Optimization Algorithms (ACM 41030)

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Exercises #6

3. Consider the half-space defined by:

Question 4

$$H_\alpha = \{\boldsymbol{x} \in \mathbb{R}^n | \boldsymbol{a} \cdot \boldsymbol{x} + \alpha \geq 0\},$$

where $\boldsymbol{a} \in \mathbb{R}^n$ is a constant non-zero vector and $\alpha \in \mathbb{R}$ is a constant scalar. Formulate and solve the OP for finding the point $\boldsymbol{x} \in H_\alpha$ with the smallest Euclidean norm.

$H_\alpha = \{ \underline{x} \in \mathbb{R}^n \mid \underline{a} \cdot \underline{x} + \alpha \geq 0 \}$, where:

- $\underline{a}$ is a constant vector,
- α is a constant scalar.

Equality: $\underline{a} \cdot \underline{x} + \alpha = 0 \Rightarrow$ Hyperplane in $\mathbb{R}^n$.

H_α: Everything to the right of the hyperplane.

Find the vector $\underline{x}$ in the half-space with the smallest Euclidean norm.

OP: $\min_{\underline{x} \in \mathbb{R}^n} \frac{1}{2} \|\underline{x}\|_2^2$ subject to: $c_1(\underline{x}) \geq 0$

where $c_1(\underline{x}) = \underline{a} \cdot \underline{x} + \alpha$

OR

$$\min_{\underline{x}\in\mathbb{R}^n} f(\underline{x}) \quad , \quad f(\underline{x}) = \frac{1}{2}\sum_{i=1}^{n} x_i^2$$

subject to: $\sum_{i=1}^{n} a_i x_i + \alpha \geq 0$.

Solve

KKT conditions.

$$\mathcal{L}(\underline{x}, \lambda) = \frac{1}{2}\sum_{i=1}^{n} x_i^2 - \lambda\left(\sum_{i=1}^{n} a_i x_i + \alpha\right)$$

KKT 1: $\nabla_x \mathcal{L} = 0$ at optimality.

$$\nabla_x \mathcal{L} = x_i - \lambda a_i$$

KKT 1 $\Rightarrow$ $x_i = \lambda a_i$, $i = 1, \dots, n$.

KKT 2: no equality constraints

KKT 3: $\sum_{i=1}^{n} a_i x_i + \alpha \geq 0$

KKT 4: $\lambda \geq 0$

KKT 5: $\lambda\left(\sum_{i=1}^{n} a_i x_i + \alpha\right) = 0$

KKT 1 $\Rightarrow$ $x_i = \lambda a_i$

KKT 3 $\Rightarrow$ $\lambda \sum_i a_i^2 + \alpha \geq 0$

KKT 4 $\Rightarrow$ $\lambda \geq 0$

KKT 5 $\Rightarrow$ $\lambda\left(\lambda \sum_i a_i^2 + \alpha\right) = 0$.

$$\lambda \Big(\underbrace{\lambda \sum_i a_i^2 + \alpha}_{} \Big) = 0.$$

Solutions:

$\lambda = 0$ is a solution, valid when $\alpha > 0$.

Or

$$\lambda \sum_i a_i^2 + \alpha = 0 \Rightarrow \lambda = -\frac{\alpha}{\sum_i a_i^2}, \quad \alpha < 0.$$

Case 1 : $\lambda = 0, \quad \alpha > 0$

Case 2 : $\lambda = -\dfrac{\alpha}{\sum_i a_i^2}, \quad \alpha < 0.$

Back to KKT 1 : $x_i = \lambda a_i$

Case 1 : $\lambda = 0, \alpha > 0, \quad x_i = 0 \Rightarrow \boxed{\underline{x}_* = 0}$

Case 2 : $\lambda = -\dfrac{\alpha}{\sum_i a_i^2}, \quad x_i = \lambda a_i \Rightarrow \boxed{\underline{x}_* = -\dfrac{\alpha \underset{\sim}{a}}{\sum_i a_i^2}}$

This is the required solution. ∎

Interpretation in $\mathbb{R}^2$.

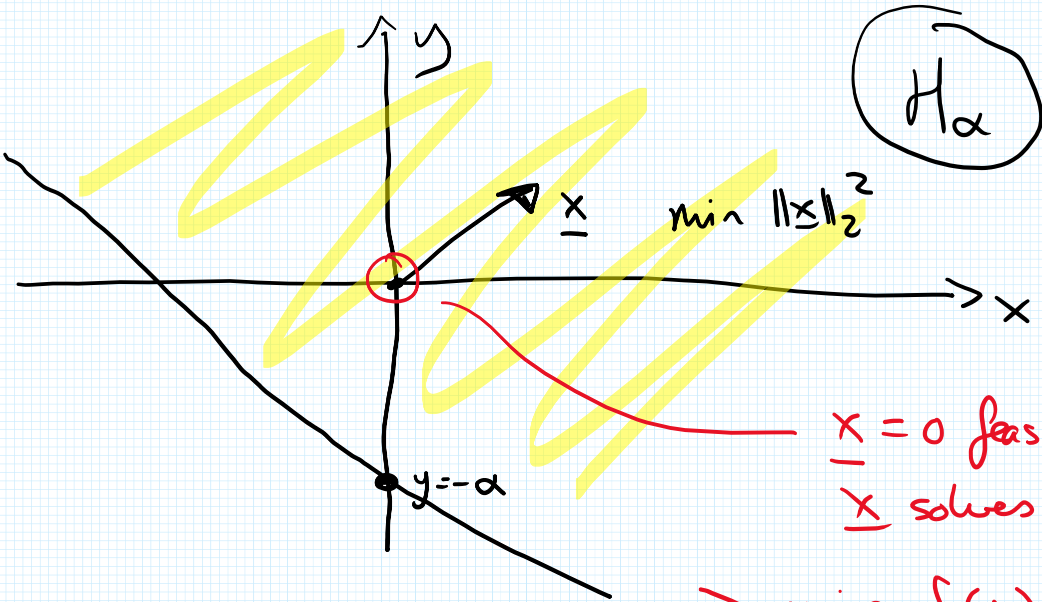
$\underline{a} \cdot \underline{x} + \alpha \geq 0$, take $\underset{\sim}{a} = (a_1, 1)$.

$x a_1 + y + \alpha \geq 0.$

$$x a_1 + y + \alpha \geq 0.$$

$$\Rightarrow \boxed{y = -a_1 x - \alpha} \quad (*) \qquad \text{Equality}$$

Case 1: $\alpha > 0 \Rightarrow$ y-intercept is at $y = -\alpha$ (neg).

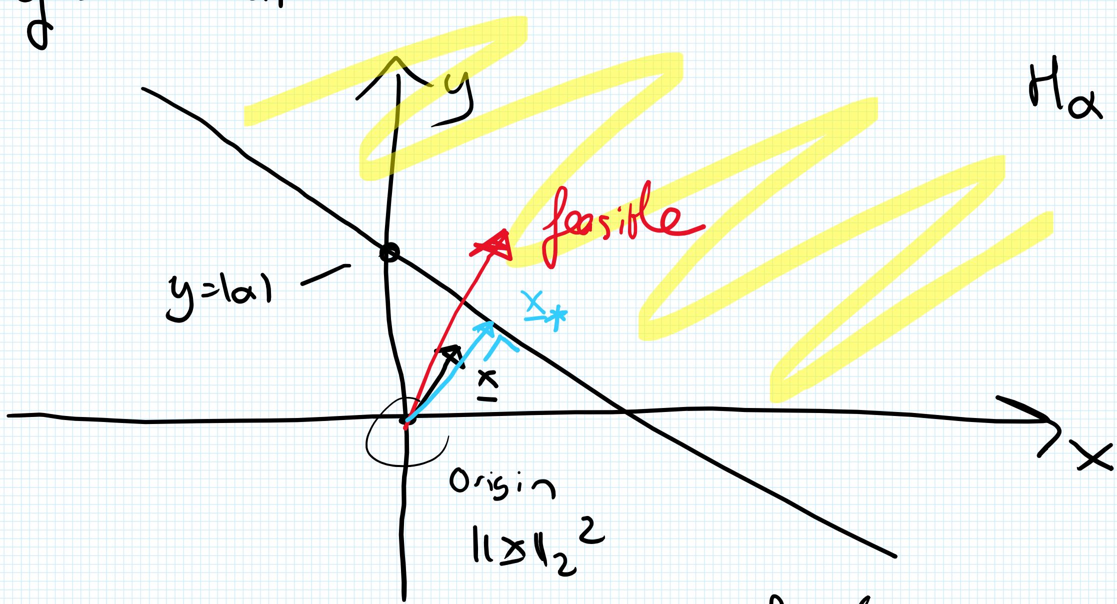


$\underline{x} = 0$ feasible,
$\underline{x}$ solves the OP

$\Rightarrow \min f(\underline{x}) = 0$

<u>Case 2</u> $\quad \alpha < 0.$

$$y = -a_1 x - \alpha = -a_1 x + |\alpha|$$



$\|\underline{x}\|_2^2$

$\underline{x} = 0$ not feasible.

Optimum vector $\underline{x}_*$ lies on the line forming the half-

Optimum vector $\underline{x}_*$ lies on the line forming the half-space boundary, and is orthogonal to same.

$$\Rightarrow \underline{x}_* = -\frac{\alpha\,\underline{a}}{\sum_i a_i^2}, \text{ same as KKT.}$$

Question 4.

4. Consider the following modification of the example in class notes. Here, t is a parameter that is fixed prior to solving the problem:

$$\min_{x\in\mathbb{R}^2} f(x),$$

where

$$f(x) = \left(x - \tfrac{3}{2}\right)^2 + (y-t)^4,$$

t is fixed for the problem.

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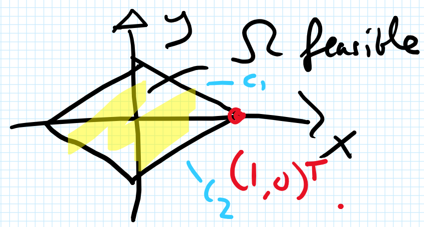
subject to:

$$\begin{bmatrix} 1-x-y \\ 1-x+y \\ 1+x-y \\ 1+x+y \end{bmatrix} \geq 0.$$

(a) For what values of t does the point $x_* = (1,0)^T$ satisfy the KKT conditions?

(b) Show that when $t=1$, only the first constraint is active at the solution and find the solution.



t

a) Sln:

$$\mathcal{L}(\underline{x},\lambda_1,\lambda_2) = \left(x-\tfrac{3}{2}\right)^2 + (y-t)^4 - \lambda_1(1-x-y) - \lambda_2(1-x+y)$$

KKT 1: $\dfrac{\partial \mathcal{L}}{\partial x} = 2\left(x-\tfrac{3}{2}\right) + \lambda_1 + \lambda_2$

$\dfrac{\partial \mathcal{L}}{\partial y} = 4(y-t)^3 + \lambda_1 - \lambda_2$

$$\frac{\partial \mathcal{L}}{\partial y} = 4(y-t)^3 + \lambda_1 - \lambda_2$$

$\underline{x}_* = (1,0)^T$. Look at KKT1 evaluated at $\underline{x}_*$.

$$\left.\frac{\partial \mathcal{L}}{\partial x}\right|_{\underline{x}_*} = 0 \implies 2\underbrace{\left(1-\frac{3}{2}\right)}_{-1/2} + \lambda_1 + \lambda_2 = 0.$$

$$\implies \lambda_1 + \lambda_2 = 1$$

$$\left.\frac{\partial \mathcal{L}}{\partial y}\right|_{\underline{x}_*} = 0 \implies 4(-t)^3 + \lambda_1 - \lambda_2 = 0$$

$$\implies \lambda_1 - \lambda_2 = -4\,(-t)^3.$$

Gather up:

$$\lambda_1 + \lambda_2 = 1$$
$$\lambda_1 - \lambda_2 = -4\,(-t)^3.$$

Add: $\quad 2\lambda_1 = 1 - 4(-t)^3.$

$\lambda_1 \geq 0$. If $t > 0$, this is ok.

If $t < 0$, ...

Take $t = -|t|$.

$$2\lambda_1 = 1 - 4|t|^3 \geq 0$$
$$\implies |t| \leq \frac{1}{4^{1/3}} \implies t \geq -\frac{1}{4^{1/3}}.$$

$$\lambda_2 = 1 - \lambda_1$$
$$= 1 - \left[\frac{1}{2} - \frac{1}{2}\cdot 4(-t)^3\right]$$
$$= \frac{1}{2} + 2(-t)^3.$$

$$= \frac{1}{2} + 2(-t)^3 .$$

Require $\lambda_2 \geq 0$. Same reasoning as before:

$$t \leq \frac{1}{4^{1/3}} .$$

Putting the two limits together, we get:

$$\boxed{-\frac{1}{4^{1/3}} \leq t \leq \frac{1}{4^{1/3}}} .$$

b) When $t=1$, only the first constraint is active. <u>Find the sl^n of the OP</u>.

Elementary method. Given: only the first constraint is active. From class notes / geometric reasoning, we know that the sl^n is on the constraint boundary $(c_1 = 0)$, hence $y = 1 - x$.

Sub into cost function and re-parametrize:

$$\tilde{f}(x) = f(x, 1-x) \qquad y-t$$

$$= \left(x - \frac{3}{2}\right)^2 + (y-t)^4$$

$$= \left(x - \frac{3}{2}\right)^2 + \left(\cancel{1} - x - \cancel{1}\right)^4$$

$$\left(x - \frac{3}{2}\right)^2 \quad x^4$$

$$= \left(x - \tfrac{3}{2}\right)^2 + x^4 .$$

$$= x^4 + x^2 - 3x + 9/4 , \qquad x \in [0,1] .$$

$$\frac{d\tilde{f}}{dx} = 4x^3 + 2x - 3$$

$$\frac{d\tilde{f}}{dx} = 0 \implies \boxed{2x^3 + x - \tfrac{3}{2} = 0} , \quad x \in [0,1] .$$

<u>Rootfinding</u> : Sl$^{\text{n}}$ of eq$^{\text{n}}$ at $x_* \simeq 0.728$.

<u>Also</u>, $\quad y_* = 1 - x_*$.

OP is solved.

The same problem, using the KKT conditions:

One active constraint $\quad (\lambda_2 = 0)$

$$\frac{\partial \mathcal{L}}{\partial x} = 2\left(x - \tfrac{3}{2}\right) + \lambda_1 + \cancel{\lambda_2} \overset{\text{KKT 1}}{=} 0$$

$$\frac{\partial \mathcal{L}}{\partial y} = 4(y-1)^3 + \lambda_1 - \cancel{\lambda_2} \overset{\text{KKT 2}}{=} 0$$

Eliminate λ_1 :

$$2\left(x - \tfrac{3}{2}\right) = -\lambda_1 = 4(y-1)^3$$

$$\Rightarrow \quad \boxed{-2\left(x-\tfrac{3}{2}\right)+4(y-1)^3=0.}$$

Complementarity condition KKTS

$$\lambda_1(1-x-y) = 0.$$

If $\lambda_1 = 0 \Rightarrow 2(x-3/2) = 0 \Rightarrow x = 3/2$ Infeasible.

$\lambda_1 \neq 0 \Rightarrow 1-x-y = 0$ (Active constraint)

$$y = 1-x.$$

$$y-1 = -x.$$

$$\Rightarrow (y-1)^3 = (-x)^3.$$

Back to highlighted equation:

$$-2\left(x-\tfrac{3}{2}\right)+4(y-1)^3=0.$$

$$-2(x-3/2) + 4(-x)^3 = 0.$$

$$\Rightarrow -4x^3 - 2x + 3 = 0.$$

$$\Rightarrow +4x^3 + 2x - 3 = 0.$$

$$\Rightarrow \boxed{2x^3 + x - \tfrac{3}{2} = 0.}$$

Same as eq^n^ obtained via elementary method.

Sol^n^: $x = x_* \simeq 0.728$

SP: $x = x_* \simeq 0.728$

$y_* = 1 - x_*$