Here, again, $\eta = \eta_0 \mathrm{e}^{\mathrm{i}kx+\sigma t}$ and η_0 is the phase, determined by the kinematic condition

$$\sigma\eta_0 = -\mathrm{i}k\Psi(0). \tag{3.11g}$$

Finally, the following boundary conditions apply:

$$\psi_L, \partial_z\psi_L \to 0 \text{ as } z \to \infty, \qquad \psi_G, \partial_z\psi_G \to 0 \text{ as } z \to -\infty. \tag{3.11h}$$

These are the equations we now solve. Although it is possible to solve these equations semi-analytically (e.g. [Cha61]), it is arguably more instructive to solve them numerically. We introduce a convenient numerical method in the next chapter – numerical spectral methods.

3.4 Rayleigh–Taylor instability – Further Examples

We postpone numerical exercises on the Rayleigh–Taylor instability until the next chapter. However, in order to develop a more in-depth understanding of the Rayleigh–Taylor instability, we present below some analytical exercises on example systems where the same basic instability mechanism is in evidence.

Exercise 3.1: Rayleigh–Taylor instability in a long-wave model

We consider a thin liquid film of mean thickness h_0 which adheres to the top of a solid wall, as in Figure 3.3. The film is initially perturbed by a sinusoidal wave of wavelength k. In the so-called long-wave limit where $kh \ll 1$, it is possible to reduce the Navier–Stokes equations for both phases down to a single equation for the evolution of the film height, given here as

$$\mu\frac{\partial h}{\partial t} + \tfrac{1}{3}\rho g\frac{\partial}{\partial x}\left(h^3\frac{\partial h}{\partial x}\right) + \tfrac{1}{3}\gamma\frac{\partial}{\partial x}\left(h^3\frac{\partial^3 h}{\partial x^3}\right) = 0, \tag{3.12}$$

where μ is the liquid velocity, ρ is the liquid density, and γ is the surface tension between the liquid and the gas. For a full derivation of Equation (3.12), see Reference [ODB97]; the equation can be used as a model for pendant drops such as in Figure 3.4.

Identify the mean film height h_0 as the base state. Identify a solution $h(x,t) = h_0 + \delta h(x,t)$, where $|\delta h| \ll h_0$. Perform the normal-mode decomposition $\delta h(x,t) \propto \mathrm{e}^{\mathrm{i}kx+\sigma t}$. Hence, show that in linear theory, the dispersion relation $\sigma(k)$ is given by

$$\sigma(k) = \tfrac{1}{3\mu}h_0^3\left(\rho g - \gamma k^2\right)k^2.$$

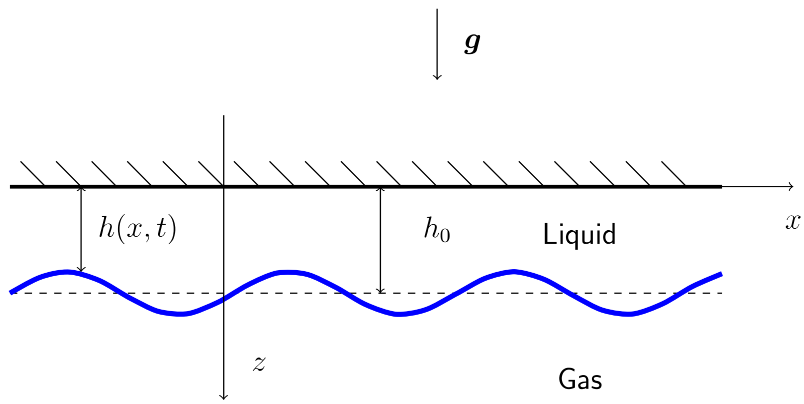

Figure 3.3: Definition sketch for Exercise 3.4

Figure 3.4: An every day example of pendant drops

Exercise 3.2: Superposed fluids confined in a vertical channel

Consider an inviscid incompressible fluid of density ρ_1 at rest beneath a similar fluid of density ρ_2, the fluids being confined by a long vertical rectangular channel of sides of length L. Surface tension γ acts at the interface with equation $z = 0$. Here standard Cartesian coordinates (x, y, z) are used and Oz is the upward vertical. Then show, much as in the analysis throughout this chapter, that small irrotational disturbances of the state of rest may be found in terms of the normal modes of the form

$$\phi = \cos[(m\pi/L)x]\cos[(n\pi/L)y]\mathrm{e}^{-k|z|+\sigma t}.$$

Here, ϕ is the velocity potential of the disturbance, where $m, n = 0, 1, 2, \cdots$ (but not both $m = 0$ and $n = 0$), and where the wavenumber k is determined by the condition $k^2 =$

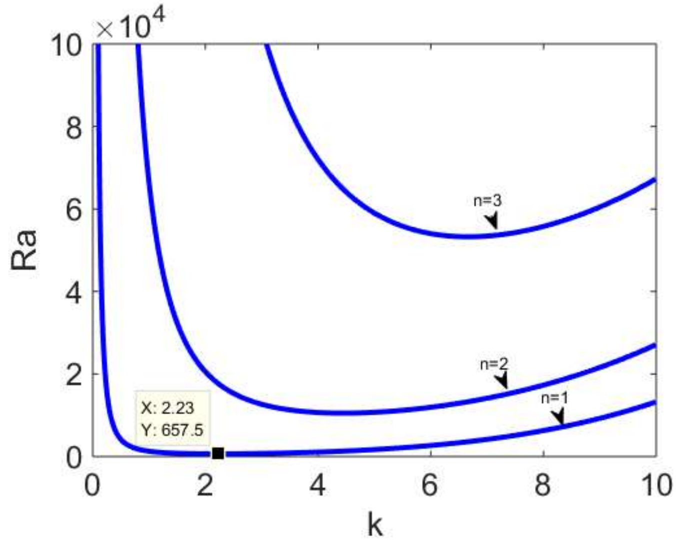

Figure A.2: The family of curves $\mathrm{Ra}\, k^2 = (q_n^2 + k^2)^3$ for $n = 1, 2, 3$.

only the minimum of the neutral curve:

$$1 = \frac{(\pi^2 + k^2)^3}{3k^2(\pi^2 + k^2)^2}.$$

This gives $2k^2 = \pi^2$. Back-substitution into Equation (A.21) then gives $\mathrm{Ra} = (27/4)\pi^4$ as the minimum of the neutral curve – i.e. the critical Rayleigh number for the onset of convection in the system bounded by two free surfaces, as required.

A.3 Solutions of exercises in Chapter 3

1. We consider a thin liquid film of mean thickness h_0 which adheres to the top of a solid wall, as in Figure 3.3 in the main text. The film is initially perturbed by a sinusoidal wave of wavelength k. In the so-called long-wave limit where $kh \ll 1$, it is possible to reduce the Navier–Stokes equations for both phases down to a single equation for the evolution of the film height, given here as

$$\mu \frac{\partial h}{\partial t} + \tfrac{1}{3}\rho g \frac{\partial}{\partial x}\left(h^3 \frac{\partial^2 h}{\partial x}\right) + \tfrac{1}{3}\gamma \frac{\partial}{\partial x}\left(h^3 \frac{\partial^3 h}{\partial x^3}\right) = 0, \qquad \text{(A.23)}$$

where μ is the liquid velocity, ρ is the liquid density, and γ is the surface tension between the liquid and the gas (For a full derivation of Equation (A.23), see Reference [ODB97]).
We are asked to: identify the mean film height h_0 as the base state; identify a solution $h(x, t) = h_0 + \delta h(x, t)$, where $|\delta h| \ll h_0$; perform the normal-mode decomposition $\delta h(x, t) \propto$

$\mathrm{e}^{\mathrm{i}kx+\sigma t}$. Hence, we are asked to show that in linear theory, the dispersion relation $\sigma(k)$ is given as follows:

$$\sigma(k) = \tfrac{1}{3\mu} h_0^3 \left(\rho g - \sigma k^2\right) k^2.$$

We linearize Equation (A.23) as instructed and obtain the following evolution equation:

$$\mu \frac{\partial}{\partial t}\delta h + \tfrac{1}{3}\rho g \frac{\partial}{\partial x}\left(h_0^3 \frac{\partial}{\partial x}\delta h\right) + \tfrac{1}{3}\gamma \frac{\partial}{\partial x}\left(h_0^3 \frac{\partial^3}{\partial x^3}\delta h\right) = 0. \tag{A.24}$$

We substitute in the proposed trial solution $\delta h(x,t) = \epsilon \mathrm{e}^{\mathrm{i}kx+\sigma t}$, where ϵ is a small amplitude. The action of the derivatives $\partial/\partial t$ and $\partial/\partial x$ on the perturbation δh can be written as multiplication by σ and $\mathrm{i}k$ respectively. Hence, we are left with

$$\left\{\mu\sigma + \tfrac{1}{3}\rho g(\mathrm{i}k)\left[h_0^3(\mathrm{i}k)\right] + \tfrac{1}{3}\gamma(\mathrm{i}k)\left[h_0^3(\mathrm{i}k)^3\right]\right\}\epsilon \mathrm{e}^{\mathrm{i}kx+\sigma t} = 0. \tag{A.25}$$

The factor $\epsilon \mathrm{e}^{\mathrm{i}kx+\sigma t}$ can be cancelled to yield the required solution for σ:

$$\sigma = \tfrac{1}{3\mu} h_0^3 \left(\rho g - \gamma k^2\right) k^2.$$

This gives the required functional form for the dependence of σ on k, i.e. the dispersion relation.

Notice that the most-dangerous mode occurs when $\mathrm{d}\sigma/\mathrm{d}k = 0$, hence

$$k = \sqrt{\frac{\rho g}{2\gamma}},$$

which for water ($\rho = 1000\,\mathrm{kg}\cdot\mathrm{m}^{-3}$, $g = 9.8\,\mathrm{m}\cdot\mathrm{s}^{-2}$, $\gamma = 0.072\,\mathrm{N}\cdot\mathrm{m}^{-1}$) works out as $k = 260.87\ldots\,\mathrm{m}^{-1}$, hence $\lambda = 2\pi/k \approx 2.4\,\mathrm{mm}$.

2. Consider an inviscid incompressible fluid of density ρ_1 at rest beneath a similar fluid of density ρ_2, the fluids being confined by a long vertical rectangular channel of sides of length L. Surface tension γ acts at the interface with equation $z = 0$. Here standard Cartesian coordinates (x, y, z) are used and Oz is the upward vertical. Assuming the flow is irrotational, the small-amplitude disturbances have a velocity potential; we are asked to how that this is given in terms of normal modes as follows:

$$\phi = \cos[(m\pi/L)x]\cos[(n\pi/L)y]\mathrm{e}^{-k|z|+\sigma t}.$$

Here, $m, n = 0, 1, 2, \cdots$ (but not both $m = 0$ and $n = 0$), and the wavenumber k is determined by the condition $k^2 = (\pi/L)^2(m^2 + n^2)$. We are asked also to show that σ