Fredholm Integral Equations

$$y(x) = f(x) + \lambda \int_a^b k(x,s)\, y(s)\, ds \quad , \quad |k(x,s)| \le M \qquad (1)$$

Solution via series:

$$y(x) = \underbrace{f(x)}_{\text{Inhomogeneous}} + \sum_{n=1}^{\infty} \underbrace{\lambda^n K^n f(x)}_{u_n(x)} \quad , \quad \boxed{|\lambda| < \frac{1}{M|b-a|}}$$

Today:

- Check the series is uniformly convergent
- Check limit satisfies the FIE

To show uniform convergence we use the Weierstras M-test.

We put bounds on the u_n's:

- $u_n(x) = \lambda^n K^n f(x)$ continuous
- $|u_n(x)| \le \lambda^n M^n |b-a|^n \cdot \overbrace{C}^{|f(x)| \le C} = M_n$

We sum the M_n's:

$$\sum_{n=1}^{\infty} M_n = \sum_{n=1}^{\infty} \underbrace{\lambda^n M^n |b-a|^n}_{\lambda M|b-a| \le r \ , \ r \in [0,1)} \cdot C$$

$$= \left(\sum_{n=1}^{\infty} r^n \right) \cdot C$$

$$= \left(\sum_{n=0}^{\infty} r^n - 1 \right) \cdot C$$

$$= \Big(\sum_{n=0} \qquad - 1 \Big) \cdot C$$

$$\overset{\text{g.p.}}{=} \left(\frac{1}{1-r} - 1 \right) \cdot C < \infty$$

Hence, $\sum_{n=1}^{\infty} M_n < \infty$, since $r < 1$, so

by the Weierstrass M-test,

$$f(x) + \sum_{n=1}^{\infty} \underbrace{\lambda^n K^n f(x)}_{u_n(x)} \qquad (2)$$

converges uniformly to a continuous function.

We also have to show that the limit (2) satisfies the FIE (1).

Sl^{n} of FIE:

$$y(x) = f(x) + \lambda \int_a^b K(x,s)\, y(s)\, ds$$

$$\left\{ \begin{aligned} y_{n+1}(x) &= f(x) + \lambda \int_a^b K(x,s)\, y_n(s)\, ds, \\ y_0(x) &= f(x) \end{aligned} \right\} \quad (3)$$

$$y_{n+1}(x) - y(x) = \cancel{f(x)} - \cancel{f(x)} + \lambda \int_a^b K(x,s) \left[y_n(s) - y(s) \right] ds$$

$$|y_{n+1}(x) - y(x)| = |\lambda| \left| \int_a^b k(x,s) \, [y_n(s) - y(s)] \, ds \right|$$

Suppose the max of the LHS is realised at $x = x_*$:

$$\| y_{n+1}(x) - y(x) \|_\infty = |\lambda| \left| \int_a^b k(x_*, s) \, [y_n(s) - y(s)] \, ds \right|$$

— OR —

$$\| y_{n+1} - y \|_\infty = |\lambda| \left| \int_a^b k(x_*, s) \, [y_n(s) - y(s)] \, ds \right|$$

$$\overset{\Delta}{\leq} |\lambda| \int_a^b |k(x_*, s)| \, |y_n(s) - y(s)| \, ds$$

$$\leq |\lambda| \cdot M \cdot \| y_n - y \|_\infty \, |b-a|$$

$$= \underbrace{|\lambda| \cdot M \cdot |b-a|}_{r, \quad 0 \leq r < 1} \, \| y_n - y \|_\infty$$

$$\Rightarrow \| y_{n+1} - y \|_\infty \leq r \, \| y_n - y \|_\infty$$

"Telescope this result":

$$\| y_{n+1} - y \|_\infty \leq r^n \| y - \underbrace{f}_{= y_0} \|_\infty$$

or

$\underline{n = 0}$ $\quad \| y_1 - y \|_\infty \leq r \, \| f - y \|_\infty$

$\underline{n = 1}$ $\quad \| y_2 - y \|_\infty \leq r \, \| y_1 - y \|_\infty$

$$\leq r \cdot r \, \| f - y \|_\infty$$

$$\leq r \cdot r \, \|f - y\|_\infty$$

$$\Rightarrow \|y_2 - y\|_\infty \leq r^2 \|f - y\|_\infty$$

General pattern:

$$\|y_n - y\|_\infty \leq r^n \underbrace{\|f - y\|_\infty}_{\text{Finite}}, \quad |r| < 1$$

Take the limit as $n \to \infty$:

$$\|y_n - y\|_\infty \longrightarrow 0 \quad \text{as} \quad n \to \infty$$

Hence,

$$\underbrace{y_n}_{\text{Iterative method (3)}} \longrightarrow \underbrace{y}_{\text{Sl}^{\text{n}} \text{ of FIE (1)}}, \quad \text{as } n \to \infty$$

Interpretation (§ 9.1.2)

$$y = f + \lambda K y$$

$$\Rightarrow (\mathbb{I} - \lambda K) y = f, \qquad \mathbb{I} y = y.$$

Invert:

$$y = (\mathbb{I} - \lambda K)^{-1} f \qquad (4)$$

The inverse on the RHS has already been given

The inverse on the RHS has already been given two equivalent meanings:

- Eigenfunction expansion,

$$y = \sum_n \frac{\lambda}{\lambda_i - \lambda} f_i y_i \quad , \quad \lambda \neq \lambda_i$$

- Iterative method, Eq$^{\underline{n}}$ (3).

The iterative method agrees with a purely algebraic, formal manipulation of Eq$^{\underline{n}}$ (4):

$$y = (\mathbb{I} - \lambda K)^{-1} f$$

$$\overset{\text{Binomial}}{=} f + \sum_{n=1}^{\infty} \lambda^n K^n f$$

The iterative method (3) shows us that this purely algebraic, formal manipulation is correct, provided

$$|\lambda| \, M \, |b-a| < 1$$

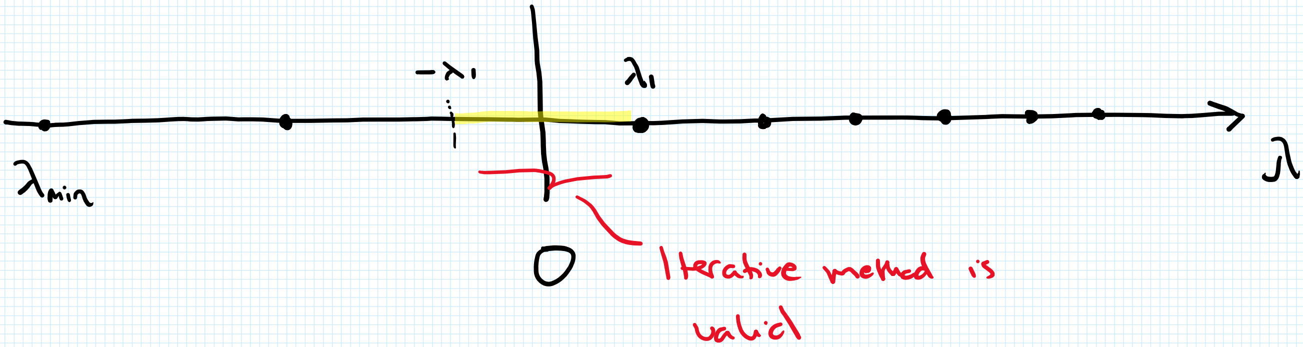
Binomial, $x \in \mathbb{R}$	$(\mathbb{I} - \lambda K)^{-1} f$
$(1-x)^{-1} = 1 + \sum_{n=1}^{\infty} x^n$	$= f + \sum_{n=1}^{\infty} \lambda^n K^n f$
$\lvert x \rvert < 1$	$\lvert\lambda\rvert \, M \, \lvert b-a\rvert < 1$

The convergence criterion $|\lambda| < \dfrac{1}{M|b-a|}$

is equivalent to:

$$|\lambda| < |\lambda_1|$$

where λ_1 is the eigenvalue of the FIE with the smallest absolute value.



Eigenfunction expansion is valid for all $\lambda \neq \lambda_i$.

Example showing the connection between eigenvalue expansion and iterative method (§ 9.1.3)

$$y(x) = \underbrace{1}_{f=1} + \lambda \int_0^1 (1 - 3xs)\, y(s)\, ds$$

Solution via eigenvalue/eigenfunctions:

$$y(x) = \frac{4(\lambda+1)}{4-\lambda^2} - \frac{6\lambda}{4-\lambda^2}\, x, \qquad \lambda \neq \pm 2.$$

The same sl$\underline{n}$ now, via the iterative method:

$$y = f + \sum^{\infty} K^n \cdot f$$

$$y = f + \sum_{n=1}^{\infty} K^n \cdot f$$

For now, $f = 1$ (const. function 1).

$$Kf(x) = \int_0^1 (1-3xs)\cdot 1 \cdot ds = 1 - \tfrac{3}{2}x$$

$$K^2 f(x) = \int_0^1 (1-3xs)\left(1-\tfrac{3}{2}s\right) ds = \tfrac{1}{4}$$

$$K^3 f(x) = \int_0^1 (1-3xs)\, \tfrac{1}{4}\, ds = \tfrac{1}{4}\left(1-\tfrac{3}{2}x\right)$$

Guess the pattern:

$$K^{2n+1} f = \frac{1}{4^n}\left(1 - \frac{3}{2}x\right), \quad n = 0, 1, 2, \dots$$

$$K^{2n} f = \frac{1}{4^n}, \quad n = 1, 2, \dots$$

$$y(x) = 1 + \sum_{n \text{ even}} \lambda^n K^n \cdot f + \sum_{n \text{ odd}} \lambda^n K^n f$$

$$= 1 + \sum_{n=1}^{\infty} \lambda^{2n} K^{2n} f + \sum_{n=0}^{\infty} \lambda^{2n+1} K^{2n+1} f$$

$$= 1 + \boxed{\sum_{n=1}^{\infty} \frac{1}{4^n} \lambda^{2n}} + \sum_{n=0}^{\infty} \lambda^{2n+1} \boxed{\frac{1}{4^n}\left(1 - \frac{3}{2}x\right)}$$

$$= 1 + \left(1 - \tfrac{3}{2}x\right)\lambda \sum_{n=0}^{\infty} \underbrace{\left(\frac{\lambda}{2}\right)^{2n}}_{\rho = (\lambda/2)^2} + \sum_{n=1}^{\infty} \left(\frac{\lambda}{2}\right)^{2n}$$

$$\underbrace{\qquad}_{\rho = (\lambda/2)^2} \qquad n=1$$

$$\overset{\text{G.P.}}{=} 1 + \left(1 - \tfrac{3}{2}x\right)\lambda \frac{1}{1-\frac{\lambda^2}{4}} + \left[\sum_{n=0}^{\infty}\left[\left(\frac{\lambda}{2}\right)^2\right]^n - 1\right]$$

$$= \cancel{1} + \left(1 - \tfrac{3}{2}x\right)\frac{\lambda}{1-\frac{\lambda^2}{4}}\frac{4}{4} + \frac{1}{1-\frac{\lambda^2}{4}}\frac{4}{4} - \cancel{1}$$

$$= \frac{4}{4-\lambda^2}\left[(\lambda+1) - \tfrac{3}{2}x\lambda\right] \quad (5a)$$

Agrees with eigenvalue analysis:

- $\lambda^2/4 < 1 \Rightarrow |\lambda| < 2$. G.P.

Same expression via eigenvalue analysis

$$y = \frac{4}{4-\lambda^2}(\lambda+1) - \frac{6\lambda}{4-\lambda^2}x \qquad |\lambda| \neq 2. \quad (5b)$$

Two expressions are the same; validity of (5a) is extended to all $|\lambda| \neq 2$ (i.e. (5b)) via analytic continuation.

Applied Analysis (ACM30020)

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Exercises #5

5. Solve for $\phi(x)$ in the integral equation

$$\phi(x) = f(x) + \lambda \int_0^1 \left[\left(\frac{x}{y}\right)^n + \left(\frac{y}{x}\right)^n \right] \phi(y)\mathrm{d}y,$$

where $f(x)$ is bounded for $0 < x < 1$ and $-1/2 < n < 1/2$, expressing your answer in terms of the quantities $F_m = \int_0^1 f(y) y^m \mathrm{d}y$.

(a) Give the explicit solution when $\lambda = 1$.

(b) For what values of λ are there no solutions unless $F_{\pm n}$ are in a particular ratio? What is this ratio?

First, pick out the kernel:

$$K(x,y) = \underbrace{x^n}_{u_1(x)} \underbrace{y^{-n}}_{v_1(y)} + \underbrace{x^{-n}}_{u_2(x)} \underbrace{y^n}_{v_2(y)}$$

Separable kernel, $K(x,y) = \sum_{i=1}^{2} u_i(x) v_i(y)$

$u_1(s) = s^n \qquad u_2(s) = s^{-n}$

$v_1(s) = s^{-n} \qquad v_2(s) = s^n$

Question asks to find the sl$^{\underline{n}}$ of the FIE when $\lambda = 1$.

We first of all check that λ is not an eigenvalue.

Let

$$c_1 = \int_0^1 v_1(s)\phi(s)\,ds = \int_0^1 s^{-n}\phi(s)\,ds \qquad \text{UNKNOWN}$$

$$c_2 = \int_0^1 v_2(s)\phi(s)\,ds = \int_0^1 s^n \phi(s)\,ds \qquad \text{UNKNOWN}$$

$$c_2 = \int_0^1 v_2(s)\,\phi(s)\,ds = \int_0^1 s^n \phi(s)\,ds \qquad \text{UNKNOWN}$$

Inhomogeneous terms:

$$b_1 = \int_0^1 v_1(s) f(s)\,ds = \int_0^1 s^{-n} f(s)\,ds = F_{-n} \text{ (notation)}$$

$$b_2 = \int_0^1 v_2(s) f(s)\,ds = \int_0^1 s^n f(s)\,ds = F_n$$

Class notes:

$$c_i = b_i + \lambda \sum_{j=1}^{2} A_{ij} c_j$$

$$A_{ij} = \int_0^1 v_i(s) u_j(s)\,ds$$

$$A_{11} = \int_0^1 v_1(s) u_1(s)\,ds = \int_0^1 1 \cdot ds = 1$$

$$A_{12} = \int_0^1 s^{-n} \cdot s^{-n}\,ds = \int_0^1 s^{-2n}\,ds = \frac{1}{1-2n}, \quad |n| < 1/2$$

$$A_{21} = \int_0^1 s^n \cdot s^n\,ds = \int_0^1 s^{2n}\,ds = \frac{1}{1+2n}, \quad |n| < 1/2$$

$$A_{22} = \int_0^1 s^n s^{-n}\,ds = \int_0^1 1\,ds = 1$$

Linear problem:

$$\begin{pmatrix} 1 & 0 \end{pmatrix} \underline{c} = \underline{b} + \lambda \begin{pmatrix} 1 & \frac{1}{1-2n} \\ \end{pmatrix} \underline{c}$$

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \underline{c} = \underline{b} + \lambda \begin{pmatrix} 1 & 1-2n \\ \frac{1}{1+2n} & 1 \end{pmatrix} \underline{c}$$

<u>OR</u>

$$\underbrace{\begin{pmatrix} 1-\lambda & \frac{-\lambda}{1-2n} \\ \frac{-\lambda}{1+2n} & 1-\lambda \end{pmatrix}}_{M(\lambda)} \underline{c} = \underline{b}$$

Characteristic polynomial of $M(\lambda)$: $\det[M(\lambda)] = 0$

$$(1-\lambda)^2 - \frac{\lambda^2}{(1+2n)(1-2n)} = 0$$

$$\Rightarrow (1-\lambda)^2 = \frac{\lambda^2}{1-4n^2}$$

$$\Rightarrow (1-\lambda) = \pm \frac{\lambda}{\sqrt{1-4n^2}} \qquad |n| < 1/2$$

(+) $$\frac{1}{\lambda_+} - 1 = \frac{1}{\sqrt{1-4n^2}}$$

(−) $$\frac{1}{\lambda_-} - 1 = \frac{-1}{\sqrt{1-4n^2}}$$

$$\boxed{\frac{1}{\lambda_\pm - 1} = \pm \frac{1}{\sqrt{1-4n^2}}}$$

Back to: $1-\lambda = \pm \dfrac{\lambda}{\sqrt{1-4n^2}}$

Uach to: $1-\lambda = \pm \frac{\quad}{\sqrt{1-4n^2}}$

$$\oplus \qquad 1-\lambda = \frac{\lambda}{\sqrt{1-4n^2}}$$

$$\Rightarrow \quad 1 = \lambda\left(1+\frac{1}{\sqrt{1-4n^2}}\right)$$

$$\Rightarrow \quad 1 = \lambda\,\frac{\sqrt{1-4n^2}+1}{\sqrt{1-4n^2}}$$

$$\Rightarrow \quad \lambda_+ = \frac{\sqrt{1-4n^2}}{1+\sqrt{1-4n^2}}$$

$$\ominus \qquad 1-\lambda = -\frac{\lambda}{\sqrt{1-4n^2}}$$

$$\Rightarrow \quad 1 = \lambda\left(1-\frac{1}{\sqrt{1-4n^2}}\right)$$

$$\Rightarrow \quad 1 = \lambda\left(\frac{\sqrt{1-4n^2}-1}{\sqrt{1-4n^2}}\right)$$

$$\Rightarrow \quad \lambda = \frac{\sqrt{1-4n^2}}{-1+\sqrt{1-4n^2}}$$

$$\lambda_+ = \frac{\sqrt{1-4n^2}}{1+\sqrt{1-4n^2}}, \qquad \lambda_- = \frac{\sqrt{1-4n^2}}{-1+\sqrt{1-4n^2}}.$$

Question: Part (a) is about $\lambda = 1$.

When $\lambda = 1$ it is definitely true that $\lambda \neq \lambda_+, \lambda_-$.

$\lambda \neq$ eigenvalue $\Rightarrow$ unique sl^n of FIE.

Back to $M(\lambda=1)\,\underline{c} = \underline{b}$.

$$M(\lambda) = \begin{pmatrix} 1-\lambda & \dfrac{-\lambda}{1-2n} \\ \dfrac{-\lambda}{1+2n} & 1-\lambda \end{pmatrix}$$

$$\Rightarrow M(\lambda=1) = \begin{pmatrix} 0 & -\dfrac{1}{1-2n} \\ -\dfrac{1}{1+2n} & 0 \end{pmatrix}$$

$$\det M = -\frac{1}{1-4n^2} = -\frac{1}{(1-2n)(1+2n)}$$

$$M^{-1} = \begin{pmatrix} 0 & \dfrac{1}{1-2n} \\ \dfrac{1}{1+2n} & 0 \end{pmatrix} \underbrace{(-1)(1-2n)(1+2n)}_{(\det(M))^{-1}}$$

$$= -\begin{pmatrix} 0 & 1+2n \\ 1-2n & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & -(2n+1) \\ 2n-1 & 0 \end{pmatrix}$$

$$M\underline{c} = \underline{b} \Rightarrow \underline{c} = M^{-1}\underline{b}$$

$$\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 0 & -(2n+1) \\ (2n-1) & 0 \end{pmatrix} \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$$

$$\Rightarrow c_1 = -(2n+1) b_2 = -(2n+1) F_n \quad ,$$

$$c_2 = (2n-1) b_1 = (2n-1) F_{-n}$$

Class notes:

$$\phi(x) = f(x) + c_1 u_1(x) + c_2 u_2(x)$$

$\Rightarrow$

$$\phi(x) = f(x) - (2n+1) F_n x^n + (2n-1) F_{-n} x^{-n}$$

$\Rightarrow$

$$\phi(x) = f(x) - (2n+1) F_n x^n - (1-2n) F_{-n} x^{-n}$$

(b) For what values of λ are there no solutions unless $F_{\pm n}$ are in a particular ratio? What is that ratio?

<u>Answer</u>: $\lambda = \lambda_+$ or λ_-.

<u>Solvability condition</u>: $M(\lambda_\pm) \underline{c} = \underline{b}$

$$\Bigg(\quad 1 - \lambda_\pm \qquad - \frac{\lambda_\pm}{1-2n} \quad \Bigg)$$

$$\begin{pmatrix} 1-\lambda_\pm & -\dfrac{\lambda_\pm}{1-2n} \\ -\dfrac{\lambda_\pm}{1+2n} & 1-\lambda_\pm \end{pmatrix} \underline{c} = \underline{b}$$

$$\lambda_\pm \begin{pmatrix} \dfrac{1}{\lambda_\pm}-1 & -\dfrac{1}{1-2n} \\ -\dfrac{1}{1+2n} & \dfrac{1}{\lambda_\pm}-1 \end{pmatrix} \underline{c} = \underline{b}$$

$$\lambda_\pm \begin{pmatrix} \pm\dfrac{1}{\sqrt{1-4n^2}} & -\dfrac{1}{1-2n} \\ -\dfrac{1}{1+2n} & \pm\dfrac{1}{\sqrt{1-4n^2}} \end{pmatrix} \underline{c} = \underline{b} \,/\, \lambda_\pm$$

Rows:

$$\pm\frac{1}{\sqrt{1-4n^2}}c_1 - \frac{1}{1-2n}c_2 = b_1/\lambda_\pm$$

$$\left[-\frac{1}{1+2n}c_1 \pm \frac{1}{\sqrt{1-4n^2}}c_2 = b_2/\lambda_\pm\right](\mp)\frac{1+2n}{\sqrt{1-4n^2}}$$

Second row becomes:

$$\pm\frac{1}{\sqrt{1-4n^2}}c_1 - \frac{1+2n}{1-4n^2}c_2 = \frac{b_2/\lambda_\pm}{\sqrt{1-4n^2}}$$

OR

$$\pm\frac{1}{\sqrt{1-4n^2}}c_1 - \frac{1}{1-2n}c_2 = \frac{b_2/\lambda_\pm}{\sqrt{1-4n^2}}(1+2n)$$

First row:

First row:

$$\pm \frac{1}{\sqrt{1-4n^2}} c_1 - \frac{1}{1-2n} c_2 = b_1/\lambda_\pm$$

LHS's are identical.

For the equations to be consistent, the RHS's have to be identical too:

$$b_1/\cancel{\lambda_\pm} = \frac{b_2/\cancel{\lambda_\pm}}{\sqrt{1-4n^2}}(1+2n)$$

$$\Rightarrow \frac{b_1}{b_2} = \frac{1+2n}{\sqrt{1-4n^2}} = \sqrt{\frac{(1+\cancel{2n})(1+2n)}{(1+\cancel{2n})(1-2n)}}$$

OR $$\frac{F_{-n}}{F_n} = \sqrt{\frac{1+2n}{1-2n}}$$

or finally, $$\frac{F_n}{F_{-n}} = \sqrt{\frac{1-2n}{1+2n}}, \quad |n| < 1/2.$$

This is Case 2 of the FIE, $\lambda = \lambda_+$ or λ_-:

- Infinitely many sl^{n}s if $\frac{F_n}{F_{-n}} = \sqrt{\frac{1-2n}{1+2n}}$

- Two linear equations are inconsistent (therefore

- Two linear equations are inconsistent (therefore no solutions) otherwise.