Advanced Fluid Mechanics (ACM 40740) – Assignment 1

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1. The linearized equations of motion for the Rayleigh–Bénard convection problem read

$$
\begin{aligned}
\frac{\partial u_i}{\partial t} &= -\frac{\partial}{\partial x_i}\left(\frac{\delta p}{\rho_0}\right) + \delta_{i,z} g\alpha\theta + \nu\nabla^2 u_i, && \text{(1a)}\\
\frac{\partial u_i}{\partial x_i} &= 0, && \text{(1b)}\\
\frac{\partial \theta}{\partial t} &= w\beta + \kappa\nabla^2\theta. && \text{(1c)}
\end{aligned}
$$

Here, δp is the perturbation pressure. Prove Equation (1) by carrying out the relevant linearization.

Solution: Start with the Boussinesq approximation:

$$
\rho_0\left(\frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{u}\cdot\nabla\boldsymbol{u}\right) = -\nabla p + \mu\nabla^2\boldsymbol{u} - g\left[\rho_0 - \rho_0\alpha\left(T - T_0\right)\right]\hat{\boldsymbol{z}}
$$

These are the 'full' equations of motion. However, we now consider a solution that represents a perturbation from the base state. As such, we henceforth let $\boldsymbol{u}$ denote a small-amplitude perturbation velocity. Correspondingly,

$$
p = \underbrace{-\rho_0 g\left(z + \tfrac{1}{2}\alpha\beta z^2\right)}_{\text{base state}} + \underbrace{\delta p}_{\text{perturbation}}
$$

is the pressure, and

$$
T = \underbrace{\left(T_0 - \beta z\right)}_{\text{base state}} + \underbrace{\theta}_{\text{perturbation}}
$$

is the temperature. To simplify the following presentation, we will write $p = p_0 + \delta p$ for the pressure, where p_0 is the base pressure and δp is the perturbed pressure. Then, the Boussinesq equation becomes

$$
\rho_0\left(\frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{u}\cdot\nabla\boldsymbol{u}\right) = -\nabla\delta p + \mu\nabla^2\boldsymbol{u} + g\rho_0\alpha\theta\hat{\boldsymbol{z}} + \left[-\nabla p_0 - g\rho_0\hat{\boldsymbol{z}}\right].
$$

By hydrostatic balance, $-\nabla p_0 - g\rho_0\hat{\boldsymbol{z}} = 0$. Also, the term $\boldsymbol{u}\cdot\nabla\boldsymbol{u}$ is omitted, by linearization. Hence, the linearized Boussinesq equation reads

$$
\frac{\partial \boldsymbol{u}}{\partial t} = -\nabla(\delta p/\rho_0) + \nu\nabla^2\boldsymbol{u} + g\alpha\theta\hat{\boldsymbol{z}}, \qquad \nu = \mu/\rho_0. \tag{2a}
$$

The incompressibility condition is trivial:

$$\nabla \cdot \boldsymbol{u} = 0. \tag{2b}$$

The full advection-diffusion for the temperature $T' = (T_0 - \beta z) + \theta$ is

$$\frac{\partial \theta}{\partial t} + \boldsymbol{u} \cdot \nabla \left[(T_0 - \beta z) + \theta\right] = \kappa \nabla^2 [(T_0 - \beta z) + \theta].$$

The base-state temperature distribution solves Laplace's equation, $\nabla^2 (T_0 - \beta z) = 0$ Also, by linearization, the term $\boldsymbol{u} \cdot \nabla \theta$ is omitted. Thus, we are left with

$$\frac{\partial \theta}{\partial t} + w \frac{dT_0}{dz} = \kappa \nabla^2 \theta,$$

or

$$\frac{\partial \theta}{\partial t} = w\beta + \kappa \nabla^2 \theta, \tag{2c}$$

2. Show that the Rayleigh number is dimensionless.

   Start with

$$\mathrm{Ra} = \frac{g\alpha\beta d^4}{\nu\kappa}.$$

   Only α and β have somewhat unusual units. From $\delta\rho = -\rho_0 \alpha (T - T_0)$, the constant α has units of [Temperature]$^{-1}$. From $T = T_0 - \beta z$, β has units of [Temperature]$/L$. Thus, $\alpha\beta$ has units of $1/L$. So,

$$\begin{aligned} [\mathrm{Ra}] &= \frac{(L/T^2)(1/L)L^4}{(L^2/T)^2}, && (3) \\ &= \frac{L^4/T^2}{L^4/T^2}, && (4) \\ &= 1. && (5) \end{aligned}$$