Week 1, Lecture 3 24/01/2024

We resume our discussion of Picard's Theorem.
We have an approximate solution of the ODE

$$y'(x) = F(x, y(x)), \quad y(x_0) = y_0.$$

This is:

$$y_{n+1}(x) = y_0 + \int_{x_0}^{x} F(x, y_n(x))\,dx$$

with starting value

$$y_0(x) = \text{Constant } f^{n} = y_0.$$

The aim is to show:

- $y_n(x)$ converges to a limit as $n \to \infty$
- The limit solves the ODE
- This is the unique sl^{n} of the ODE.

The trick is to introduce

$$u_0(x) = y_0$$
$$u_n(x) = y_n(x) - y_{n-1}(x) \quad n \in \{1, 2, \ldots\}.$$

The aim is to put a bound

$$|u_n(x)| \leq M_n$$

on each u_n and hence to apply the Weierstrass M-test.

We have:

$$|u_1(x)| = |y(x) - y_0|$$

$$= \left| \int_{x_0}^{x} F(x, y_0)\, dx \right|$$

$$\overset{\Delta}{\leq} \int_{x_0}^{x} |F(x, y_0)|\, dx$$

Since F is continuous in each of its variables, we have:

$$|F(x, y)| \leq M \quad \forall x \in [a, b] \ \forall y \in \mathbb{R}$$

Hence,

$$|u_1(x)| \leq \int_{x_0}^{x} M\, dx$$

Without loss of generality, choose $x > x_0$ giving:

$$|u_1(x)| \leq M|x - x_0|$$

Next:

$$|u_2(x)| = \cancel{\Big| \int_{x_0}^{x} F(x,}$$

$$= |y_2(x) - y_1(x)|$$

$$= \left| y_0 + \int_{x_0}^{x} F(x, y_1(x))\, dx - \left(y_0 + \int_{x_0}^{x} F(x, y_0)\, dx \right) \right|$$

$$= \left| \int_{x_0}^{x} [F(x, y_1(x)) - F(x, y_0)]\, dx \right|$$

Hence:

$$|u_2(x)| = \left| \int_{x_0}^{x} [F(x, y_1(x)) - F(x, y_0)]\,dx \right|$$

$$\overset{\Delta}{\leq} \int_{x_0}^{x} |F(x, y_1(x)) - F(x, y_0)|\,dx$$

$$\overset{\text{Lipschitz}}{\leq} \int_{x_0}^{x} K\,|y_1(x) - y_0|\,dx$$

$$= \int_{x_0}^{x} K\,|u_1(x)|\,dx$$

$$\leq \int_{x_0}^{x} K \cdot M\,|x - x_0|$$

$$= \frac{1}{2} K M\,|x - x_0|^2$$

$$\therefore \boxed{|u_2(x)| \leq \frac{1}{2} M\,K\,|x - x_0|^2}$$

Guess the pattern:

$$|u_n(x)| \leq \frac{1}{n!} M\,K^{n-1}\,|x - x_0|^n$$

Since $|x - x_0| \leq |b - a|$ we have:

$$|u_n(x)| \leq \overbrace{\frac{1}{n!} M\,K^{n-1}\,|b - a|^n}^{=M_n}$$

Hence:

$$\sum_{n=0}^{\infty} M_n = \frac{M}{K} \sum_{n=0}^{\infty} K^n |b-a|^n / n!$$

$$= \frac{M}{K} e^{K|b-a|} < \infty.$$

So, in conclusion:

- $|u_n(x)| \leq M_n$
- $\sum_{n=0}^{\infty} M_n < \infty$

Hence, by the Weierstrass M-test,

$$\sum_{r=0}^{n} u_r(x) \longrightarrow \text{(Limit function)}$$

uniformly, as $n \to \infty$.

But

$$\sum_{r=0}^{n} u_r(x) \overset{\text{telescoping}}{=} y_n(x).$$

Hence,

$$\sum_{r=0}^{n} u_r(x) = y_n(x) \xrightarrow{\text{uniformly}} y(x).$$

<u>We have shown the first</u> part.

The aim next is to show that the limit function $y(x)$ solves the ODE.

We have: which we want to show is zero

$$\left| y(x) - y_0 - \int_{x_0}^{x} F(x, y(x)) \right|$$

$$= \left| y(x) - \cancel{y_0} - \int_{x_0}^{x} F(x, y(x)) - \left(y_{N+1}(x) - \cancel{y_0} - \int_{x_0}^{x} F(x, y_N(x))\,dx \right) \right|$$

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$$= \left| y(x) - y_{N+1}(x) - \int_{x_0}^{x} [F(x, y(x)) - F(x, y_N(x))]\, dx \right|$$

$$\overset{\Delta}{\leq} |y(x) - y_{N+1}(x)|$$

$$+ \int_{x_0}^{x} |F(x, y(x)) - F(x, y_N(x))|\, dx$$

$$\leq \underbrace{\epsilon}_{\text{TAIL}} + \int_{x_0}^{x} K\, |y(x) - y_N(x)|\, dx$$

$$\leq \epsilon + \epsilon \cdot K \int_{x_0}^{x} 1\, dx$$

$$= \epsilon + \epsilon \cdot K \cdot |x - x_0|$$

$$\leq \epsilon + \epsilon \cdot K \cdot |b - a|$$

Since ϵ can be made arbitrarily small, it follows that $|\ldots|$ which we started with is zero, hence:

$$\left| y(x) - y_0 - \int_{x_0}^{x} F(x, y(x))\, dx \right| = 0$$

hence $y(x)$ solves the ODE.

Uniqueness:

Suppose that $y(x)$ and $Y(x)$ both solve the ODE with the same initial conditions.

We have:

$$|Y(x)-y(x)| = \left|y_0 + \int_{x_0}^{x} F(x, Y(x))\,dx - \left(y_0 + \int_{x_0}^{x} F(x, y(x))\,dx\right)\right|$$

$$= \left|\int_{x_0}^{x} [F(x, Y(x)) - F(x, y(x))]\,dx\right|$$

$$\overset{\Delta}{\leq} \int_{x_0}^{x} |F(x, Y(x)) - F(x, y(x))|\,dx$$

$$\overset{\text{Lipschitz}}{\leq} \int_{x_0}^{x} K\,|Y(x) - y(x)|\,dx \quad (*)$$

Now, $y(x)$ and $Y(x)$ must be continuous (indeed, differentiable), so the difference $|Y(x)-y(x)|$ is bounded:

$$|Y(x) - y(x)| \leq C \quad \forall\, x \in [a,b].$$

Hence, going back to $(*)$, we have:

$$|Y(x)-y(x)| \leq K \cdot C\,|x-x_0|$$

We iterate this procedure by plugging back into $(*)$:

$$|Y(x) - y(x)| \leq K \int_{x_0}^{x} K C |x - x_0|^2 \, dx$$

$$= \frac{1}{2} K^2 C |x - x_0|^2$$

Continue iterating, to get:

$$|Y(x) - y(x)| \leq \frac{1}{n!} K^n |x - x_0|^n \cdot C$$

$$\leq \frac{1}{n!} K^n |b - a|^n \cdot C$$

Everything on the RHS is constant and bounded and x- and y- independent, so taking $n \to \infty$ we get:

$$|Y(x) - y(x)| \leq 0$$

Hence:

$Y(x) = y(x)$ and the solution is unique.