} &\text{Minimize} \quad q(\lambda) = -\frac{1}{2}\lambda^2 \underbrace{\underline{a}\cdot\underline{a}}_{\|\underline{a}\|_2^2} - \lambda\alpha \\ &\text{Subject to} \quad \lambda \geq 0. \end{aligned}}$$



Solution : $\frac{dg}{d\lambda} = -\lambda \|\underline{a}\|_2^2 - \alpha$

Attempt : $\frac{dg}{d\lambda} = 0 \Rightarrow -\lambda \|\underline{a}\|_2^2 - \alpha = 0$

$\Rightarrow \lambda \|\underline{a}\|_2^2 = -\alpha \ , \quad \lambda \geq 0.$

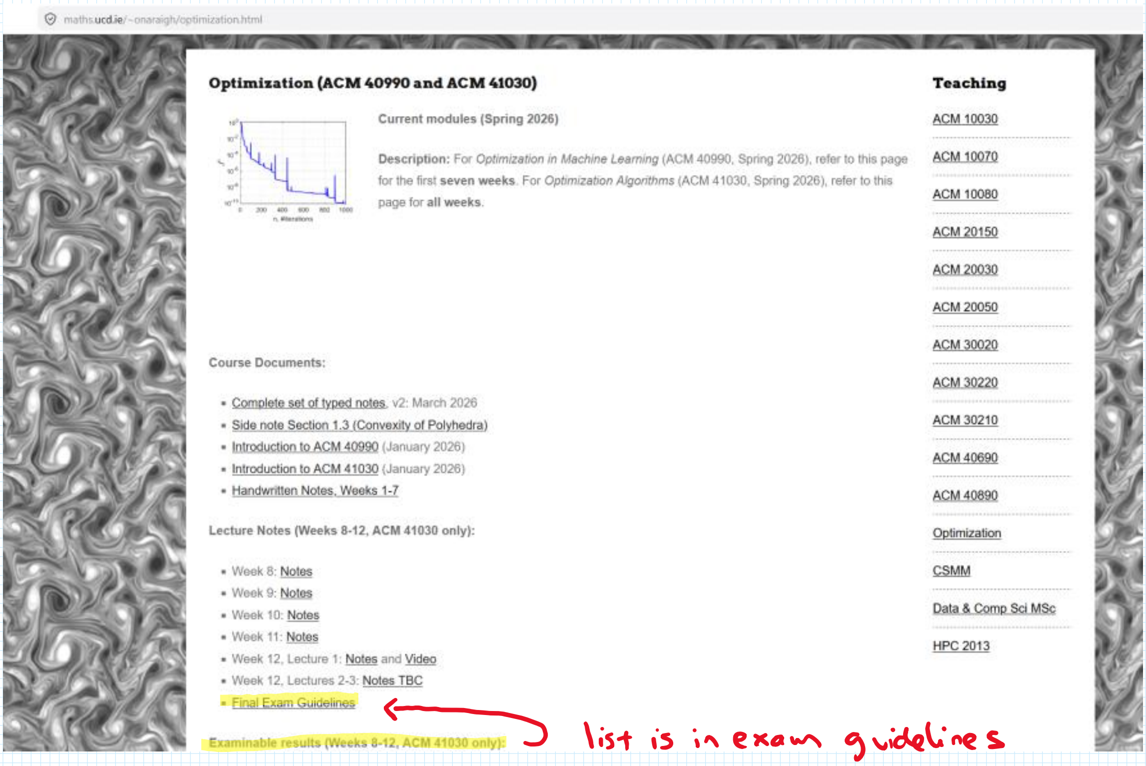
Case 2 : $\alpha < 0, \quad \lambda = -\frac{\alpha}{\|\underline{a}\|_2^2}.$

Back to $\underline{x} = \lambda \underline{a} \Rightarrow \boxed{\underline{x} = -\frac{\alpha\, \underline{a}}{\|\underline{a}\|_2^2}}.$

Case 1 : $\alpha > 0$ $\quad g(\lambda)$ is maximized at $\lambda = 0$

$\Rightarrow \boxed{\underline{x} = \underline{0}}$

STRUCTURE OF FINAL EXAM



maths.ucd.ie/~onaraigh/optimization.html

Optimization (ACM 40990 and ACM 41030)

Current modules (Spring 2026)

Description: For *Optimization in Machine Learning* (ACM 40990, Spring 2026), refer to this page for the first **seven weeks**. For *Optimization Algorithms* (ACM 41030, Spring 2026), refer to this page for **all weeks**.

Course Documents:

- Complete set of typed notes, v2: March 2026
- Side note Section 1.3 (Convexity of Polyhedra)
- Introduction to ACM 40990 (January 2026)
- Introduction to ACM 41030 (January 2026)
- Handwritten Notes, Weeks 1-7

Lecture Notes (Weeks 8-12, ACM 41030 only):

- Week 8: Notes
- Week 9: Notes
- Week 10: Notes
- Week 11: Notes
- Week 12, Lecture 1: Notes and Video
- Week 12, Lectures 2-3: Notes TBC
- Final Exam Guidelines

Examinable results (Weeks 8-12, ACM 41030 only):

Teaching

- ACM 10030
- ACM 10070
- ACM 10080
- ACM 20150
- ACM 20030
- ACM 20050
- ACM 30020
- ACM 30220
- ACM 30210
- ACM 40690
- ACM 40890
- Optimization
- CSMM
- Data & Comp Sci MSc
- HPC 2013

list is in exam guidelines

Optimization Algorithms (ACM 41030)

Second Written Exam

23/04/2026

The second written exam is worth 50% of the module grade. This will take place as a full end-of-trimester exam in the RDS. Note the duration: the exam will last **60 minutes**.

The exam will contain 4 questions, For maximum marks, all 4 questions must be answered. The exam format is **closed book**. Non-programmable calculators are permitted.

The questions will involve a mixture of theory and calculations / exercises. The following theory is examinable:

- The projection operator – Section 12.4
- First-order optimality conditions in case of a single inequality constraint – Section 13.2
- Showing that if the LICQs are satisfied, then the Lagrange Multipliers are unique - Section 15.1, Theorem 15.2
- Showing that the tangant cone is inside the set of LFDDs; showing that the tangant cone and the LFDDs are the same when the LICQs are satisfied – Section 15.2, Theorem 15.3.

  NOTE: you can assume without proof that the matrix $\begin{pmatrix} A \\ Z^T \end{pmatrix}$ has full row rank.

- Assuming Farkas's Lemma, prove the necessary conditions (KKT conditions) for a feasible point $\boldsymbol{x}_*$ to be a minimizer - Section 16.4

The calculations / exercises will come exclusively from Exercises 5 and 6:

- Everything on Exercises 5 is examinable.
- Everything on Exercises 6 is examinable except for:
  - Question 5, which requires a computer for the solution.