Optimization Algorithms
(ACM 41030)

Dr Lennon Ó Náraigh

Exercises #6

3. Consider the half-space defined by:

Question 4

$$H_\alpha = \{\boldsymbol{x} \in \mathbb{R}^n | \boldsymbol{a} \cdot \boldsymbol{x} + \alpha \geq 0\},$$

where $\boldsymbol{a} \in \mathbb{R}^n$ is a constant non-zero vector and $\alpha \in \mathbb{R}$ is a constant scalar. Formulate and solve the OP for finding the point $\boldsymbol{x} \in H_\alpha$ with the smallest Euclidean norm.

$H_\alpha = \{ \underline{x} \in \mathbb{R}^n \mid \underline{a} \cdot \underline{x} + \alpha \geq 0 \}$, where:

- $\underline{a}$ is a constant vector,
- α is a constant scalar.

Equality: $\underline{a} \cdot \underline{x} + \alpha = 0 \Rightarrow$ Hyperplane in $\mathbb{R}^n$.

H_α: Everything to the right of the hyperplane.

Find the vector $\underline{x}$ in the half-space with the smallest Euclidean norm.

OP: $\min_{\underline{x} \in \mathbb{R}^n} \frac{1}{2} \|\underline{x}\|_2^2$ subject to: $c_1(\underline{x}) \geq 0$

where $c_1(\underline{x}) = \underline{a} \cdot \underline{x} + \alpha$

OR

$$\min_{\underline{x} \in \mathbb{R}^n} f(\underline{x}) \quad , \quad f(\underline{x}) = \frac{1}{2} \sum_{i=1}^{n} x_i^2$$

subject to: $\displaystyle\sum_{i=1}^{n} a_i x_i + \alpha \geq 0$.

Solve

KKT conditions.

$$\mathcal{L}(\underline{x}, \lambda) = \frac{1}{2} \sum_{i=1}^{n} x_i^2 - \lambda \left(\sum_{i=1}^{n} a_i x_i + \alpha \right)$$

KKT 1: $\nabla_x \mathcal{L} = 0$ at optimality.

$$\nabla_x \mathcal{L} = x_i - \lambda a_i$$

KKT 1 $\Rightarrow$ $x_i = \lambda a_i \quad , \quad i = 1, \dots, n$.

KKT 2: no equality constraints

KKT 3: $\displaystyle\sum_{i=1}^{n} a_i x_i + \alpha \geq 0$

KKT 4: $\lambda \geq 0$

KKT 5: $\displaystyle\lambda \left(\sum_{i=1}^{n} a_i x_i + \alpha \right) = 0$

KKT 1 $\Rightarrow$ $x_i = \lambda a_i$

KKT 3 $\Rightarrow$ $\displaystyle\lambda \sum_i a_i^2 + \alpha \geq 0$

KKT 4 $\Rightarrow$ $\lambda \geq 0$

KKT 5 $\Rightarrow$ $\displaystyle\lambda \left(\lambda \sum_i a_i^2 + \alpha \right) = 0$.

$$\lambda\Big(\underbrace{\lambda\sum_i a_i^2 + \alpha}\Big) = 0.$$

Solutions:

$\lambda = 0$ is a solution, valid when $\alpha > 0$.

<u>or</u> $\lambda\sum_i a_i^2 + \alpha = 0 \Rightarrow \lambda = -\dfrac{\alpha}{\sum_i a_i^2}$, $\alpha < 0$.

<u>Case 1</u> : $\lambda = 0$, $\alpha > 0$

<u>Case 2</u> : $\lambda = -\dfrac{\alpha}{\sum_i a_i^2}$, $\alpha < 0$.

Back to KKT 1 : $x_i = \lambda a_i$

<u>Case 1</u> : $\lambda = 0$, $\alpha > 0$, $x_i = 0 \Rightarrow \boxed{\underline{x}_* = 0}$

<u>Case 2</u> : $\lambda = -\dfrac{\alpha}{\sum_i a_i^2}$, $x_i = \lambda a_i \Rightarrow \boxed{\underline{x}_* = -\dfrac{\alpha\, \underset{\sim}{a}}{\sum_i a_i^2}}$

This is the required solution. ∎

<u>Interpretation</u> in $\mathbb{R}^2$.

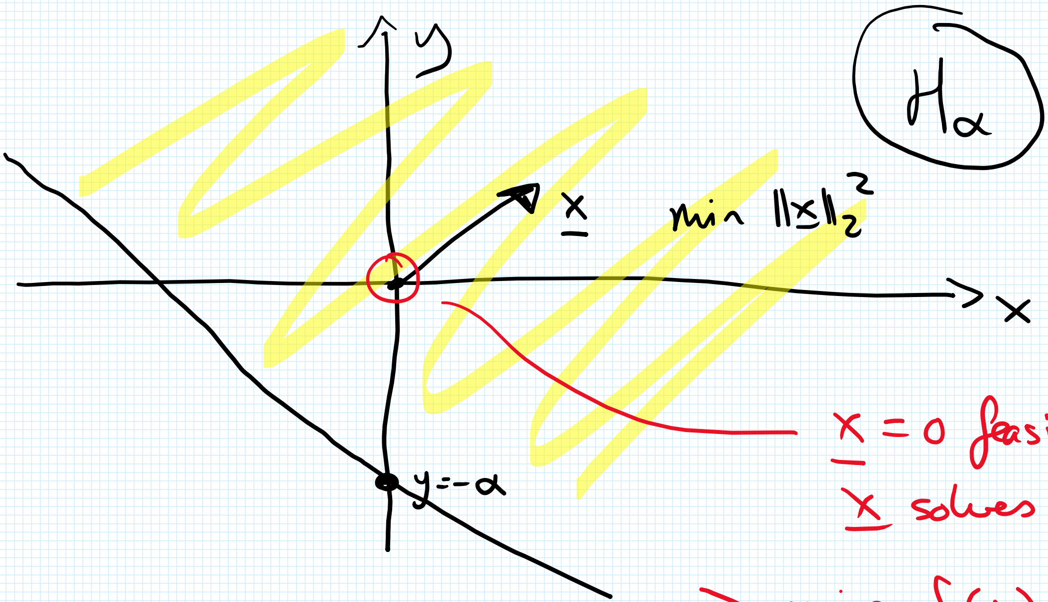
$\underline{a}\cdot\underline{x} + \alpha \geq 0$, take $\underset{\sim}{a} = (a_1, 1)$.

$x a_1 + y + \alpha \geq 0$.

$x a_1 + y + \alpha \geq 0.$

$\Rightarrow \boxed{y = -a_1 x - \alpha}$ (*) Equality

Case 1: $\alpha > 0 \Rightarrow$ y-intercept is at $y = -\alpha$ (neg).

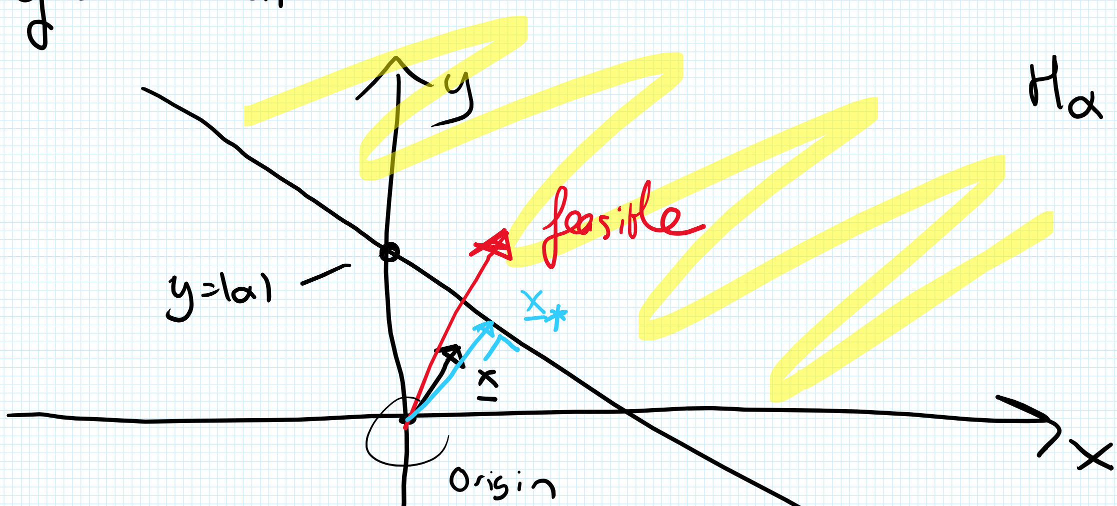


$\underline{x} = 0$ feasible,
$\underline{x}$ solves the OP

$\Rightarrow \min f(\underline{x}) = 0$

<u>Case 2</u> $\alpha < 0.$

$$y = -a_1 x - \alpha = -a_1 x + |\alpha|$$



$\|\underline{x}\|_2^2$

$\underline{x} = 0$ not feasible.

Optimum vector $\underline{x}_*$ lies on the line forming the half-

Optimum vector $\underline{x}_*$ lies on the line forming the half-space boundary, and is orthogonal to same.

$$\Rightarrow \underline{X}_* = -\frac{\alpha\, \underline{a}}{\sum_i a_i^2}, \text{ same as KKT.}$$

Question 4.

4. Consider the following modification of the example in class notes. Here, t is a parameter that is fixed prior to solving the problem:

$$\min_{x \in \mathbb{R}^2} f(x),$$

where

$$f(x) = \left(x - \tfrac{3}{2}\right)^2 + (y - t)^4,$$

t is fixed for the problem.

1

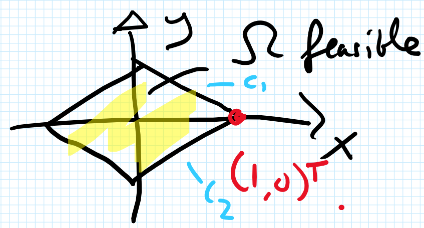
ACM 41030 More on Constrained Optimization

subject to:

$$\begin{bmatrix} 1 - x - y \\ 1 - x + y \\ 1 + x - y \\ 1 + x + y \end{bmatrix} \geq 0.$$

(a) For what values of t does the point $x_* = (1,0)^T$ satisfy the KKT conditions?

(b) Show that when $t = 1$, only the first constraint is active at the solution and find the solution.



t

a) Sln:

$$\mathcal{L}(\underline{x}, \lambda_1, \lambda_2) = \left(x - \tfrac{3}{2}\right)^2 + (y-t)^4 - \lambda_1 (1 - x - y) - \lambda_2 (1 - x + y)$$

KKT 1: $\dfrac{\partial \mathcal{L}}{\partial x} = 2\left(x - \tfrac{3}{2}\right) + \lambda_1 + \lambda_2$

$\dfrac{\partial \mathcal{L}}{\partial y} = 4(y-t)^3 + \lambda_1 - \lambda_2$

$$\frac{\partial \mathcal{L}}{\partial y} = 4(y-t)^3 + \lambda_1 - \lambda_2$$

$\underline{x}_* = (1,0)^T$. Look at KKT1 evaluated at $\underline{x}_*$.

$$\left.\frac{\partial \mathcal{L}}{\partial x}\right|_{\underline{x}_*} = 0 \Rightarrow \quad 2\big(\underbrace{1-\tfrac{3}{2}}_{-1/2}\big) + \lambda_1 + \lambda_2 = 0.$$

$$\Rightarrow \lambda_1 + \lambda_2 = 1$$

$$\left.\frac{\partial \mathcal{L}}{\partial y}\right|_{\underline{x}_*} = 0 \quad \Rightarrow \quad 4(-t)^3 + \lambda_1 - \lambda_2 = 0$$

$$\Rightarrow \lambda_1 - \lambda_2 = -4(-t)^3.$$

Gather up:

$$\lambda_1 + \lambda_2 = 1$$
$$\lambda_1 - \lambda_2 = -4(-t)^3.$$

Add: $\quad 2\lambda_1 = 1 - 4(-t)^3$.

$\lambda_1 \geq 0$. If $t > 0$, this is ok.

If $t < 0$, ...

Take $t = -|t|$.

$$2\lambda_1 = 1 - 4|t|^3 \geq 0$$
$$\Rightarrow |t| \leq \frac{1}{4^{1/3}} \Rightarrow t \geq -\frac{1}{4^{1/3}}.$$

$$\lambda_2 = 1 - \lambda_1$$
$$= 1 - \left[\tfrac{1}{2} - \tfrac{1}{2}\cdot 4(-t)^3\right]$$
$$= \tfrac{1}{2} + 2(-t)^3.$$

$$= \frac{1}{2} + 2(-t)^3 .$$

Require $\lambda_2 \geq 0$. Same reasoning as before:

$$t \leq \frac{1}{4^{1/3}} .$$

Putting the two limits together, we get:

$$\boxed{-\frac{1}{4^{1/3}} \leq t \leq \frac{1}{4^{1/3}}} .$$

b) When $t=1$, only the first constraint is active. <u>Find the sl^n of the OP</u>.

Elementary method. Given: only the first constraint is active. From class notes / geometric reasoning, we know that the sl^n is on the constraint boundary ($c_1 = 0$), hence $y = 1-x$.

Sub into cost function and re-parametrize:

$$\tilde{f}(x) = f(x, 1-x) \qquad\qquad y-t$$

$$= \left(x - \frac{3}{2}\right)^2 + (y-t)^4$$

$$= \left(x - \frac{3}{2}\right)^2 + \left(\cancel{1} - x \quad - \cancel{1}\right)^4$$

$$= \left(x-\tfrac{3}{2}\right)^2 + x^4 .$$

$$= x^4 + x^2 - 3x + 9/4, \quad x \in [0,1] .$$

$$\frac{d\tilde{f}}{dx} = 4x^3 + 2x - 3$$

$$\frac{d\tilde{f}}{dx} = 0 \Rightarrow \boxed{2x^3 + x - \tfrac{3}{2} = 0}, \quad x \in [0,1] .$$

Rootfinding : Sl^{n} of eq^{n} at $x_* \simeq 0.728$.

Also, $y_* = 1 - x_*$.

OP is solved.

The same problem, using the KKT conditions:

One active constraint $(\lambda_2 = 0)$

$$\frac{\partial \mathcal{L}}{\partial x} = 2\left(x-\tfrac{3}{2}\right) + \lambda_1 + \cancel{\lambda_2} \overset{\text{KKT 1}}{=} 0$$

$$\frac{\partial \mathcal{L}}{\partial y} = 4(y-1)^3 + \lambda_1 - \cancel{\lambda_2} \overset{\text{KKT 2}}{=} 0$$

Eliminate λ_1:

$$2\left(x-\tfrac{3}{2}\right) = -\lambda_1 = 4(y-1)^3$$

$$\Rightarrow \quad \boxed{-2\left(x-\tfrac{3}{2}\right) + 4(y-1)^3 = 0.}$$

Complementarity condition KKTS

$$\lambda_1 (1-x-y) = 0.$$

If $\lambda_1 = 0 \Rightarrow 2(x-3/2) = 0 \Rightarrow x = 3/2$ Infeasible.

$\lambda_1 \neq 0 \Rightarrow 1-x-y = 0$ (Active constraint)

$$y = 1-x.$$

$$y-1 = -x.$$

$$\Rightarrow (y-1)^3 = (-x)^3.$$

Back to highlighted equation:

$$-2\left(x-\tfrac{3}{2}\right) + 4(y-1)^3 = 0.$$

$$-2(x-3/2) + 4(-x)^3 = 0.$$

$$\Rightarrow -4x^3 - 2x + 3 = 0.$$

$$\Rightarrow +4x^3 + 2x - 3 = 0.$$

$$\Rightarrow \boxed{2x^3 + x - \tfrac{3}{2} = 0.}$$

Same as eq$^{\underline{n}}$ obtained via elementary method.

Sol$^{\underline{n}}$: $x = x_* \simeq 0.728$

SP: $x = x_* \simeq 0.728$

$$y_* = 1 - x_* \qquad \blacksquare$$

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Exercises #6

6. Formulate the dual problem for the following OPs:

   (a) Minimize:
   $$\min_{x\in\mathbb{R}^n}\langle c, x\rangle, \text{ subject to } Ax - b \geq 0.$$
   Here, $c \in \mathbb{R}^n$ is a constant vector, $b \in \mathbb{R}^m$ is a constant vector, and $A \in \mathbb{R}^{m\times n}$ is a contant matrix.

   (b) Minimize:
   $$\min_{x\in\mathbb{R}^n}\tfrac{1}{2}\langle x, Gx\rangle, \text{ subject to } Ax - b \geq 0.$$
   Here, A and b are as before, and $G \in \mathbb{R}^{n\times n}$ is a constant symmetric positive-definite matrix.

6a) Linear cost function $f(\underline{x}) = \langle \underline{c}, \underline{x}\rangle$

m-dimensional

$$\mathcal{L}(\underline{x}, \underline{\lambda}) = \langle \underline{c}, \underline{x}\rangle - \sum_{i=1}^{m} \lambda_i \,(A\underline{x} - \underline{b})_i$$

$$= \langle \underline{c}, \underline{x}\rangle_{\mathbb{R}^n} - \langle \underline{\lambda}, A\underline{x} - \underline{b}\rangle_{\mathbb{R}^m}$$

$$= \langle \underline{c}, \underline{x}\rangle_{\mathbb{R}^n} - \langle A^T\underline{\lambda}, \underline{x}\rangle_{\mathbb{R}^n} + \langle \underline{\lambda}, \underline{b}\rangle_{\mathbb{R}^m}$$

Step 1: $$q(\underline{\lambda}) = \inf_{\underline{x}\in\mathbb{R}^n} \mathcal{L}(\underline{x}, \underline{\lambda})$$

So we minimize $\mathcal{L}(\underline{x}, \underline{\lambda})$ w.r.t. $\underline{x}$:

$$\nabla_x \mathcal{L}(\underline{x}, \underline{\lambda}) = 0.$$

$$\nabla_x \mathcal{L} = \underline{c} - A^T\underline{\lambda} + 0$$

Set $\nabla_x \mathcal{L} = 0$ to obtain $q(\underline{\lambda})$. I.e.

$$\underline{c} = A^T\underline{\lambda}.$$

Sub back into $\mathcal{L}(\underline{x}, \underline{\lambda})$:

$$\mathcal{L}(x, \lambda)\Big| = \Big[\langle c, x\rangle - \langle \lambda, Ax\rangle + \langle \underline{\lambda}, \underline{b}\rangle\Big]$$

$$\mathcal{L}(\underline{x},\underline{\lambda})\Big|_{\underline{c}=A^T\underline{\lambda}} = \Big[\langle \underline{c},\underline{x}\rangle - \langle \underline{\lambda}, A\underline{x}\rangle + \langle \underline{\lambda},\underline{b}\rangle\Big]_{\underline{c}=A^T\underline{\lambda}}$$

$$= \cancel{\langle A^T\underline{\lambda},\underline{x}\rangle} - \cancel{\langle A^T\underline{\lambda},\underline{x}\rangle} + \langle \underline{\lambda},\underline{b}\rangle$$

$$= \langle \underline{\lambda},\underline{b}\rangle$$

$$= \langle \underline{\lambda},\underline{b}\rangle_{\mathbb{R}^m}$$

$$\therefore \quad q(\underline{\lambda}) = \langle \underline{\lambda},\underline{b}\rangle$$

Dual problem:

$$\boxed{\max_{\underline{\lambda}\in\mathbb{R}^m} \langle \underline{\lambda},\underline{b}\rangle \quad \text{subject to:} \begin{cases} \underline{\lambda} \geq 0 \\ A^T\underline{\lambda} = \underline{c} \end{cases}}$$

6b) Quadratic primal problem:

$$f(\underline{x}) = \frac{1}{2}\langle \underline{x}, G\underline{x}\rangle$$

Constraint: $A\underline{x} - \underline{b} \geq 0$, m constraints

Given: $G \in \mathbb{R}^{n\times n}$ is symmetric P.D.

Dual formulation:

$$\mathcal{L}(\underline{x},\underline{\lambda}) = \frac{1}{2}\langle \underline{x}, G\underline{x}\rangle_{\mathbb{R}^n} - \langle \underline{\lambda}, A\underline{x}-\underline{b}\rangle_{\mathbb{R}^m}$$

$$= \frac{1}{2}\langle \underline{x}, G\underline{x}\rangle - \langle A^T\underline{\lambda}, \underline{x}\rangle_{\mathbb{R}^n} + \langle \underline{\lambda},\underline{b}\rangle_{\mathbb{R}^m}$$

$$= \frac{1}{2}\langle \underline{x}, G\underline{x}\rangle - \langle A^T\underline{\lambda}, \underline{x}\rangle_{\mathbb{R}^n} + \langle \underline{\lambda}, \underline{b}\rangle_{\mathbb{R}^m}$$

Minimize over $\underline{x}$:

$$\nabla_x \mathcal{L} = G\underline{x} - A^T\underline{\lambda}$$

$$\nabla_x \mathcal{L} = 0 \Rightarrow G\underline{x} = A^T\underline{\lambda} \Rightarrow \boxed{\underline{x} = G^{-1}A^T\underline{\lambda}}$$

$$\mathcal{L}(\underline{x} = G^{-1}A^T\underline{\lambda}, \underline{\lambda}) = \left[\frac{1}{2}\langle \underline{x}, G\underline{x}\rangle - \langle \underline{\lambda}, A\underline{x}\rangle + \langle \underline{\lambda}, \underline{b}\rangle\right]_{\underline{x} = G^{-1}A^T\underline{\lambda}}$$

$$= \frac{1}{2}\langle G^{-1}A^T\underline{\lambda}, \cancel{G}\cancel{G^{-1}}A^T\underline{\lambda}\rangle - \langle \underline{\lambda}, A\,G^{-1}A^T\underline{\lambda}\rangle + \langle \underline{\lambda}, \underline{b}\rangle$$

$$= \frac{1}{2}\langle A^T\underline{\lambda}, G^{-1}A^T\underline{\lambda}\rangle - \langle A^T\underline{\lambda}, G^{-1}A^T\underline{\lambda}\rangle + \langle \underline{\lambda}, \underline{b}\rangle$$

$$= -\frac{1}{2}\langle A^T\underline{\lambda}, \underset{\sim}{G^{-1}}A^T\underline{\lambda}\rangle_{\mathbb{R}^n} + \langle \underline{\lambda}, \underline{b}\rangle_{\mathbb{R}^m}$$

$$= q(\underline{\lambda})$$

Dual problem :

$$\boxed{\begin{array}{l} \text{Maximize} \\ \quad q(\underline{\lambda}) = -\frac{1}{2}\langle A^T\underline{\lambda}, G^{-1}A^T\underline{\lambda}\rangle + \langle \underline{\lambda}, \underline{b}\rangle \\ \text{Subject to} \\ \quad \underline{\lambda} \geq 0 \end{array}}$$

Back to Q.3

3. Consider the half-space defined by:

$$H_\alpha = \{\boldsymbol{x} \in \mathbb{R}^n | \boldsymbol{a} \cdot \boldsymbol{x} + \alpha \geq 0\},$$

where $\boldsymbol{a} \in \mathbb{R}^n$ is a constant non-zero vector and $\alpha \in \mathbb{R}$ is a constant scalar. Formulate and solve the OP for finding the point $\boldsymbol{x} \in H_\alpha$ with the smallest Euclidean norm.

$$f(\underline{x}) = \frac{1}{2}\langle \underline{x}, \underline{x} \rangle \equiv \frac{1}{2}\,\underline{x}\cdot\underline{x}$$

Single constraint: $\underline{a}\cdot\underline{x} + \alpha \geq 0$.

$$\mathcal{L}(\underline{x}, \lambda) = \frac{1}{2}\,\underline{x}\cdot\underline{x} - \lambda(\underline{a}\cdot\underline{x} + \alpha)$$

Step 1: Minimize wrt $\underline{x}$:

$$\nabla_x \mathcal{L} = \underline{x} - \lambda \underline{a}$$

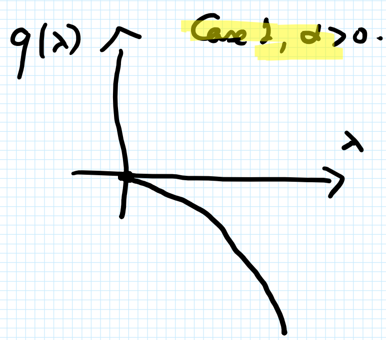
$$\nabla_x \mathcal{L} = 0 \Rightarrow \underline{x} = \lambda \underline{a}.$$

$$\mathcal{L}(\underline{x} = \lambda\underline{a}, \lambda) = \frac{1}{2}(\lambda\underline{a})\cdot(\lambda\underline{a}) - \lambda\,\underline{a}\cdot(\lambda\underline{a}) - \lambda\alpha$$

$$= \frac{1}{2}\lambda^2\,\underline{a}\cdot\underline{a} - \lambda^2\,\underline{a}\cdot\underline{a} - \lambda\alpha$$

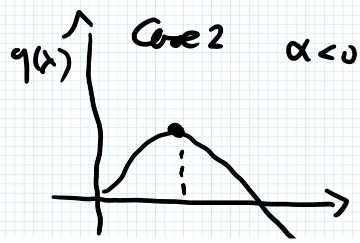
$$= -\frac{1}{2}\lambda^2\,\underline{a}\cdot\underline{a} - \lambda\alpha$$

$$= q(\lambda)$$



Dual problem:

$$\boxed{\begin{aligned} &\text{Minimize} \quad q(\lambda) = -\frac{1}{2}\lambda^2\,\underbrace{\underline{a}\cdot\underline{a}}_{\|\underline{a}\|_2^2} - \lambda\alpha \\ &\text{Subject to} \quad \lambda \geq 0. \end{aligned}}$$



Solution : $\dfrac{dg}{d\lambda} = -\lambda \|\underline{a}\|_2^2 - \alpha$

Attempt : $\dfrac{dg}{d\lambda} = 0 \Rightarrow -\lambda \|\underline{a}\|_2^2 - \alpha = 0$

$\Rightarrow \lambda \|\underline{a}\|_2^2 = -\alpha \quad , \quad \lambda \geq 0 .$

Case 2 : $\alpha < 0, \quad \lambda = -\dfrac{\alpha}{\|\underline{a}\|_2^2} .$

Back to $\underline{x} = \lambda \underline{a} \Rightarrow \boxed{\underline{x} = -\dfrac{\alpha \, \underline{a}}{\|\underline{a}\|_2^2}} .$

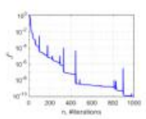
Case 1 : $\alpha > 0 \qquad g(\lambda)$ is maximized at $\lambda = 0$

$\Rightarrow \boxed{\underline{x} = \underline{0}}$

STRUCTURE OF FINAL EXAM

maths.ucd.ie/~onaraigh/optimization.html

Optimization (ACM 40990 and ACM 41030)



Current modules (Spring 2026)

Description: For *Optimization in Machine Learning* (ACM 40990, Spring 2026), refer to this page for the first **seven weeks**. For *Optimization Algorithms* (ACM 41030, Spring 2026), refer to this page for **all weeks**.

Course Documents:

- Complete set of typed notes, v2: March 2026
- Side note Section 1.3 (Convexity of Polyhedra)
- Introduction to ACM 40990 (January 2026)
- Introduction to ACM 41030 (January 2026)
- Handwritten Notes, Weeks 1-7

Lecture Notes (Weeks 8-12, ACM 41030 only):

- Week 8: Notes
- Week 9: Notes
- Week 10: Notes
- Week 11: Notes
- Week 12, Lecture 1: Notes and Video
- Week 12, Lectures 2-3: Notes TBC
- Final Exam Guidelines

Examinable results (Weeks 8-12, ACM 41030 only):

list is in exam guidelines

Teaching

- ACM 10030
- ACM 10070
- ACM 10080
- ACM 20150
- ACM 20030
- ACM 20050
- ACM 30020
- ACM 30220
- ACM 30210
- ACM 40690
- ACM 40890
- Optimization
- CSMM
- Data & Comp Sci MSc
- HPC 2013

Optimization Algorithms (ACM 41030)

Second Written Exam

23/04/2026

The second written exam is worth 50% of the module grade. This will take place as a full end-of-trimester exam in the RDS. Note the duration: the exam will last **60 minutes**.

The exam will contain 4 questions, For maximum marks, all 4 questions must be answered. The exam format is **closed book**. Non-programmable calculators are permitted.

The questions will involve a mixture of theory and calculations / exercises. The following theory is examinable:

- The projection operator – Section 12.4
- First-order optimality conditions in case of a single inequality constraint – Section 13.2
- Showing that if the LICQs are satisfied, then the Lagrange Multipliers are unique - Section 15.1, Theorem 15.2
- Showing that the tangent cone is inside the set of LFDDs; showing that the tangent cone and the LFDDs are the same when the LICQs are satisfied – Section 15.2, Theorem 15.3.

  NOTE: you can assume without proof that the matrix $\begin{pmatrix} A \\ Z^T \end{pmatrix}$ has full row rank.

- Assuming Farkas's Lemma, prove the necessary conditions (KKT conditions) for a feasible point $\boldsymbol{x}_*$ to be a minimizer - Section 16.4

The calculations / exercises will come exclusively from Exercises 5 and 6:

- Everything on Exercises 5 is examinable.
- Everything on Exercises 6 is examinable except for:
  - Question 5, which requires a computer for the solution.