Chapter 1 – Lecture notes

Ordinary Differential Equations (ODEs)

Generic Initial Value Problem (IVP):

- One independent variable x
- One dependant variable y, $y(x)$

Generic IVP:

$$y'(x) = F(x, y(x)) \qquad (1)$$

Eq. (1) is posed on an interval $I = [a,b]$, with initial condition $y(x_0) = y_0$, $x_0 \in I$.

Question: Can we solve (1) by Taylor Series?

Attempt:

$$y(x) = \overbrace{y(x_0)}^{y_0} + y'(x_0)(x - x_0) + \tfrac{1}{2} y''(x_0)(x-x_0)^2 + \dots \qquad (2)$$

$$y'(x) \overset{\text{Eq.(1)}}{=} F(x, y(x))$$

$$\Rightarrow y'(x_0) = F(x_0, y_0) \quad \checkmark$$

$$y''(x) = \frac{d}{dx} y'(x)$$

$$\overset{\text{Eq.(1)}}{=} \frac{d}{dx} F(x, y(x))$$

$$= \frac{\partial F}{\partial x} + \frac{\partial F}{\partial y} \frac{dy}{dx} \xrightarrow{\text{ODE}}$$

$$\Rightarrow y''(x_0) \overset{\text{Chain Rule}}{=} \frac{\partial F}{\partial x}(x_0, y_0) + \frac{\partial F}{\partial y}(x_0, y_0)\, F(x_0, y_0) \quad \checkmark$$

Iterate for $y'''(x_0)$ etc. CLOSED PROBLEM.

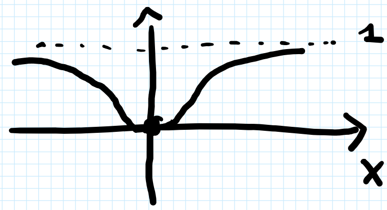
Problem: We don't know what is the radius of convergence of the Taylor series (2).

Even worse, the radius of convergence can be zero.

§1.1.1 — Pathological Example

$$y'(x) = F(x), \quad \text{where}$$

$$F(x) = \begin{cases} e^{-1/x^2}, & x \neq 0 \\ 0, & x = 0 \end{cases}.$$



By direct differentiation,

$$y^{(n)}(0) = 0, \quad n = 0, 1, 2, \ldots$$

So, by setting $x_0 = 0$ and $y_0 = 0$ corresponding to a particular initial condition, we get from (2) that

$$y(x) = 0 \quad (3)$$

is a solution of the IVP.

But we know that

$$y(x) = \begin{cases} xe^{-1/x^2} + \sqrt{\pi}\left(\operatorname{erf}\left(\tfrac{1}{x}\right) - 1\right) & x \neq 0 \\ 0, & x = 0 \end{cases} \quad (4)$$

The only way to avoid a contradiction between (3) and (4) is to conclude that the radius of convergence of the Taylor series is zero.

Conclusion: A general theory of ODEs can't be based on Taylor series.

We take a step back and define what it means for an IVP to be well posed (Def^{n} 1.1 in typed notes):

Given $y'(x) = F(x, y(x))$ on an interval I, with $x \in I$, the IVP is to find a differentiable solution $y(x)$ on a subinterval $J \subseteq I$ such that $y(x_0) = x_0$, for $x_0 \in J$ and $y_0 \in \mathbb{R}$. We say that the IVP is well posed if:

strong / classical solution (PDEs)

- A solution exists and is differentiable
- The solution is unique
- The solution depends continuously on the initial conditions.

Example: Ill-posed IVP:

$I = [-1, 1]$

$$y'(x) = \begin{cases} 0, & -1 \le x \le 0 \end{cases}$$

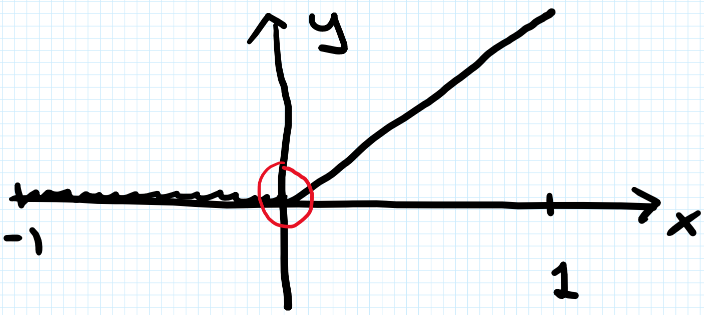
$$y'(x) = \begin{cases} 0, & -1 \le x \le 0 \\ 1, & 0 \le x \le 1 \end{cases}$$

IC: $y(0) = 0$, $x \in I$

Solution by integration:

$$y(x) = \begin{cases} 0, & -1 \le x \le 0 \\ x, & 0 \le x \le 1 \end{cases}$$

... constant of integration $= 0$ for continuity, to satisfy the IC



Discontinuity in first derivative at $x = 0$.

Solution fails to be differentiable at $x = 0$

Solution fails first criterion of well posedness.

Remark: This is a solution but it's not well posed.

A weak solution.

Example: $y'(x) = 3y^{2/3}$

Example of $y'(x) = F(\cancel{x}, y)$

$-1 \le x \le 1$, $I = [-1,1]$

$-1 \le x \le 1, \quad I = [-1, 1]$

$y(0) = 0 \quad \ldots \quad x_0 = 0, \; y_0 = 0.$

Separation of variables:

$$\frac{1}{3} \frac{1}{y^{2/3}} \frac{dy}{dx} = 1$$

$$\left\{ \begin{array}{l} \frac{1}{3} y^{-2/3} \frac{dy}{dx} \\ = \frac{d}{dx}\left(y^{1/3}\right) \end{array} \right.$$

$$\Rightarrow \frac{d}{dx}\left(y^{1/3}\right) = 1$$

$$\Rightarrow y^{1/3} = x + C \overset{\text{I.C.}}{=} x$$

$$\Rightarrow \boxed{y = x^3}$$

Other solutions:

$$y(x) = 0 \quad \forall x \in I$$

$$y(x) = \begin{cases} 0, & -1 \le x \le x_1 \\ (x - x_1)^3, & x_1 \le x \le 1 \end{cases}$$

for any parameter $x_1 \in (-1, 1)$.

Infinitely many solutions.