Optimization in Machine Learning (ACM 40990)

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Constrained Optimization #1

1. Consider the OP

$$\min(x+y) \qquad \text{subject to: } \begin{cases} c_1(\boldsymbol{x}) \geq 0, \\ c_2(\boldsymbol{x}) \geq 0, \end{cases}$$

   where $c_1(\boldsymbol{x}) = 1 - x^2 - (y-1)^2$ and $c_2 = -y$. Show that the LICQ does not hold at $\boldsymbol{x}_* = (0,0)^T$.

2. Consider the feasible set:

$$\Omega = \{\boldsymbol{x} \in \mathbb{R}^2 | y \geq 0,\, y \leq x^2\}.$$

   (a) For $\boldsymbol{x}_* = (0,0)^T$, write down $T_\Omega(\boldsymbol{x}_*)$ and $\mathcal{F}_\Omega(\boldsymbol{x}_*)$.
   (b) Is the LICQ satisfied at $\boldsymbol{x}_*$?
   (c) If the objective function is $f(\boldsymbol{x}) = -y$, verify that the KKT conditions are satisfied at $\boldsymbol{x}_*$.
   (d) Find a feasible sequence $\{\boldsymbol{z}_k\}_{k=0}^{\infty}$ approaching $\boldsymbol{x}_*$ with $f(\boldsymbol{z}_k) < f(\boldsymbol{x}_*)$, for all k.

3. Consider the half-space defined by:

$$H_\alpha = \{\boldsymbol{x} \in \mathbb{R}^n | \langle \boldsymbol{a}, \boldsymbol{x} \rangle + \alpha \geq 0\},$$

   where $\boldsymbol{a} \in \mathbb{R}^n$ is a constant non-zero vector and $\alpha \in \mathbb{R}$ is a constant scalar. Formulate and solve the OP for finding the point $\boldsymbol{x} \in H_\alpha$ with the smallest Euclidean norm.

4. Consider the following modification of the example in class notes. Her, t is a parameter that is fixed prior to solving the problem:

$$\min_{\boldsymbol{x} \in \mathbb{R}^2} f(\boldsymbol{x}),$$

   where

$$f(\boldsymbol{x}) = \left(x - \tfrac{3}{2}\right)^2 + (y-t)^4,$$

subject to:

$$\begin{bmatrix} 1-x-y \\ 1-x+y \\ 1+x-y \\ 1+x+y \end{bmatrix} \geq 0.$$

(a) For what values of t does the point $\boldsymbol{x}_* = (1,0)^T$ satisfy the KKT conditions?

(b) Show that when $t = 1$, only the first constraint is active at the solution and find the solution.

5. Solve the OP in Question 4 (part (ii)) numerically, using Matlab or Python. Compare your answer with the answer obtained previously.