Polyhedron:

$$S = \{ \underline{x} \in \mathbb{R}^n \mid A\underline{x} = \underline{b}, \quad C\underline{x} \leq \underline{d} \}$$

Let $\underline{x}$ and $\underline{y}$ be any two points in S:

$$A\underline{x} = \underline{b} \qquad (1a)$$

$$A\underline{y} = \underline{b} \qquad (1b)$$

Let $t \in [0,1)$ and take:

$$t\,(1a) + (1-t)(1b):$$

$$tA\underline{x} + (1-t)A\underline{y} = t\underline{b} + (1-t)\underline{b} = \underline{b}.$$

By linearity:

$$A\left[t\cdot\underline{x} + (1-t)\underline{y}\right] = \underline{b}.$$

Hence,

$$\left\{ \begin{aligned} & A\underline{x}(t) = \underline{b}, \\ & \underline{x}(t) = t\underline{x} + (1-t)\underline{y}, \\ & t \in [0,1) \end{aligned} \right\}$$

The second condition is entry-wise:

$$\sum_j C_{ij} x_j \leq d_i \qquad (2)$$

where i and j have "appropriate ranges".

Again, take $\underline{x}$ and $\underline{y}$ to be any two points in S satisfying (2):

$$\sum_j C_{ij} x_j \leq d_i \qquad (3a)$$

$$\sum_j C_{ij} y_j \leq d_i \qquad (3b)$$

Take $t(3a) + (1-t)(3b)$, where $t \in [0,1]$:

$$t \sum_j C_{ij} x_j + (1-t) \sum_j C_{ij} y_j \leq d_i$$

Hence, by linearity,

$$\sum_j C_{ij} \left[t x_j + (1-t) y_j \right] \leq d_i$$

hence

$$\sum_j C_{ij} [x(t)]_j \leq d_i .$$

Or in vector notation,

$$\underline{c} \leq \underline{x}(t) \leq \underline{d}, \quad t \in [0,1]$$

Hence $\underline{x}(t) \in S$, for all $t \in [0,1]$.

Hence finally, S is <u>convex</u>. $\blacksquare$