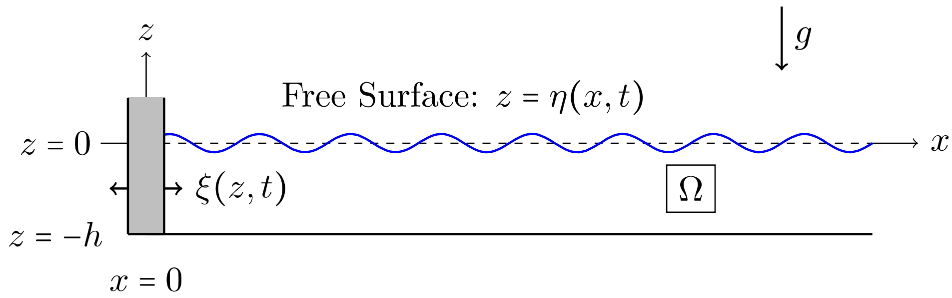


FIG. 1. Schematic diagram showing the generation of small-amplitude water waves by a piston wavemaker located at $x = 0$

## II. SPATIO-TEMPORAL ANALYSIS OF SMALL-AMPLITUDE WATER WAVES

In this section we develop the spatio-temporal theory of small-amplitude water waves. This theory describes the linear response of the free surface to a localized forcing corresponding to a wavemaker and as such, forms the basis of wavemaker theory. The theory has already been presented in the standard reference [5] and is included here for completeness, and to provide the proper context for the subsequent experimental and computational investigations.

For this purpose, we refer to the set-up in Figure 1, and take the direction of propagation along the $x$-axis, and the direction of oscillation along the $z$-axis. The free surface is therefore denoted by $z = \eta(x,t)$, where $z = 0$ represents the undisturbed free-surface height. Standard undergraduate texts describe a temporal theory [3], where the free surface is initialized to have a monochromatic sinusoidal profile $\eta(x, t = 0) \propto \sin(kx + \varphi)$ everywhere (here, $\varphi$ is a constant phase term). Here, we describe in detail the spatio-temporal theory, wherein the free surface is assumed to be undisturbed initially, but to undergo a localized forcing at $x = 0$ corresponding to the impact of a piston wavemaker.

To understand the setup of the spatio-temporal wave propagation, we refer to Figure 1. A piston located at $x = 0$ generates localized, impulsive forcing. The piston oscillates according to:

$$\xi(z,t) = \Re\left[-\frac{1}{\mathrm{i}\omega} f(z) \mathrm{e}^{-\mathrm{i}\omega t}\right]. \tag{2}$$

where $f(z)$ is a shape function describing the details of the back-and-forth motion of the piston. This can be left unspecified for now. Inside the domain $\Omega$, the flow is inviscid and irrotational, so potential theory applies:

$$\nabla^2 \Phi = 0, \qquad \boldsymbol{x} \in \Omega. \tag{3}$$

Here, $\Phi$ is the velocity potential, such that $\boldsymbol{u} = \nabla\Phi$. Also, the vector $\boldsymbol{x} = (x, z)$ is a two-dimensional vector. The boundary condition at $z = -h$ is the no-penetration condition, $w = 0$, hence:

$$\frac{\partial \Phi}{\partial z} = 0, \qquad z = -h. \tag{4}$$

### A. Conditions at the free surface

We next look at the boundary condition at the free surface $z = \eta$. Bernoulli's equation gives the pressure on the free surface as:

$$p = -\rho\frac{\partial\phi}{\partial t} - \tfrac{1}{2}\rho\boldsymbol{u}^2 - \rho g\eta + f(t), \tag{5}$$

where $f(t)$ is a parameter associated with Bernoulli's principle. We assume that the wave amplitude is small in comparison to the water depth $h$. This introduces a small parameter $\epsilon = \max(\eta)/h$ into the problem. Thus, disturbances, whether of amplitude, pressure, velocity or streamfunction are proportional to $\epsilon$, whereas products of disturbances (such as $\boldsymbol{u}^2$) are proportional to $\epsilon^2$ and can be neglected in a small-amplitude approximation. Thus, the pressure on the free surface can be approximated as:

$$p = -\rho\frac{\partial\Phi}{\partial t} - \rho g\eta + f(t). \tag{6}$$

From Reference [3], the pressure condition at the interface for an inviscid flow is:

$$p_{atm} - p = \gamma\kappa, \tag{7}$$

where $\kappa = \eta_{xx}/(1+\eta_x^2)^{3/2}$ is the mean curvature and $p_{atm}$ is the atmospheric pressure. In the small-amplitude approximation, this reduces to:

$$p_{atm} - p = \gamma\eta_{xx}. \tag{8}$$

Using Equation (6), this becomes:

$$\rho\frac{\partial\Phi}{\partial t} + \rho g\eta + [p_{atm} - f(t)] = \gamma\eta_{xx}, \qquad z = \eta. \tag{9}$$

Since $f(t)$ is arbitrary, we set $f(t) = p_{atm}$, leaving:

$$\rho\frac{\partial\Phi}{\partial t} + \rho g\eta = \gamma\eta_{xx}, \qquad z = \eta. \tag{10}$$

However, we may expand $\Phi(z = \eta) = \Phi(z = 0) + (\partial\Phi/\partial z)_{z=0}\eta + O(\eta^2)$. Because of the small-amplitude approximation, we can replace $\Phi(z = \eta)$ with $\Phi(z = 0)$, and similarly for derivatives, giving

$$\rho\frac{\partial\Phi}{\partial t} + \rho g\eta = \gamma\eta_{xx}, \qquad z = 0. \tag{11}$$

The difference between Equations (10) and (11) is subtle but it enables a great simplification in the foregoing analysis.

We now make the standard transformations:

$$\Phi = \Re\left[\phi(\boldsymbol{x})\mathrm{e}^{-\mathrm{i}\omega t}\right], \tag{12a}$$

$$\eta = \Re\left[\widehat{\eta}(x)\mathrm{e}^{-\mathrm{i}\omega t}\right]. \tag{12b}$$

We henceforth drop the hat on $\widehat{\eta}$. Thus, we use the same symbol for $\eta$ (which depends on $x$ and $t$), and $\widehat{\eta}$ (which depends on $x$ only). It should be clear from context which variable is being used. In this way, Equation (11) becomes:

$$\rho\mathrm{i}\omega\phi = \rho g\eta - \frac{\gamma}{\rho}\eta_{xx}, \qquad z = 0. \tag{13}$$

A second interfacial condition is the kinematic condition. In the small-amplitude approximation, which states that the interface moves with the flow, hence:

$$\frac{\partial \eta}{\partial t} + u\frac{\partial \eta}{\partial x} = w, \qquad z = \eta. \tag{14}$$

As with Equation (11), we linearize this identity on to the surface $z = 0$, which gives:

$$\frac{\partial \eta}{\partial t} = w \qquad z = 0, \tag{15}$$

or $\partial_t \eta = \partial_z \Phi$ on $z = 0$, hence:

$$-\mathrm{i}\omega\eta = \frac{\partial \phi}{\partial z}, \qquad z = 0. \tag{16}$$

We combine Equations (13)–(16). First, Equation (16) gives $\eta = -1/(\mathrm{i}\omega)\phi_z$. We substitute this into Equation (13) to obtain a single boundary condition at $z = 0$:

$$\omega^2 \phi = g\frac{\partial \phi}{\partial z} - \frac{\gamma}{\rho}\partial_{xx}\frac{\partial \phi}{\partial z}, \qquad z = 0. \tag{17}$$

### B. Solving Laplace's Equation

We solve $\nabla^2 \phi = 0$ in the linearized domain $\Omega_L = \{(x,z)| - h < z < 0\}$. We do separation of variables to get $\phi(x,z) = X(x)Z(z)$. Following standard steps, we get:

$$\frac{X''}{X} = -\frac{Z''}{Z} = k^2. \tag{18}$$

We look at the boundary conditions at $z = 0$ next. The boundary condition (17) gives:

$$\omega^2 X(x) Z(0) = \left(gX(x) - \frac{\gamma}{\rho}X''(x)\right) Z'(0). \tag{19}$$

We use the separation-of-variables condition (18) to reduce this to:

$$\omega^2 Z(0) = \left(g - \frac{\gamma}{\rho}k^2\right) Z'(0). \tag{20}$$

We further re-write this as:

$$Z'(0) = \alpha_k Z(0), \qquad \alpha_k = \frac{\omega^2}{g - \frac{\gamma}{\rho}k^2}. \tag{21}$$

Putting it all together, we have to solve:

$$Z'' + k^2 Z = 0, \tag{22a}$$

$$Z'(-h) = 0, \tag{22b}$$

$$Z'(0) = \alpha_k Z(0). \tag{22c}$$

The solution is:

$$Z = \frac{\cos[k(z+h)]}{\cos kh}, \tag{23}$$

with solvability condition $k\tan(kh) = -\alpha_k$, or:

$$k\tan(kh) = -\frac{\omega^2}{g - \frac{\gamma}{\rho}k^2}. \tag{24}$$

We label the solutions of Equation (25) as $k_n$, where $n \in \{0, 1, 2, \cdots\}$.

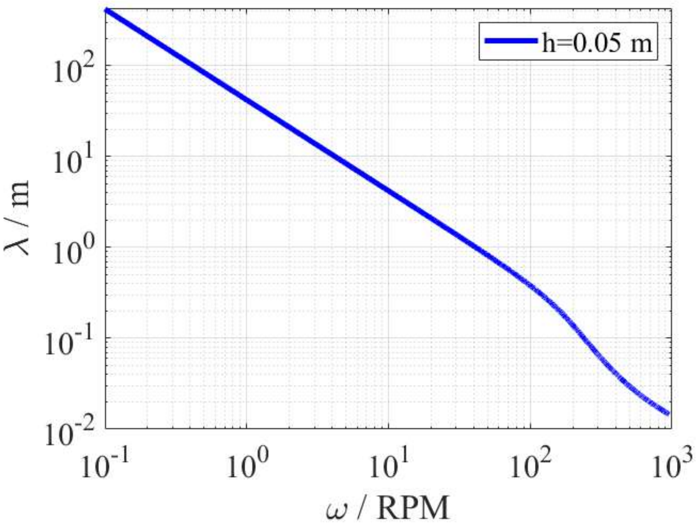


FIG. 2. The dispersion relation (25). For a given $\omega$, there is a uniquely determined $k$-value, hence a uniquely determined wavelength $\lambda = 2\pi/k$. Parameter values: $h = 0.05\,\mathrm{m}$, $\rho = 1000$, $\mathrm{kg}\cdot\mathrm{m}^{-3}$, $g = 9.8\,\mathrm{m}\cdot\mathrm{s}^{-2}$, $\gamma = 0.072\,\mathrm{N}\cdot\mathrm{m}^{-1}$.

### C. Dispersion Relation

Equation (24) has two solution types:

- Case 1. This corresponds to $n = 0$, so we are dealing with $k_0$. In this case, $k_0$ is purely imaginary, and we write $k_0 = \pm \mathrm{i}\kappa$, where $\kappa$ is real. Using the properties of trigonometric functions, Equation (25) reduces to:

$$\kappa \tanh(\kappa h) = \frac{\omega^2}{g + \frac{\gamma}{\rho}\kappa^2}, \tag{25}$$

  which is precisely Equation (1). In this case, however, $\omega$ is known, and $\kappa$ has to be obtained by inversion. A sample dispersion curve is shown in Figure 2.

- Case 2. In this case, we look at $k_n$, where $n \geq 1$. A standard graphical eigenvalue analysis shows in this case there are infinitely many real positive roots, confirming that $n \in \{1, 2, \cdots\}$.

Putting the two cases together, we have the following set of eigenfunctions, with $Z(z)$ being replaced by $\chi_n(z)$:

$$\chi_n(z) = \begin{cases} \frac{\cos[k_n(z+h)]}{\cos k_n h}, & n \geq 1 \\ \frac{\cosh[\kappa(z+h)]}{\cosh \kappa h}, & n = 0. \end{cases} \tag{26}$$

As these are eigenfunctions of a self-adjoint operator, we have an orthogonality relation

$$\int_{-h}^{0} \chi_m(z)\chi_n(z)\mathrm{d}z = C_n \delta_{nm}. \tag{27}$$

In particular,

$$C_0 = \frac{1}{4\kappa}\frac{1}{\cosh^2(\kappa h)}\left[2\kappa h + \sinh(2\kappa h)\right]. \tag{28}$$

### D. General Solution

The general solution for the velocity potential can now be written as:

$$\phi(x,z) = \sum_{n=1}^{\infty} a_n \chi_n(z) \mathrm{e}^{-k_n x} + a_0 \chi_0(z) \mathrm{e}^{\mathrm{i}\kappa x}. \tag{29}$$

Notice that we do not allow for a contribution proportional to $\mathrm{e}^{-\mathrm{i}\kappa x}$, as this would correspond to a wave travelling inward from positive infinity, which is not physical. Notice also that we rule out intrinsically negative eigenvalues $k_n$ ($n \geq 1$) as well. Thus, the Sommerfeld Radiation condition $\partial\phi/\partial x \sim \mathrm{i}k\phi$ is satisfied as $x \to \infty$. Furthermore, at $x = 0$, we have:

$$\left(\frac{\partial \phi}{\partial x}\right)_{(x=0,z)} = \sum_{n=1}^{\infty} a_n \chi_n(z)(-k_n) + a_0 \chi_0(z)(\mathrm{i}\kappa). \tag{30}$$

The boundary condition at $x = 0$ is $\partial_x \phi = u = \partial_t \xi$, where $\xi$ is the displacement of the wall at $x = 0$ (*cf.* Equation (2)). Thus, we obtain:

$$\sum_{n=1}^{\infty} a_n \chi_n(z)(-k_n) + a_0 \chi_0(z)(-\mathrm{i}\kappa) = f(z). \tag{31}$$

Hence, the coefficients $a_0$ and $a_n$ can be determined from:

$$\begin{aligned} a_0 &= \frac{1}{(\mathrm{i}\kappa)C_0} \int_{-h}^{0} f(z)\chi_0(z)\mathrm{d}z, \\ a_n &= \frac{1}{(-k_n)C_n} \int_{-h}^{0} f(z)\chi_n(z)\mathrm{d}z, \qquad n \geq 1. \end{aligned}$$

In particular, for a piston wavemaker with $f(z) = f_0 = \text{Const.}$, we have:

$$a_0 = \frac{f_0}{(\mathrm{i}\kappa)C_0} \frac{1}{\kappa} \frac{\sinh(\kappa h)}{\cosh(\kappa h)}. \tag{32}$$

Furthermore, in the far field, we have

$$\phi \sim a_0 \chi_0(z) \mathrm{e}^{\mathrm{i}\kappa x}, \qquad x \to \infty, \tag{33}$$

since $\mathrm{e}^{-k_n x} \to 0$ as $x \to \infty$, for $n \geq 1$. Only the oscillatory wave with dispersion relation (25) survives far downstream of the disturbance.

### E. Results

By analysing the dispersion relation (25), we can see what type of wavelengths can be expected for a given forcing frequency. The wavelengths depend sharply on depth, as shown in Table I.

A further key quantity of interest is the height-to-stroke ratio, which we derive now for the piston wavemaker as follows. We apply the kinematic condition (16) in the far field (for $x \to \infty$) to get

$$a_0 \left(\frac{\partial \chi_0}{\partial z}\right)_{z=0} = -\mathrm{i}\omega\eta_0. \tag{34}$$

| $\omega$ (RPM) | $\lambda$ ($h = 0.05\,\mathrm{m}$) | $\lambda$ ($h = 0.1\,\mathrm{m}$) |
|---|---|---|
| 10 | 4.19 | 5.91 |
| 50 | 0.820 | 1.13 |
| 100 | 0.381 | 0.484 |
| 200 | 0.139 | 0.142 |

TABLE I. Expected wavelengths (in metres), based on the dispersion relation (25). Depths: $h = 0.05\,\mathrm{m}$ and $0.1\,\mathrm{m}$. Other parameters as in Figure 2.

Here, we have decomposed $\eta(x)$ into a phase $\eta_0$ and the complex exponential $\mathrm{e}^{-\mathrm{i}\kappa x}$, corresponding to the $n = 0$ normal model. We fill in for $\chi_0(z)$ (*cf.* Equation (26)) to get

$$\frac{f_0}{-\mathrm{i}\kappa\, C_0} \frac{\sinh^2(\kappa h)}{\cosh^2(\kappa h)} = -\mathrm{i}\omega\eta_0. \tag{35}$$

For a piston wavemaker, we have $f_0 = \omega A \mathrm{e}^{\mathrm{i}\varphi}$, where $A$ is the amplitude of the back-and-forth motion of the piston (and equal to half the stroke, $2A = S$), and $\varphi$ is a constant phase. This gives:

$$\left|\frac{\eta}{A}\right| = \frac{1}{\kappa C_0} \frac{\sinh^2(\kappa h)}{\cosh^2(\kappa h)}, \tag{36}$$

and filling in for $C_0$ gives:

$$\left|\frac{\eta_0}{A}\right| = \frac{4\sinh^2(\kappa h)}{2\kappa h + \sinh(2\kappa h)}. \tag{37}$$

We identify the height of the wave $H = 2|\eta_0|$, hence $|\eta/A| = |2\eta/(2A)| = H/S$. This gives the required height-to-stroke ratio in the far field, valid for a piston wavemaker:

$$\frac{H}{S} = \frac{4\sinh^2(\kappa h)}{2\kappa h + \sinh(2\kappa h)}. \tag{38}$$