Exercises in Optimization
(ACM 40990 / ACM41030)

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Exercises #3

Exercises #3 - BFGS again and Trust-Region Methods

1. A simple way to approximate the Hessian (i.e. simpler than BFGS) is to use the so-called **symmetric rank-1** formula, defined by:

$$B_{k+1} = B_k + \frac{(\boldsymbol{y}_k - B_k\boldsymbol{s}_k)(\boldsymbol{y}_k - B_k\boldsymbol{s}_k)^T}{\langle \boldsymbol{y}_k - B_k\boldsymbol{s}_k, \boldsymbol{s}_k\rangle}$$

Unfortunately, this formula does not guarantee that the approximate Hessian is positive-definite. However, you should:

(a) Check that the update satisfies the Secant equation:

$$B_{k+1}\boldsymbol{s}_k = \boldsymbol{y}_k,$$

where

$$\boldsymbol{s}_k = \boldsymbol{x}_{k+1} - \boldsymbol{x}_k \qquad \boldsymbol{y}_k = \nabla f_{k+1} - \nabla f_k.$$

(b) Check that B_k is a symmetric matrix, for all $k \in 0, 1, 2, \cdots$.

Furthermore,

(c) You should show that the inverted Hessians $H_k := B_k^{-1}$ satisfy:

$$H_{k+1} = H_k + \frac{(\boldsymbol{s}_k - H_k\boldsymbol{y}_k)(\boldsymbol{s}_k - H_k\boldsymbol{y}_k)^T}{(\boldsymbol{s}_k - H_k\boldsymbol{y}_k)^T\boldsymbol{y}_k}$$

Hint: Use the Sherman–Morrison formula. Suppose $A \in \mathbb{R}^{n\times n}$ is an invertible square matrix and $\boldsymbol{u}, \boldsymbol{v} \in \mathbb{R}^n$ are column vectors. Then $A + \boldsymbol{u}\boldsymbol{v}^T$ is invertible if and only if $1 + \langle \boldsymbol{v}, A^{-1}\boldsymbol{u}\rangle \neq 0$. In this case,

$$\left(A + \boldsymbol{u}\boldsymbol{v}^T\right)^{-1} = A^{-1} - \frac{A^{-1}\boldsymbol{u}\boldsymbol{v}^T A^{-1}}{1 + \langle \boldsymbol{v}, A^{-1}\boldsymbol{u}\rangle}.$$

2. Write a code (in whatever programming langauge) that uses the Trust-Region method (Dogleg method) to solve the Rosenbrock problem

$$f = 10(x_2 - x_1^2)^2 + (1 - x_1)^2.$$