Week 10, Lecture 1

02 April 2026 17:05

Main aim this week:

$$T_\Omega(\underline{x}) = \mathcal{F}_\Omega(\underline{x})$$

provided the LICQs hold at $\underline{x}$ (Thursday).

Today:

- Recap
- Prove a theorem
- Look at example

Recap: How to construct a tangent vector at $\underline{x} \in \Omega$:

- Construct a feasible sequence $\{\underline{z}_k\}_{k=0}^{\infty}$ tending to $\underline{x}$ as $k \to \infty$ $(\underline{z}_k \in \Omega)$
- Find a sequence of positive scalars $\{t_k\}_{k=0}^{\infty}$ tending to zero as $k \to \infty$.
- Define: $\underline{d} = \lim_{k\to\infty} \frac{\underline{z}_k - \underline{x}}{t_k}$
- Set of all such tangent vectors is the tangent cone at $\underline{x}$, $T_\Omega(\underline{x})$.

We also review the def^n of the LFDDs:

$$\mathcal{F}_\Omega(\underline{x}) = \left\{ \underline{d} \in \mathbb{R}^n \;\middle|\; \begin{array}{ll} \underline{d} \cdot \underline{\nabla} c_i(\underline{x}) = 0 & i \in \mathcal{E} \\ \underline{d} \cdot \underline{\nabla} c_i(\underline{x}) \geq 0 & i \in \mathcal{I} \cap \mathcal{A}(\underline{x}) \end{array} \right\}$$

Active set:

$$\mathcal{A}(\underline{x}) = \mathcal{E} \cup \{ i \in \mathcal{I} \mid c_i(\underline{x}) = 0 \}$$

LICQs:

$\underline{\nabla} c_i(\underline{x})$ linearly independent, $i \in \mathcal{A}(\underline{x})$.

Theorem 15.3 (Thursday) Let $\underline{x}$ be a feasible

Theorem 15.3 (Thursday) Let $\underline{x}$ be a feasible point. Then the following statements are true:

- $T_\Omega(\underline{x}) \subset \mathcal{F}_\Omega(\underline{x})$
- If the LICQs hold at $\underline{x}$, then
  $$T_\Omega(\underline{x}) = \mathcal{F}_\Omega(\underline{x})$$

We recall also the KKT conditions :

$$\text{OP}: \quad \min_{\underline{x}\in\mathbb{R}^n} f(\underline{x}) \quad \text{subject to} \quad \begin{cases} c_i(\underline{x}) = 0 & i \in \underline{\mathcal{E}} \\ c_i(\underline{x}) \geq 0 & i \in \underline{\mathcal{I}} \end{cases}$$

$$\mathcal{L}(\underline{x}, \underline{\lambda}) = f(\underline{x}) - \sum_{i\in\mathcal{E}\cup\mathcal{I}} \lambda_i c_i(\underline{x})$$

If the vector $\underline{x}_*$ solves the OP then there exist scalars $\lambda_1^*, \dots$ with $\underline{\lambda}_* = (\lambda_1^*, \dots)$ such that :

$$\left\{ \begin{array}{lll} \nabla_x \mathcal{L}(\underline{x}_*, \underline{\lambda}_*) = 0 & & \\ c_i(\underline{x}_*) = 0 & i \in \mathcal{E} \\ c_i(\underline{x}_*) \geq 0 & i \in \underline{\mathcal{I}} \\ \lambda_i^* \geq 0 & i \in \mathcal{I} \\ \lambda_i^* c_i(\underline{x}_*) = 0 & i \in \mathcal{E}\cup\underline{\mathcal{I}} \end{array} \right\} \text{KKT conditions}$$

Inequality constraint with $c_i(\underline{x}) > 0$ (inactive constraint)

The corresponding λ_i's are zero. So KKT1 can be replaced with

$$0 = \nabla_x \mathcal{L}(\underline{x}_*, \underline{\lambda}^*) = \nabla_x f(\underline{x}_*) - \sum_{i\in\mathcal{A}(\underline{x}_*)} \lambda_i^* \nabla c_i(\underline{x}_*) .$$

$$0 = \nabla_x \mathcal{L}(\underline{x}_*, \underline{\lambda}^*) = \nabla_x f(\underline{x}_*) - \sum_{i \in \mathcal{A}(\underline{x}_*)} \lambda_i^* \nabla c_i(\underline{x}_*) .$$

Terminology: We say that strict complementarity holds if $\lambda_i^* = 0$ or $c_i(\underline{x}_*) = 0$ but not both, for $i \in \mathcal{I}$, in KKT 5.

Remark: For a given OP and minimizer $\underline{x}_*$ there may be many vectors $\underline{\lambda}_*$ which satisfy the KKT conditions. However, imposing the LICQs makes $\underline{\lambda}_*$ unique.

Theorem: If the c_i's satisfy the LICQs at the minimizer $\underline{x}_*$, then the Lagrange multipliers λ_i^* are unique (Theorem 15.2, p. 121)

Proof: Suppose that λ_i^* and μ_i^* are two sets of Lagrange multipliers satisfying the KKT conditions at $\underline{x}_*$. From KKT 1 we have:

$$0 = \nabla_x f(\underline{x}_*) - \sum_{i \in \mathcal{A}(\underline{x}_*)} \lambda_i^* \nabla c_i(\underline{x}_*)$$

$$0 = \nabla_x f(\underline{x}_*) - \sum_{i \in \mathcal{A}(\underline{x}_*)} \mu_i^* \nabla c_i(\underline{x}_*) .$$

Subtract:

$$\sum_{i \in \mathcal{A}(\underline{x}_*)} (\lambda_i^* - \mu_i^*) \nabla c_i(\underline{x}_*) = 0$$

Since the $\nabla c_i(\underline{x}_*)$'s are linearly independent,

$$\lambda_i^* - \mu_i^* = 0 , \quad i \in \mathcal{A}(\underline{x}_*)$$

$$\lambda_i^* - \mu_i^* = 0 \quad , \quad i \in \mathcal{A}(\underline{x}_*)$$

$$\Rightarrow \quad \lambda_i^* = \mu_i^* \quad , \quad i \in \mathcal{A}(\underline{x}_*)$$

For all other indices ,

$$\lambda_i^* = \mu_i^* = 0 \quad .$$

Hence, $\lambda_i^* = \mu_i^*$, for all $i \in \mathcal{E} \cup \mathcal{I}$.

<u>Example</u> :

$$\min_{\underline{x} \in \mathbb{R}^2} f(\underline{x}) = (x - \tfrac{3}{2})^2 + (y - \tfrac{1}{2})^2$$

Subject to :

$$\begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{pmatrix} = \begin{pmatrix} 1-x-y \\ 1-x+y \\ 1+x-y \\ 1+x+y \end{pmatrix} \geq 0$$

By inspection, the minimizer $\underline{x}_*$ is in the first quadrant , on the boundary of the feasible region:

$$\underline{x}_* = (x_*, y_*) \in L_1 \, ; \quad L_1 : \underline{y = 1 - x} \quad .$$

We can reparametrize the cost function:

$$f(x, y = 1-x) = (x - \tfrac{3}{2})^2 + \underbrace{(1 - x \; - \tfrac{1}{2})^2}_{(x-\frac{1}{2})^2}$$

$$= x^2 - 3x + \tfrac{9}{4} + x^2 - x + \tfrac{1}{4}$$

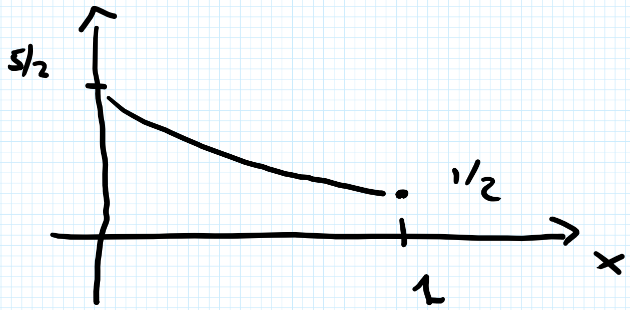
$$= 2x^2 - 4x + \tfrac{5}{2}$$

$$= \tilde{f}(x)$$

Thus,

$$x_* = \arg \min_{x \in [0,1]} \tilde{f}(x)$$

$$x_* = \arg\min_{x\in[0,1]} f(x)$$



Minimizer at $x_* = 1$, $y_* = 1 - x_* = 0$.

$$\underline{x}_* = (1,0)^T.$$

Check this, using Lagrange multipliers. i.e. we check that the KKT conditions (and LICQs) held at $\underline{x}_0 := (1,0)^T$.

$$\nabla f(\underline{x}_0) = \left[2\left(x-\frac{3}{2}\right)\underline{i} + 2\left(y-\frac{1}{2}\right)\underline{j}\right]_{\overbrace{(1,0)}^{\underline{x}_0}}$$

$$= -\underline{i} - \underline{j}$$

Constraints c_1 and c_2 are active at $\underline{x}_0$

$$\nabla c_1(\underline{x}_0) = -\underline{i} - \underline{j}$$

$$\nabla c_2(\underline{x}_0) = -\underline{i} + \underline{j}$$

$$A = \begin{pmatrix} | & | \\ \nabla c_1 & \nabla c_2 \\ | & | \end{pmatrix} = \begin{pmatrix} -1 & -1 \\ -1 & 1 \end{pmatrix}$$

$$\det(A) = -1 - (-1)(-1) = -2 \neq 0.$$

LICQs hold at $\underline{x}_0$ ☑

To show that the KKT conditions hold at $\underline{x}_0$, we

To show that the KKT conditions hold at $\underline{x}_0$, we find λ_1^* and λ_2^* satisfying KKT1 –5.

$$\nabla f(\underline{x}_0, \underline{\lambda}^*) - \lambda_1^* \nabla c_1(x_0) - \lambda_2^* \nabla c_2(\underline{x}_0) = 0$$

$$\begin{pmatrix} -1 \\ -1 \end{pmatrix} - \lambda_1^* \begin{pmatrix} -1 \\ -1 \end{pmatrix} - \lambda_2^* \begin{pmatrix} -1 \\ 1 \end{pmatrix} = 0$$

$$\begin{aligned} \lambda_1^* + \lambda_2^* &= 1 \\ \lambda_1^* - \lambda_2^* &= 1 \end{aligned}$$

$$\Rightarrow \begin{cases} \lambda_1^* = 1. \\ \lambda_2^* = 0 \end{cases}$$

$$\boxed{c_1(\underline{x}_0) = 0}, \quad \boxed{c_2(\underline{x}_0) = 0}$$ (Active inequality constraints)

No equality constraints to check.

Complementarity condition:

$\boxed{\lambda_1^* c_1(\underline{x}_0)} = 0$ ☑ $(\lambda_1^* = 1)$

$\boxed{\lambda_2^* c_2(\underline{x}_0)} = 0$ ☑ $(\lambda_2^* = 0)$

I.e. not strict complementarity.

Hence, KKT conditions are met – i.e. first-order optimality conditions are satisfied.

Previous calculation with $x_* = \underset{[0,1]}{\arg\min}\ \tilde{f}(x)$ confirms that $\underline{x}_0 = (1,0)^T$ is a minimizer.

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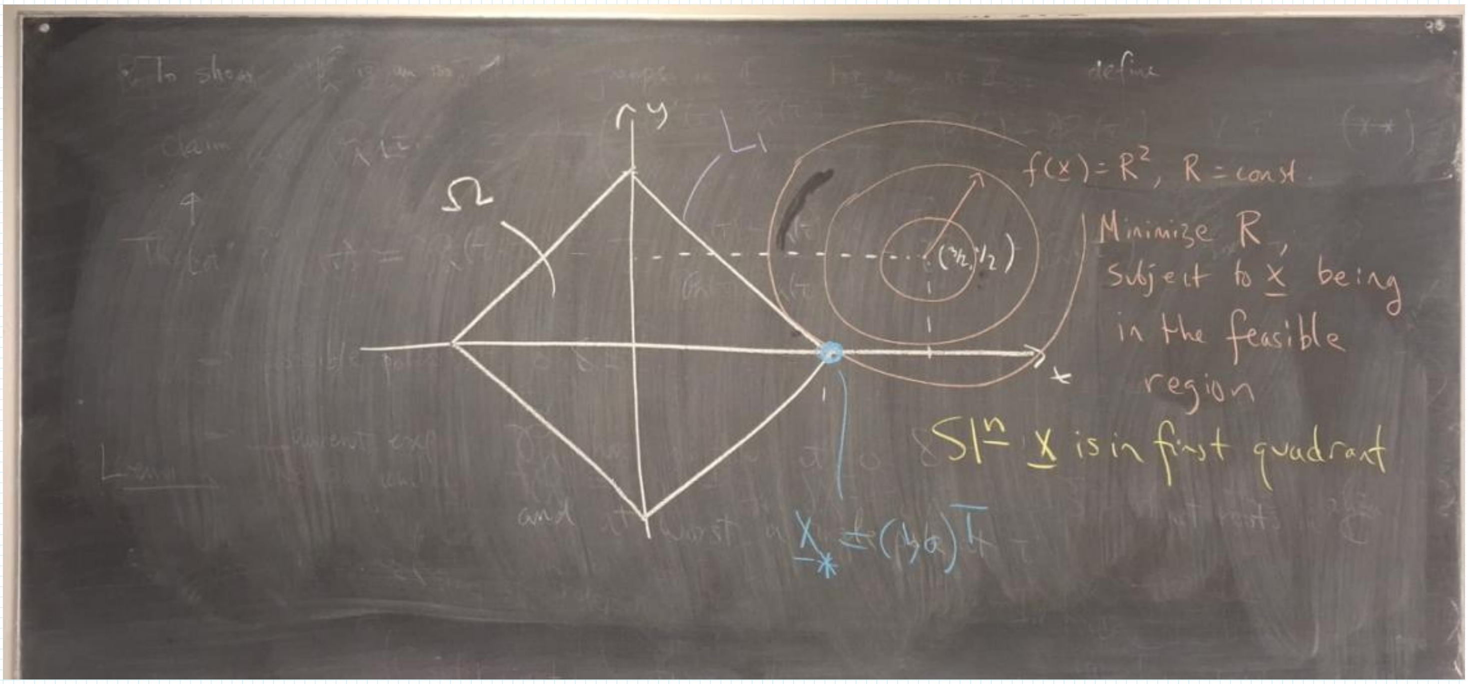

Week 10, Lectures 2-3

Thursday 9 April 2026 14:54

The main aim: We prove Theorem 15.3.

Let $\underline{x}_*$ be a feasible point. Then the following statements are true:

- $T_\Omega(\underline{x}_*) \subset \mathcal{F}_\Omega(\underline{x}_*)$
- If the LICQ holds, then $T_\Omega(\underline{x}_*) = \mathcal{F}_\Omega(\underline{x}_*)$.

Notation: The active set $\mathcal{A}(\underline{x}_*)$ is the set of indices:

$$\mathcal{A}(\underline{x}_*) = \mathcal{E} \cup \{ i \in \mathcal{I} \mid c_i(\underline{x}_*) = 0 \}$$

Label the indices in $\mathcal{A}(\underline{x}_*)$ as $i_1, \dots, i_m$.

Hence:

$$c_{i_1}(\underline{x}_*) = 0 \ , \ \dots , \ c_{i_m}(\underline{x}_*) = 0$$

$$\mathcal{A}(\underline{x}_*) = \{ i_1, \dots, i_m \} .$$

Define:

$$A(\underline{x}_*) = \begin{array}{cc} \begin{pmatrix} \frac{\partial c_{i_1}}{\partial x_1} & \cdots & \frac{\partial c_{i_1}}{\partial x_n} \\ \frac{\partial c_{i_2}}{\partial x_1} & \cdots & \frac{\partial c_{i_2}}{\partial x_n} \\ \vdots & & \\ \frac{\partial c_{i_m}}{\partial x_1} & \cdots & \frac{\partial c_{i_m}}{\partial x_n} \end{pmatrix} & \begin{array}{c} \uparrow \\ m \text{ rows} \\ \downarrow \end{array} \\ \longleftarrow n \text{ columns} \longrightarrow & \end{array}$$

hence $A(\underline{x}_*) \in \mathbb{R}^{m \times n}$

From now on : We drop the $\underline{x}_*$ on A as this is implicit:

$$A \equiv A(\underline{x}_*)$$

Assume : $m < n$, to avoid over-constraining the OP.

We look at the kernel of A :

$$\boxed{\ker(A) = \{\underline{z} \mid A\underline{z} = 0\}}$$

$$\begin{array}{cc} & \leftarrow n \text{ columns} \rightarrow \\ \begin{array}{c}\uparrow \\ m \\ \downarrow\end{array} & \underbrace{\begin{pmatrix} a_{11} & \cdots & a_{1n} \\ a_{21} & \cdots & a_{2n} \\ \vdots & & \\ a_{m1} & \cdots & a_{mn} \end{pmatrix}}_{A} \begin{pmatrix} z_1 \\ \vdots \\ z_n \end{pmatrix} = 0 \end{array}$$

Or

$$\textcircled{j} \rightarrow \begin{pmatrix} a_{11}z_1 + a_{12}z_2 + \cdots + a_{1n}z_n \\ a_{21}z_1 + a_{22}z_2 + \cdots + a_{2n}z_n \\ \vdots \\ a_{m1}z_1 + a_{m2}z_2 + \cdots + a_{mn}z_n \end{pmatrix} = 0$$

j^{th} row :

$$\sum_{i=1}^{n} a_{ji} z_i = 0, \quad j \in \{1, .., m\}$$

We have :

- $\underline{z} = (z_1, \ldots, z_n)^T$, n variables
- $\sum^{n} a_{ji} z_i = 0$, m constraints

- $\sum_{i=1}^{n} a_{ji} z_i = 0$, m constraints

- Hence, only $n-m$ free variables, hence

$$\dim(\ker(A)) = n-m .$$

- Hence, $\left\{ \underline{z}^{(1)}, \dots, \underline{z}^{(n-m)} \right\}$ are a basis of $\ker(A)$.

Form the matrix

$$Z = \underbrace{\begin{pmatrix} | & & | \\ \underline{z}^{(1)} & \cdots & \underline{z}^{(n-m)} \\ | & & | \end{pmatrix}}_{\longleftarrow\ n-m\ \longrightarrow} \begin{matrix} \uparrow \\ n \\ \downarrow \end{matrix} \in \mathbb{R}^{n \times (n-m)}$$

Kernel condition : $\boxed{AZ = 0}$

$$Z^T \in \mathbb{R}^{(n-m) \times n}$$

Recall

$$A \in \mathbb{R}^{m \times n}$$

Make a new matrix

$$\begin{array}{c c c} & m \updownarrow & \\ n \updownarrow & & \\ & n-m \updownarrow & \end{array} \underbrace{\begin{pmatrix} A \\ Z^T \end{pmatrix}}_{\longleftarrow\ n\ \longrightarrow} \in \mathbb{R}^{n \times n} .$$

Entries:

Entries:

$$\begin{pmatrix} A \\ Z^T \end{pmatrix} = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \\ a_{m1} & \cdots & a_{mn} \\ z_1^{(1)} & \cdots & z_n^{(1)} \\ \vdots & & \\ z_1^{(n-m)} & \cdots & z_n^{(n-m)} \end{pmatrix}$$

Lemma: $\begin{pmatrix} A \\ Z^T \end{pmatrix}$ has full row rank.

Proof:

- First m rows are lin. independent (LICQ)
- Last $n-m$ rows are lin independent, since these form a basis for $\ker(A)$.

So it remains to check that the first m rows are lin. independent of the last $n-m$ rows. We prove this by contradiction. Assume that the first m rows and the last $n-m$ rows are linearly dependent. Then, in particular, there exist scalars $\mu_1, \dots, \mu_m$ not all zero such that

$$\left(z_1^{(1)}, \dots, z_n^{(1)} \right)^T = \sum_{j=1}^{m} \mu_j \left(a_{j1}, \dots, a_{jn} \right)^T \qquad (1)$$

But

$$\sum_{i=1}^{n} a_{ji} z_i^{(1)} = 0 \; , \; j = 1, \dots, m \quad \left(\begin{matrix} \text{kernel} \\ \text{condition} \end{matrix} \right) \; (2)$$

From (1) :

From (1) :

$$z_i^{(1)} = \sum_{j=1}^{m} \mu_j a_{ji}$$

$$\Rightarrow z_i^{(1)} \overset{\text{re-index}}{=} \sum_{k=1}^{m} \mu_k a_{ki} \qquad (3)$$

Kernel condition (2) :

$$\sum_{i=1}^{n} a_{ji} z_i^{(1)} = 0$$

$$\overset{(3)}{\Rightarrow} \quad \sum_{i=1}^{n} a_{ji} \left(\sum_{k=1}^{m} \mu_k a_{ki} \right) = 0$$

Re-arrange:

$$\sum_{i=1}^{n} \sum_{k=1}^{m} a_{ji} a_{ki} \mu_k = 0 .$$

$$\downarrow \qquad \searrow$$

Matrix multiplication:

$$\Rightarrow \sum_{i=1}^{n} \sum_{k=1}^{m} (A)_{ji} (A^T)_{ik} \mu_k = 0$$

$$\Rightarrow \sum_{k=1}^{m} (AA^T)_{jk} \mu_k = 0$$

No indices:

$$AA^T \underline{\mu} = 0 , \quad \underline{\mu} = (\mu_1, \ldots, \mu_m)$$

Dot with $\underline{\mu}$:

$$\langle \underline{\mu}, AA^T \underline{\mu} \rangle = 0$$

$$\langle \underline{\mu}, AA' \underline{\mu} \rangle = 0$$

$$\Rightarrow \langle A^T \underline{\mu}, A^T \underline{\mu} \rangle = 0$$

Since the dot product is positive-definite,

$$A^T \underline{\mu} = 0$$

In terms of components:

$$\mu_1 (a_{11}, a_{12}, \ldots, a_{1n}) + \cdots + \mu_m (a_{m1}, \ldots, a_{mn}) = 0 \quad (4)$$

But the rows of A are lin. independent (LICQ).

So the only way for (4) to be true is if

$$\mu_1 = 0, \ldots, \mu_m = 0.$$

Back to Eq. (1):

$$(z^{(1)}_1, \ldots, z^{(1)}_n)^T = \sum_{j=1}^{m} \mu_j (a_{j1}, \ldots, a_{jn})^T = 0$$

But $\underline{z}^{(1)}$ is a basis vector, so $\underline{z}^{(1)} \neq 0$. Contradiction.

To resolve the contradiction, the rows of $\begin{pmatrix} A \\ z^T \end{pmatrix}$ must all be linearly independent. ∎

We now prove Theorem 15.3: Let x_* be a feasible point. Then the following statements are true:

Then the following statements are true :

- $T_\Omega(\underline{x}_*) \subset \mathcal{F}_\Omega(\underline{x}_*)$
- If the LICQ holds, then $T_\Omega(\underline{x}_*) = \mathcal{F}_\Omega(\underline{x}_*)$.

Proof : Start with the first part. Assume without loss of generality that all constraints are active. Hence, we use an index $i \in \{1, \dots, m\}$ to label the active constraints.

Let $\underline{d} \in T_\Omega(\underline{x}_*)$. Then we need to show that $\underline{d}$ is also in $\mathcal{F}_\Omega(\underline{x}_*)$.

Since $\underline{d} \in T_\Omega(\underline{x}_*)$ there exists a feasible sequence $\{\underline{z}_k\}_{k=0}^{\infty}$ tending to $\underline{x}_*$, and another sequence $\{t_k\}_{k=0}^{\infty}$ with $t_k > 0$ tending to zero, such that:

$$\underline{d} = \lim_{k \to \infty} \frac{\underline{z}_k - \underline{x}_*}{t_k} .$$

Re-arrange:

$$\underline{z}_k = \underline{x}_* + t_k \underline{d} + \epsilon_k$$

where ϵ_k is little-o of t_k:

$$\lim_{k \to 0} \frac{\|\epsilon_k\|_2}{t_k} = 0 . \qquad \left(\epsilon_k \sim t_k^{1+\delta}, \delta > 0 \right)$$

Since $\underline{z}_k$ is feasible:

$$\underset{t_k}{\underline{c_i(\underline{z}_k)}} = 0 \qquad i \in \mathcal{E}$$

$$\Rightarrow \quad \frac{c_i\left(\underline{x}_* + t_k \underline{d} + o(t_k)\right)}{t_k} = 0$$

$$\Rightarrow \quad \frac{\overset{i \in \mathcal{E}}{\cancel{c_i(\underline{x}_*)}} + \cancel{t_k}\, \underline{d} \cdot \nabla c_i(\underline{x}_*) + o(t_k)}{\cancel{t_k}} = 0$$

$$\Rightarrow \quad \underline{d} \cdot \nabla c_i(\underline{x}_*) + \overset{\to 0}{\cancel{\frac{o(t_k)}{t_k}}} = 0$$

Take $t_k \Rightarrow 0$.

Hence: $\underline{d} \cdot \nabla c_i(\underline{x}_*) = 0 \, , \quad i \in \mathcal{E}$.

Same for active inequality constraints:

$$\frac{\overset{i \text{ active}}{\cancel{c_i(\underline{x}_*)}} + t_k\, \underline{d} \cdot \nabla c_i(\underline{x}_*) + o(t_k)}{t_k} \geq 0,$$

for $i \in \mathcal{I} \cap \mathcal{A}(\underline{x}_*)$.

The same steps as for the equality constraints:

$$\underline{d} \cdot \nabla c_i(\underline{x}_*) + \cancel{\frac{o(t_k)}{t_k}} \geq 0$$

$$\underline{d} \cdot \nabla c_i(\underline{x}_*) + \cancel{\frac{o(t_n)}{t_n}} \geq 0$$

Take $t_n \to 0$.

$$\Rightarrow \underline{d} \cdot \nabla c_i(\underline{x}_*) \geq 0 \quad i \in \mathcal{I} \cap \mathcal{A}(\underline{x}_*).$$

Summaizing:

- Take $\underline{d} \in T_\Omega(\underline{x}_*)$.
- Shown:
  - $\underline{d} \cdot \nabla c_i(\underline{x}_*) = 0 \quad , \quad i \in \mathcal{E}$
  - $\underline{d} \cdot \nabla c_i(\underline{x}_*) \geq 0 \quad , \quad i \in \mathcal{I} \cap \mathcal{A}(\underline{x}_*)$
- Therefore, $\underline{d} \in \mathcal{F}_\Omega(\underline{x}_*)$.

Hence,

$$T_\Omega(\underline{x}_*) \subset \mathcal{F}_\Omega(\underline{x}_0) \qquad \blacksquare$$

We now move on to the second part of the theorem.

We show that $\mathcal{F}_\Omega(\underline{x}_*) \subset T_\Omega(\underline{x}_*)$ provided the LICQ holds. To start, we take $\underline{d} \in \mathcal{F}_\Omega(\underline{x}_*)$. The aim is to show that $\underline{d} \in T_\Omega(\underline{x}_*)$ as well.

We introduce a map

$$R : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$$

$$(\underline{z}, t) \longrightarrow R(\underline{z}, t)$$

$$(\underline{z}, t) \longrightarrow R(\underline{z}, t)$$

Such that:

$$R(\underline{z}, t) = \begin{pmatrix} \underline{c}(\underline{z}) - t\,\overset{\cancel{A}}{\overline{\cancel{A}(\underline{x}_*)}}\,\underline{d} \\ Z^T(\cancel{\underline{z}} - \cancel{\underline{x}_*} - \cancel{t}\underline{d}) \end{pmatrix} \begin{array}{l} \updownarrow \text{ m rows} \\ \updownarrow \text{ n−m rows} \end{array}$$

Notice:

$$R\underbrace{(\underline{x}_*, 0)}_{\text{BASE POINT IFT}} = \begin{pmatrix} \underline{c}(\underline{x}_*) \\ 0 \end{pmatrix} = \begin{pmatrix} c_1(\underline{x}_*) \\ \vdots \\ c_m(\underline{x}_*) \\ 0 \\ \vdots \\ 0 \end{pmatrix} = 0$$

$$\underset{\text{n components}}{\nabla_Z} \overset{\text{n components}}{R} = \begin{pmatrix} A(\underline{z}) \\ Z^T \end{pmatrix}$$

$$\underbrace{\nabla_Z R(\underline{x}_*, 0)}_{\text{Derivative @ base point is invertible}} = \begin{pmatrix} A(\underline{x}_*) \\ Z^T \end{pmatrix}$$ is invertible,

by Lemma 15.1.

By the Implicit Function Theorem, there exists a differentiable curve $\underline{z}(t)$ valid on an interval $I \ni 0$ such that:

$$R(\underline{z}(t), t) = 0, \quad \forall t \in I.$$

Since $\underline{z}(t)$ is a curve, tangent vectors to the curve

Since $\underline{z}(t)$ is a curve, tangent vectors to the curve are candidates for feasible sequences. Therefore, let $\{t_k\}_{k=0}^{\infty}$ with $t_k > 0$ and $t_k \to 0$ as $k \to \infty$, and with $t_k \in I$. Correspondingly, let

$$\underline{z}_k = \underline{z}(t_k).$$

We have:

Total derivative

$$0 = R_i(\underline{z}(t_k), t_k)$$

$$= R_i(\underline{z}(0), 0) + \left.\frac{dR_i}{dt}\right|_{t=0} t_k + O(t_k^2)$$

$$= \cancel{R_i(\underline{x}_*, 0)} + \left.\frac{dR_i}{dt}\right|_{t=0} t_k + O(t_k^2)$$

$$= \left.\frac{\partial R_i}{\partial t}\right|_{t=0} \cdot t_k + \sum_{j=1}^{n} \left.\frac{\partial R_i}{\partial z_j}\right|_{(\underline{x}_*,0)} \left.\frac{dz_j}{dt}\right|_{t=0} t_k + O(t_k^2)$$

$$\Rightarrow 0 = \begin{pmatrix} \overset{\downarrow}{-A\underline{d}} \\ -Z^T\underline{d} \end{pmatrix} \cdot t_k + t_k \sum_{j=1}^{n} \left.\frac{\partial R_i}{\partial z_j}\right|_{(\underline{x}_*,0)} \left(\frac{\underline{z}_k - \underline{x}_*}{t_k} + O(t_k) \right) + O(t_k^2)$$

Divide at by t_k:

$$0 = \begin{pmatrix} -A\underline{d} \\ -Z^T\underline{d} \end{pmatrix} + \begin{pmatrix} A(\underline{x}_*) \\ Z^T \end{pmatrix} \left(\frac{\underline{z}_k - \underline{x}_*}{t_k} + O(t_k) \right)$$

$$0 = \left(-z^T \underline{d} \right) + \left(z^T \right) \left(\frac{\quad}{t_k} + O(t_k) \right)$$

$$+ \; O(t_k) \, .$$

Gather up:

$$\begin{pmatrix} A \\ z^T \end{pmatrix} \left(\frac{\underline{z}_k - \underline{x}_*}{t_k} - \underline{d} \right) + O(t_k) = 0 .$$

Since $\begin{pmatrix} A \\ z^T \end{pmatrix}$ is invertible, $(\dots) = 0$, as $t_k \to 0$. Therefore:

$$\lim_{k \to \infty} \frac{\underline{z}_k - \underline{x}_*}{t_k} - \underline{d} = 0 .$$

$$\therefore \; \underline{d} = \lim_{k \to \infty} \frac{\underline{z}_k - \underline{x}_*}{t_k} .$$

Lastly, we check that $\underline{z}_k$ is a <u>feasible</u> sequence.

$$R(\underline{z}_k, t_k) = 0 .$$

First m rows:

$$c_i(\underline{z}_k) = t_k \left[A(x_*) \underline{d} \right]_i \qquad i \in \mathcal{A}(x_*) .$$

Since $\underline{d} \in$ LFDDs,

Since $\underline{d}$ ∈ LFDDs,

$$c_i(\underline{z}_k) = t_k \, (A\underline{d})_i$$

$$= t_k \, \underline{d} \cdot \nabla c_i(x_*)$$

$$= 0, \quad \text{for} \quad i \in \mathcal{E}$$

$$c_i(\underline{z}_k) = t_k \, \underline{d} \cdot \nabla c_i(\underline{x}_*)$$

$$\geq t_k(\overbrace{\geq 0}), \quad \text{for } i \in \mathcal{I} \cap \mathcal{A}(\underline{x}_*)$$

$$\geq 0 .$$

$\underline{z}_k$'s satisfy the constraints, therefore feasible.

Summarizing:

- Take $\underline{d} \in \mathcal{F}_\Omega(\underline{x}_*)$.
- Using the Implicit Function Theorem, we constructed a feasible sequence $\{\underline{z}_k\}_{k=0}^{\infty}$ and a sequence of scalars $\{t_k\}_{k=0}^{\infty}$ with $t_k > 0$ and $t_k \to 0$ as $k \to \infty$, such that

$$\underline{d} = \lim_{k \to \infty} \frac{\underline{z}_k - \underline{x}_*}{t_k}$$

- Hence, $\underline{d} \in T_\Omega(\underline{x}_*)$.

Part II: $\quad \underline{d} \in T_\Omega(\underline{x}_*) \Rightarrow \underline{d} \in \mathcal{F}_\Omega(\underline{x}_*)$

Part I : $\underline{d} \in T_\Omega(\underline{x}_*) \Rightarrow \underline{d} \in$

Part $\underline{II}$ $\underline{d} \in \mathcal{F}_\Omega(\underline{x}_*)$, LICQ $\Rightarrow \underline{d} \in T_\Omega(\underline{x}_*)$

So combining the two parts,

$T_\Omega(\underline{x}_*) = \mathcal{F}_\Omega(\underline{x}_*)$, provided the LICQ hold at $\underline{x}_*$