Plan: Looking at FIEs

- Separable kernels again §7.4
- Symmetric kernels Chapter 8

FIE:

$$y(x) = \underbrace{F(x)}_{\text{Inhomogeneous}} + \lambda \int_a^b k(x,s)\,y(s)\,ds \qquad (1)$$

Separable kernels:

$$K(x,s) = \sum_{j=1}^{n} u_j(x) v_j(s) \qquad (2)$$

Sub (2) back into (1):

$$y(x) = F(x) + \lambda \int_a^b \sum_{j=1}^{n} u_j(x)\, v_j(s)\, y(s)\,ds$$

$$= F(x) + \lambda \sum_{j=1}^{n} u_j(x) \underbrace{\int_a^b v_j(s)\, y(s)\,ds}_{c_j}$$

Multiply both sides by $v_i(x)$ and integrate:

$$\int_a^b v_i(x)\, y(x)\,dx = \underbrace{\int_a^b v_i(x) F(x)\,dx}_{} + \lambda \sum_{j=1}^{n} c_j \underbrace{\int_a^b u_j(x)\, v_i(x)\,dx}_{}$$

$$\Rightarrow \quad c_i = b_i + \lambda \sum_{j=1}^{n} A_{ij} c_j$$

where $A_{ij} = \int_a^b v_i(x) u_j(x)\,dx$

OR $\quad \underline{c} = \underline{b} + \lambda \underline{A}\, \underline{c} \qquad (3)$

Characteristic polynomial:

$$\det(\lambda \underline{A} - \mathbb{I}) = 0 \qquad (4)$$

Not $\det(A - \lambda \mathbb{I}) = 0$

Eigenvalues of characteristic polynomial (roots):

$$\lambda \in \{ \lambda^{(1)}, \dots, \lambda^{(n)} \} = S \qquad (5)$$

Cases:

1. Homogeneous case, $\quad F(x) = 0$.

FIE: $\quad y(x) = \lambda \int_a^b K(x,s)\, y(s)\, ds$

- If $\lambda \notin S$, then $y(x) = 0$ is the trivial solution
- If $\lambda \in S$, then the FIE has infinitely many solutions:

$$y(x) = \alpha \underbrace{\sum_{j=1}^{n} u_j(x)\, c_j^{(p)}}_{y^{(p)}(x)} \qquad (6)$$

where

- α is arbitrary
- $c_j^{(p)}$ is an eigenvector:

$$(\mathbb{I} - \lambda^{(p)} A)\, c^{(p)} = 0$$

$$(\mathbb{I} - \lambda^{(p)}\underline{A})\,\underline{c}^{(p)} = 0 .$$

2. Inhomogeneous case

- If $\lambda \notin S$ then the FIE has a unique solution

- If $\lambda \in S$ and a compatibility condition is met, then the FIE has infinitely many solutions:

$$y(x) = \left[F(x) - \cdots \right] + \alpha \underbrace{y^{(p)}(x)}_{\text{From (6)}} , \quad \alpha \in \mathbb{R} \qquad (7)$$

Compatibility condition:

$$\int_a^b F(s)\, y^{(p)}(s)\, ds = 0$$

3. Special inhomogeneous case, $f(x) \neq 0$ also,

$$\int_a^b F(x) V_i(x)\, dx = 0, \qquad \text{each } i \in \{1, 2, \dots, n\}$$

- If $\lambda \notin S$ there is a unique $s|^{n}$.
- If $\lambda \in S$ and a compatibility condition is met, then the FIE has infinitely many $s|^{n}s$:

$$y(x) = F(x) + \alpha y^{(p)}(x).$$

Only thing missing: the correction (highlighted) in Equation (7). Chapter (8) will help us to fill in the missing terms, as well as telling us about symmetric kernels.

We move straight to § 8.2. A symmetric kernel is one where

$$K(x,s) = K(s,x)$$

Homogeneous FIE:

$$y(x) = \lambda \int_a^b K(x,s)\, y(s)\, ds$$

Eigenvalues / Eigenfunctions:

$$y_i(x) = \lambda_i \int_a^b K(x,s)\, y_i(s)\, ds$$

Assume:

- $K(x,s)$ is symmetric and continuous in both variables on $[a,b]$
- Eigenvalues / Eigenfunctions exist.

Notation:

$$Kf(x) = \int_a^b K(x,s)\, f(s)\, ds$$

$$\mathcal{K} f(x) = \int_a^b K(x,s) f(s)\, ds$$

$$K * f(x) = \int_a^b K(x,s) f(s)\, ds$$

Properties of symmetric kernels (§8.2.2)

<u>Notation</u> :

$$y_i(x) = \lambda_i \int_a^b K(x,s) y_i(s)\, ds$$

$$y_j(x) = \lambda_j \int_a^b K(x,s) y_j(s)\, ds$$

<u>Theorem</u> : Eigenfunctions corresponding to different eigenvalues are orthogonal in the sense that:

$$\int_a^b y_i(x) y_j(x)\, dx = 0, \quad \text{for} \quad \lambda_i \neq \lambda_j$$

<u>Proof</u> : Focus on $y_j(x)$:

$$\int_a^b y_i(x) y_j(x)\, dx = \int_a^b y_i(x) \left[\lambda_j K(x,s) \int_a^b y_j(s)\, ds \right] dx$$

$$= \lambda_j \int_a^b \int_a^b K(x,s)\, y_i(x)\, y_j(s)\, ds\, dx$$

Non-trivial solutions, hence non-zero eigenvalues, hence division by λ_j is ok :

$$\frac{1}{\lambda_j} \int_a^b y_i(x) y_j(x)\, dx = \int_a^b \int_a^b K(x,s)\, y_i(x)\, y_j(s)\, ds\, dx$$

$$\boxed{\frac{1}{\lambda_j}\int_a^b y_i(x)\,y_j(x)\,dx = \int_a^b\int_a^b k(x,s)\,y_i(x)\,y_j(s)\,ds\,dx \qquad (8)}$$

Swap indices:

$$\frac{1}{\lambda_i}\int_a^b y_i(x)\,y_j(x)\,dx = \int_a^b\int_a^b k(x,s)\,y_j(x)\,y_i(s)\,ds\,dx$$

$$= \int_a^b\int_a^b \underbrace{k(s,x)\,y_j(s)\,y_i(x)}_{\text{... dummy variables}}\,ds\,dx$$

$$= \int_a^b\int_a^b k(s,x)\,y_i(x)\,y_j(s)\,ds\,dx$$

$$= \int_a^b\int_a^b \underbrace{k(x,s)\,y_i(x)\,y_j(s)}_{\text{... symmetric kernel}}\,ds\,dx$$

$$\overset{Eq(8)}{=} \frac{1}{\lambda_j}\int_a^b y_i(x)\,y_j(x)\,dx$$

$$\therefore \left(\frac{1}{\lambda_i}-\frac{1}{\lambda_j}\right)\int_a^b y_i(x)\,y_j(x)\,dx = 0$$

Since $\lambda_i \neq \lambda_j$, we have:

$$\int_a^b y_i(x)\,y_j(x)\,dx = 0$$

∎

Theorem (8.2): The eigenvalues of a FIE with a real-valued symmetric kernel are real.

real-valued symmetric kernel are real.

<u>Proof</u> : $\lambda_i = \int_a^b k(x,s)\, y_i(x)\, dx$

$$\Rightarrow \quad \lambda_i^* = \int_a^b k(x,s)\, y_i^*(x)\, dx$$

since $k(x,s)$ is real-valued.

Hence, $\lambda_i^* = \lambda_j$ and $y_i^* = y_j$ are an eigenfunction pair.

By the previous result,

$$\left(\frac{1}{\lambda_i} - \frac{1}{\lambda_j}\right) \int_a^b y_i(x)\, y_j(x)\, dx = 0$$

$$\Rightarrow \left(\frac{1}{\lambda_i} - \frac{1}{\lambda_i^*}\right) \int_a^b y_i(x)\, y_i^*(x)\, dx = 0$$

$$\Rightarrow \frac{(\lambda_i^* - \lambda_i)}{|\lambda_i|^2} \underbrace{\int_a^b |y_i(x)|^2\, dx}_{\neq 0} = 0$$

$$\Rightarrow \quad \lambda_i^* = \lambda_i$$

$$\Rightarrow \quad \lambda_i \text{ is real.} \qquad \blacksquare$$

Fredholm integral Equations, Chapter 8

FIE:

$$y(x) = \underbrace{F(x)}_{\text{Inhomogeneous term}} + \lambda \int_a^b \underbrace{K(x,s)}_{\text{symmetric kernel}} y(s)\,ds \qquad (1)$$

Eigenvalues / Eigenfunctions satisfy:

$$y_i(x) = \lambda_i \int_a^b K(x,s) y_i(s)\,ds \qquad (2)$$

For a symmetric kernel:

- The λ_i's are real
- The y_i's are orthogonal

We choose a normalization such that:

$$\int_a^b y_i(x) y_j(x)\,dx = \delta_{ij} = \begin{cases} 1 & \text{if } i=j \\ 0 & \text{otherwise} \end{cases}$$

Today, we state an important fact, that any function of the form

$$g(x) = \int_a^b K(x,s) r(s)\,ds \qquad (3)$$

← "g is in the range of k"

can be expanded in terms of the eigenfunctions of the corresponding FIE.

Why this is important: It will help us to solve the

This is important : It will help us to solve the

inhomogeneous FIE .

For the statement after (3) to hold it must be true that :

- $k(x,s)$ is continuous , real-valued and symmetric
- $r(s)$ is a continuous function .

If the statement after (3) is true , we can write :

$$g(x) = \sum_i a_i y_i(x) \quad , \quad x \in [a,b]$$

$$\int_a^b g(x) y_j(x) dx = \int_a^b \sum_i a_i y_i(x) y_j(x) dx$$

$$= \sum_i a_i \underbrace{\int_a^b y_i(x) y_j(x) dx}_{\delta_{ij}}$$

$$= a_j$$

$$\therefore \quad a_j = \int_a^b g(x) y_j(x) dx$$

In cases where only a finite number of eigenfunctions exist, the functions generated by

$$g(x) = \int_a^b k(x,s) r(s) ds$$

is very limited . E.g. $k(x,s) = 1 - 3xs$

$$g(x) = \int_a^b (1-3xs)\, r(s)\, ds$$

$$= \int_a^b r(s)\,ds \; - \; 3x \underbrace{\int_a^b s \cdot r(s)\,ds}$$

$$= A + Bx$$

Thus, for the kernel $K(x,s) = 1-3xs$, only linear functions can be generated.

Referring back to the worked example in Week 7, there were two eigenfunctions for this kernel:

$$y_2(x) = 1-x$$

$$y_{-2}(x) = 1-3x$$

Look at:

$$\begin{array}{rcr} 3y_2(x) & = & 3-3x \\ -\; y_{-2}(x) & & -1+3x \\ \hline \end{array}$$

$$\Rightarrow \quad \tfrac{1}{2}(3y_2 - y_{-2}) = 1$$

$$\begin{array}{rcr} y_2(x) & = & 1-x \\ -\; y_{-2}(x) & & -1+3x \\ \hline \tfrac{1}{2}(y_2 - y_{-2}) & = & x \end{array}$$

$$g(x) = A + Bx$$

$$
\begin{aligned}
g(x) &= A + Bx \\
&= A\left[\tfrac{1}{2}(3y_2 - y_{-2})\right] + B\left[\tfrac{1}{2}(y_2 - y_{-2})\right] \\
&= \sum_{i=2,-2} a_i y_i
\end{aligned}
$$

We use this knowledge to solve the inhomogeneous FIE

$$y(x) = F(x) + \lambda \int_a^b k(x,s)\, y(s)\, ds$$

(Section 8.4).

$$y(x) - F(x) = \lambda \int_a^b k(x,s)\, y(s)\, ds$$

Hence, $y(x) - F(x)$ is in the range of K, so:

$$y(x) - F(x) = \sum_i a_i y_i(x)$$

$$\int_a^b \left[y(x) - F(x)\right] y_j(x)\, dx = \int_a^b \sum_i a_i y_i(x)\, y_j(x)\, dx$$

$$
\Rightarrow \int_a^b y(x) y_j(x)\, dx - \int_a^b F(x) y_j(x)\, dx = \sum_i a_i \underbrace{\int y_i(x) y_j(x)\, dx}_{\delta_{ij}}
$$

$$\Rightarrow \int^b \ldots \int^b F(x) y_j(x)\, dx = a_j \quad (4)$$

$$\Rightarrow \underbrace{\int_a^b y(x)y_j(x)\,dx}_{c_j} - \underbrace{\int_a^b F(x)y_j(x)\,dx}_{f_j} = a_j \quad (4)$$

Back to :

$$\int_a^b \left[y(x) - F(x)\right] y_j(x)\,dx = \int_a^b \lambda \int_a^b k(x,s)\,y(s)\,ds \; y_j(x)\,dx$$

$\downarrow$ Eq. (4)

$$c_j - f_j = \lambda \int_a^b \int_a^b k(x,s)\,y(s)\,y_j(x)\,ds\,dx$$

$\downarrow$ Symmetric kernel

$$= \lambda \int_a^b \int_a^b K(s,x)\,\underline{y(s)}\,y_j(x)\,ds\,dx$$

$$= \lambda \int_a^b \left[\int_a^b K(s,x)\,y_j(x)\right] y(s)\,ds$$

$\downarrow$ Dummy variables

$$= \lambda \int_a^b \left[\int_a^b K(x,s)\,y_j(s)\right] y(x)\,dx$$

$$= \frac{\lambda}{\lambda_j} \int_a^b \Big[\underbrace{\lambda_j \int_a^b k(x,s)\,y_j(s)}_{\text{Eigenvalue problem}}\Big]\, y(x)\,dx$$

$$= \frac{\lambda}{\lambda_j} \int_a^b y_j(x)\,y(x)\,dx$$

$$= \frac{\lambda}{\lambda_j} \int_a y_j(x)\, y(x)\, dx$$

$$\Rightarrow c_j - f_j = \frac{\lambda}{\lambda_j} c_j$$

Solve for c_j:

$$\Rightarrow \lambda_j c_j - f_j \lambda_j = \lambda c_j$$

$$\Rightarrow \lambda_j c_j - \lambda c_j = f_j \lambda_j$$

$$\Rightarrow (\lambda_j - \lambda) c_j = f_j \lambda_j$$

Two cases.

<u>Case 1</u> λ is not an eigenvalue. Hence:

$$c_j = \frac{f_j \lambda_j}{\lambda_j - \lambda}$$

Back to:

$$y(x) - F(x) = \sum_i a_i \, y_i(x)$$

$$\overset{\text{Eq. 4}}{=} \sum_i (c_i - f_i)\, y_i(x)$$

$$= \sum_i \left(\frac{f_i \lambda_i}{\lambda_i - \lambda} - f_i \right) y_i(x)$$

$$= \sum_i \left(\frac{\cancel{f_i \lambda_i} - f_i \cancel{(\lambda_i} - \lambda)}{\lambda_i - \lambda} \right) y_i(x)$$

$$\Rightarrow y(x) - F(x) = \sum_i \frac{f_i \lambda}{\lambda_i - \lambda} y_i(x)$$

$$\Rightarrow \boxed{y(x) = F(x) + \sum_i \frac{f_i \lambda}{\lambda_i - \lambda} y_i(x)}$$

CASE 1

Case 2, when $\lambda = \lambda_{i_0}$

$$y(x) = F(x) + \sum_{i \neq i_0} \frac{f_i \lambda}{\lambda_i - \lambda} y_i(x) + \frac{f_{i_0} \lambda}{\lambda_{i_0} - \lambda} y_{i_0}(x)$$

We take the limit as $\lambda \to \lambda_{i_0}$.

$$y(x) = F(x) + \sum_{i \neq i_0} \frac{f_i \lambda_{i_0}}{\lambda_i - \lambda_{i_0}} y_i(x)$$

$$+ \lim_{\lambda \to \lambda_{i_0}} \left(\frac{f_{i_0} \lambda}{\lambda_{i_0} - \lambda} \right) y_{i_0}(x)$$

To make the limit well-defined, we require

To make the limit well-defined, we require

$f_{i_0} = C \cdot (\lambda_{i_0} - \lambda)$, hence $f_{i_0} \to 0$ as $\lambda \to \lambda_{i_0}$

Thus:

$$y(x) = F(x) + \sum_{i \neq i_0} \frac{\lambda_{i_0} f_i}{\lambda_i - \lambda_{i_0}} Y_i(x)$$

$$+ \lambda_{i_0} \cdot C \; Y_{i_0}(x)$$

↖ arbitrary constant

The Fredholm alternative:

Case 1 λ is not an eigenvalue,

the inhomogeneous FIE has a unique solution

Case 2 λ is an eigenvalue $(=\lambda_{i_0})$

- Either $f_{i_0} = 0$ and there are infinitely many sl$^{\underline{ns}}$ of the FIE
- Or $f_{i_0} \neq 0$ and there is no solution of the FIE.

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Week 8, Lecture 3 27/03/2026

FIE

$$y(x) = F(x) + \lambda \int_a^b K(x,s)\, y(s)\, ds \qquad (1)$$

- F is the inhomogeneous term
- λ is a free parameter
- K is a symmetric kernel

Eigenfunctions / Eigenvalues:

$$y_i(x) = \lambda_i \int_a^b K(x,s)\, y_i(s)\, ds \qquad (2)$$

- Eigenvalues real
- Eigenvalues orthogonal

Solutions of (1):

$$y(x) = F(x) + \sum_i \frac{\lambda f_i}{\lambda_i - \lambda}\, y_i(x)$$

when λ is not an eigenvalue — CASE 1

When $\lambda = \lambda_{i_0}$ (eigenvalue), either $f_{i_0} = 0$ and infinitely many sl$^{\underline{n}}$s

$$y(x) = F(x) + \sum_{i \neq i_0} \frac{\lambda_{i_0} f_i}{\lambda_i - \lambda_{i_0}}\, y_i(x) + C \cdot y_{i_0}(x)$$

CASE 2a

$\left\{ \begin{array}{l} \underline{\text{Or}} \quad f_{i_0} \neq 0 \\ \text{and there are no solutions} \end{array} \right\}$ CASE 2b.

Today:

- Worked example (§ 8.5)
- Another type of FIE (§ 8.6)
- Existence theory — Chapter 9.

<u>Worked example</u> $K(x,s) = 1 - 3xs$

$$y(x) = x^2 + \lambda \int_0^1 (1-3xs)\, y(s)\, ds \qquad \underline{(3)}$$

Eigenfunctions (normalized)

$y_2(x) = \sqrt{3}\,(1-x) \quad \ldots \quad \lambda = 2$

$y_{-2}(x) = 1 - 3x \quad \ldots \quad \lambda = -2.$

Only linear functions can be generated by $K(x,s) = 1 - 3xs$, i.e.

$$A + Bx = \int_0^1 K(x,s)\, r(s)\, ds$$

However, we can still use the general theory to solve (3), even though Eq$^{=}$ (3) has a quadratic term.

General theory

$$f_1 = \int_0^1 \underbrace{x^2}_{F(x)} \underbrace{y_2}_{\text{Index } \# 1}(x)\,dx$$

$$= \int_0^1 x^2 \sqrt{3}(1-x)\,dx$$

$$= \frac{1}{4\sqrt{3}}$$

$$f_2 = \int_0^1 x^2 \underbrace{y_{-2}}_{\text{Index } \# 2}(x)\,dx$$

$$= -\frac{5}{12}.$$

Read off from general formula:

$$y(x) = x^2 + \sum_{i=2,-2} \frac{\lambda}{\lambda_i - \lambda} f_i y_i(x)$$

$$= x^2 + \frac{\lambda}{2-\lambda} \frac{1}{4\cancel{\sqrt{3}}} \cancel{\sqrt{3}}(1-x)$$

$$+ \frac{\lambda}{-2-\lambda}\left(-\frac{5}{12}\right)(1-3x)$$

$$\Rightarrow y(x) = x^2 + \frac{1}{4}\frac{\lambda}{2-\lambda}(1-x)$$

$$+ \frac{5}{12}\frac{\lambda}{2+\lambda}(1-3x)$$

$$\lambda \neq 2, -2.$$

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Another type of FIE: (§8.6)

$$0 = F(x) + \lambda \int_a^b K(x,s)\, y(s)\, ds \qquad (4)$$

Eq$^{\underline{n}}$ (4) is a FIE of <u>the first kind</u>.

We take $\lambda \neq 0$.

Since $F(x) = -\lambda \int_a^b K(x,s)\, y(s)\, ds$

$$= \int_a^b K(x,s)\, r(s)\, ds,$$

F has an expansion in eigenfunctions.

We can write:

$$F(x) = \sum_i f_i\, Y_i(x)$$

Back to (4):

$$0 = \sum_i f_i\, y_i(x) + \lambda \int_a^b K(x,s)\, y(s)\, ds$$

EIGENFUNCTION

$$= \sum_i f_i\, \lambda_i \int_a^b K(x,s)\, y_i(s)\, ds + \lambda \int_a^b K(x,s)\, y(s)\, ds$$

$$= \int_a^b K(x,s) \left[\sum_i f_i\, \lambda_i\, y_i(s) + \lambda\, y(s) \right] ds$$

$$= \int_a^b K(x,s) \left[\lambda\, Y(s) \right] ds$$

Equate

$$[\ldots] = [\ldots] \; ,$$

such that

$$\sum_i f_i \lambda_i y_i(s) + \lambda y(s) = \lambda Y(s)$$

$$\Rightarrow \boxed{y(s) = -\frac{1}{\lambda} \sum_i f_i \lambda_i y_i(s) + Y(s) \quad (5)}$$

where $Y(s)$ is any function such that

$$\int_a^b K(x,s) Y(s) \, ds = 0 .$$

What is $Y(s)$?

$$\int_a^b y_i(x) \int_a^b K(x,s) Y(s) \, ds \, dx = 0$$

$$\Rightarrow \int_a^b \int_a^b K(x,s) y_i(x) Y(s) \, ds dx = 0 .$$

Dummy index

$$\Rightarrow \int_a^b \int_a^b \underline{K(s,x) y_i(s)} \, Y(x) \, ds dx = 0 .$$

Symmetric

$$\Rightarrow \int_a^b \int_a^b K(x,s) y_i(s) Y(x) \, ds dx = 0$$

$$\Rightarrow \int_a^b \left[\int_a^b K(x,s) y_i(s) \, ds \right] Y(x) \, dx = 0 .$$

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Eigenfunction $\Rightarrow$ $\int_a^b [\lambda_i y_i(x)] Y(x)\,dx = 0$.

$\lambda_i \neq 0 \Rightarrow$ $\int_a^b y_i(x) Y(x)\,dx = 0$.

Back to the FIE of the first kind (Eq. (5)):

$$y(x) = -\frac{1}{\lambda} \sum_i f_i \lambda_i y_i(x) + Y(x)$$

where $\int_a^b Y(x) y_i(x)\,dx = 0 \quad \forall i$.

Chapter 9 : We show using iterative methods the existence of solutions of FIES.

FIE

$$y(x) = F(x) + \lambda \int_a^b k(x,s) y(s)\,ds$$

Suggests an iterative scheme:

$$y_{n+1}(x) = F(x) + \lambda \int_a^b k(x,s) y_n(s)\,ds$$

with $y_0(x) = F(x)$.

We have:

$$y_1(x) = F(x) + \lambda \int_a^b k(x,s)\, y_0(s)\, ds$$

$$y_2(x) = F(x) + \lambda \int_a^b k(x,s) \left[F(s) + \lambda \int_a^b k(x,s')\, y_0(s')\, ds' \right] ds$$

$$\Longrightarrow \quad y_2(x) = F(x) + \lambda \int_a^b k(x,s) F(s) + \lambda^2 \int_a^b \int_a^b k(x,s)\, k(s,s')\, y_0(s')\, ds'\, ds$$

$$= F(x) + \lambda \mathcal{K} F(x) + \lambda^2 \int_a^b \left[\int_a^b k(x,s)\, k(s,s')\, ds \right] y_0(s')\, ds'$$

$$= F(x) + \lambda \mathcal{K} F(x) + \lambda^2 \mathcal{K}^2 y_0(x)$$

Here:

$$\mathcal{K}^2 y_0(x) = \int_a^b \left[\int_a^b k(x,s)\, k(s,s')\, ds \right] y_0(s')\, ds'$$

$$|K y_0(x)| = \left| \int_a^b K(x,s)\, y_0(s)\, ds \right|$$

$$\leq M \cdot D \cdot |b-a|.$$

$$|K^2 y_0(x)| = \left| \int_a^b \int_a^b K(x,s) K(s,s') y_0(s')\, ds\, ds' \right|$$

$$\leq M^2 \cdot D \cdot |b-a|^2$$

Guess pattern:

$$|K^n y_0(x)| \leq M^n \cdot D\, |b-a|^n$$

We see that the remainder term is bounded by:

$$D \lambda^n \cdot M^n |b-a|^n$$

$$\Rightarrow |\lambda^n K^n y_0(x)| \leq D \lambda^n M^n |b-a|^n$$

If we restrict λ such that

$$\lambda \leq \frac{r}{M|b-a|}, \quad 0 \leq r < 1$$

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Guess pattern:

$$y_n(x) = F(x) + \lambda K F(x) + \lambda^2 K^2 F(x) + \dots \lambda^{n-1} K^{n-1} F(x) + \underbrace{\lambda^n K^n y_0(x)}_{\text{Remainder}}. \qquad (1)$$

We will show that under suitable conditions the remainder goes to zero, giving

$$y(x) = F(x) + \sum_{n=1}^{\infty} \lambda^n K^n F(x) \qquad (2)$$

We require:

$$|F(x)| \le C, \quad |y_0(x)| \le D,$$

$$|K(x,s)| \le M$$

for all x and s in $[a,b]$.

Also, these must be continuous functions in their arguments.

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$$\boxed{\lambda < \frac{1}{M|b-a|}}$$

and

$$\underbrace{|\lambda^n k^n y_0(x)|}_{\text{remainder}} < D r^n \quad , \; 0 \le r < 1$$

so the remainder goes to zero, as $n \to \infty$.

so finally,

$$\boxed{y(x) = F(x) + \sum_{n=1}^{\infty} \lambda^n K^n f(x)} \, .$$