Boundary Value Problems (BVPs) <u>Chapter 6</u>

<u>Idea</u> : $\qquad L[y] = r, \qquad x \in [a,b] \qquad (1)$

where

$$L[y] = p_2(x)\frac{d^2y}{dx^2} + p_1(x)\frac{dy}{dx} + p_0(x)y$$

Boundary Conditions

Instead of initial conditions, we apply BCs at $x=a$ and $x=b$.

Different types of BCs, starting with homogeneous BCs:

- Homogeneous Dirichlet BCs:

$$y(a) = y(b) = 0$$

- Homogeneous Neumann BCs:

$$y'(a) = y'(b) = 0$$

- Robin BCs:

$$\left.\begin{aligned} \alpha_a y(a) + \beta_a y'(a) &= 0 \\ \alpha_b y(b) + \beta_b y'(b) &= 0 \end{aligned}\right\} \text{ General case}$$

There are some applications when a and b (or both) are replaced by ∞, e.g. $[a,b] \to [a,\infty)$. Then (continuing with the example), we would impose a <u>behavioural</u> BC on the right-hand boundary, e.g. $y(x) \to 0$ as $x \to \infty$. Examples include solving Schrödinger's Equation in Quantum Mechanics.

We can get a solution to Eq. (1) by looking back to Equation (4.1) in the typed notes:

$$y(x) = \alpha\, \cancel{u}(x) + \beta\, \cancel{v}(x) - u(x)\int_{x_{\alpha \to b}}^{x} \frac{v(s)\,r(s)}{p_2(s)W(s)}\,ds$$

$$+\, v(x)\int_{x_\beta}^{x} \frac{u(s)\,r(s)}{p_2(s)W(s)}\,ds$$

<u>Pick</u> : $\alpha, \beta = 0, \quad x_\alpha = b, \quad x_\beta = a$

Hence:

$$y(x) = v(x)\int_{a}^{x} \frac{u(s)\,r(s)}{p_2(s)W(s)}\,ds + u(x)\int_{x}^{b} \frac{v(s)\,r(s)}{p_2(s)W(s)}\,ds \quad (2)$$

Define:

$$G(x,s) = \begin{cases} \dfrac{v(x)u(s)}{p_2(s)W(s)}, & a \le s < x \le b \\[2ex] \dfrac{v(x)v(s)}{p_2(s)W(s)}, & a \le x \le s \le b \end{cases} \quad (3)$$

Hence,

$$y(x) = \int_a^b G(x,s)\,r(s)\,ds \quad . \quad (4)$$

$G(x,s)$ is the Green's Function of the problem (1).

<u>Remember</u> : For this to work, we need $u(x)$ and $v(x)$ to be linearly independent.

Boundary conditions : We aim to put conditions on u and v such that, not only does $y(x) = \int_a^b G(x,s) r(s)$ solve the ODE (1), but it also solves the BCs.

Back to Eq$\underline{\text{n}}$ (2) :

$$y'(x) = -u'(x) \int_b^x \frac{v(s) r(s)}{p_2(s) W(s)} ds + v'(x) \int_a^x \frac{u(s) r(s)}{p_2(s) W(s)} ds$$

We have :

$$\left.\begin{aligned}
y(a) &= u(a) \int_a^b \frac{v(s) r(s)}{p_2(s) W(s)} ds \\
y'(a) &= u'(a) \int_a^b \frac{v(s) r(s)}{p_2(s) W(s)} ds \\
y(b) &= v(b) \int_a^b \frac{u(s) r(s)}{p_2(s) W(s)} ds \\
y'(b) &= v'(b) \int_a^b \left(\text{---}\,''\,\text{---} \right)
\end{aligned}\right\} \quad (5)$$

We require: $\alpha_a y(a) + \beta_a y'(a) = 0$

$\alpha_b y(b) + \beta_b y'(b) = 0$

We sub (5) into the BCs and get :

$$\left[\alpha_a u(a) + \beta_a u'(a) \right] \int_a^b \frac{v(s) r(s)}{p_2(s) W(s)} ds = 0$$

$$\left[\alpha_b v(b) + \beta_b v'(b)\right] \int_a^b \frac{u(s)\, r(s)}{p_2(s) W(s)}\, ds = 0$$

So we require:

$$\left\{ \begin{array}{ll} u(a) = \beta_a, & u'(a) = -\alpha_a \\ v(b) = \beta_b, & u'(b) = -\alpha_b \end{array} \right\}$$

Summarizing, to solve the BVP (1), we construct a solution out of two linearly ind. solutions of the homogeneous problem satisfying

$$L[u] = 0, \quad L[v] = 0,$$

such that:

$$u(a) = \beta_a, \quad u'(a) = -\alpha_a$$
$$v(b) = \beta_b, \quad u'(b) = -\alpha_b$$

the good case!

- If we can find such a u and a v, we are done, and Eqⁿ (1) has a solution ("L is invertible") – the good case. $Ax = b \Rightarrow x = A^{-1}b$

- If we can't find such a u and a v, then Equation (1) will have either no solution or infinitely many solutions ("L is not invertible").

A^{-1} not invertible

$b \in \mathrm{Im}(A)$ | $b \notin \mathrm{Im}(A)$

$A b_0 = b$ | $Ax = b$

$b \in \mathrm{Im}(A)$ | $b \notin \mathrm{Im}(A)$

$A b_0 = b$ | $Ax = b$

$x = \lambda x_0 + b_0$ | No solution

$x_0 \in \ker(A)$

Theorem 6.1 : Properties of the Green's Function

1. Defined on a square $\begin{cases} a \le x \le b \\ a \le s \le b \end{cases}$

2. $L_x[G(x,s)] = 0$

3. $G(x, x_-) = G(x, x_+)$

4. $\frac{\partial G}{\partial x}(x, x_-) - \frac{\partial G}{\partial x}(x, x_+) = \frac{1}{P_2(x)}$

Check #4.

$$\frac{\partial G}{\partial x} = \begin{cases} \frac{v'(x)u(s)}{P_2(s)W(s)} & s < x \\ \frac{u'(x)v(s)}{P_2(s)W(s)} & s \ge x \end{cases}$$

$$\frac{\partial G}{\partial x}(x,x_-) = \frac{v'(x)u(x)}{P_2(x)W(x)}$$

$$\frac{\partial G}{\partial x}(x,x_+) = \frac{u'(x)v(x)}{P_2(x)W(x)}$$

$$\frac{\partial G}{\partial x}(x,x_-) - \frac{\partial G}{\partial x}(x,x_+) = \frac{v'(x)u(x) - u'(x)v(x)}{P_2(x)W(x)}$$

$$= \frac{1}{p_2(x)}.$$

Property #5:

$$\alpha_a G(a,s) + \beta_a \frac{\partial G}{\partial x}(a,s) = 0$$

$$\alpha_b G(b,s) + \beta_b \frac{\partial G}{\partial x}(b,s) = 0$$

... follows by construction of the homogeneous s[n]s $u(x)$ and $v(x)$.

Example – § 6.2.1

$$\underset{p_2=1}{y''(x)} + y(x) = r(x), \qquad x \in [0, \pi/2]$$

BCs: $\quad y(0)=0, \quad y(\pi/2) = 0$

$u(x) = \sin(x)$ ... satisfies LHBC

$v(x) = \cos(x)$ ... satisfies RHBC

$$W(x) = \begin{vmatrix} u & v \\ u' & v' \end{vmatrix} = \begin{vmatrix} \sin(x) & \cos(x) \\ \cos(x) & -\sin(x) \end{vmatrix} = -1$$

$$p_2(x) W(x) = -1.$$

Solution:

$$y(x) = v(x) \int_a^x \frac{u(s)\, r(s)}{p_2(s) W(s)}\, ds - u(x) \int_b^x \frac{v(s)\, r(s)}{p_2(s) W(s)}\, ds$$

$$= \cos(x) \int_0^x \frac{\sin(s)\, r(s)}{(-1)}\, ds - \sin(x) \int_{\pi/2}^x \frac{\cos(s)\, r(s)}{(-1)}\, ds$$

$\pi/2$

$$\Rightarrow \boxed{y(x) = -\sin(x)\int_x^{\pi/2} \cos(s) r(s)\, ds - \cos(x)\int_0^x \sin(s) r(s)\, ds}$$

E.g. $r(s) = \sin(s)$

Then,

$$y(x) = -\sin(x)\int_x^{\pi/2} \sin(s)\cos(s)\, ds - \cos(x)\int_0^x \sin(s)\sin(s)\, ds$$

$(-2\sin(s)\cos(s))$

$$= -\sin(x)\left[-\tfrac{1}{2}\cos^2(s)\right]_x^{\pi/2} - \cos(x)\left[\tfrac{1}{2}s - \tfrac{1}{4}\sin 2s\right]_0^x$$

$$= -\sin(x)\left[+\tfrac{1}{2}\cos^2(x)\right] - \cos(x)\left[\tfrac{1}{2}x - \tfrac{1}{4}\sin 2x\right]$$

$$= -\tfrac{1}{2}\sin(x)\cos^2(x) - \tfrac{1}{2}x\cos(x)$$
$$+ \tfrac{1}{4}\cos(x)\sin(2x)$$

$$= \cancel{-\tfrac{1}{2}\sin(x)\cos^2(x)} - \tfrac{1}{2}x\cos(x)$$
$$\cancel{+ \tfrac{1}{4}\cos(x)\cdot 2\cdot\sin(x)\cos(x)}$$

$$\therefore \boxed{y(x) = -\tfrac{1}{2}x\cos(x)}$$

- Satisfies ODE $y'' + y = r$
- Satisfies the BCs $y(0) = 0,\ y(\pi/2) = 0$

Today: "The good case", in some sense, the linear operator L is invertible.

Wednesday: "The interesting case", when L is not invertible.

BVPs, Chapter 6

$$\left\{ \begin{array}{l} L[y] = r, \quad x \in [a,b] \\ \alpha_a y(a) + \beta_a y'(a) = 0 \\ \alpha_b y(b) + \beta_b y'(b) = 0 \end{array} \right\} \quad (1)$$

$$y(x) = -u(x)\int_b^x \frac{v(s)r(s)}{p_2(s)W(s)}\,ds + v(x)\int_a^x \frac{u(s)r(s)}{p_2(s)W(s)}\,ds$$

When things go wrong in the sl^{n} to (1).

Suppose that $u(x)$ satisfies the ODE and the LHBC and the RHBC. Suppose that $v(x)$ solves the ODE. Then $v(x)$ can't satisfy the RHBC.

Proof: Since u and v are lin. independent,

$$0 \neq W(b) = u(b)v'(b) - u'(b)v(b)$$

$$\Rightarrow 0 \neq \underbrace{\alpha_b u(b)}_{-\beta_b u'(b)} v'(b) - \alpha_b u'(b) v(b)$$

$$\Rightarrow 0 \neq -\beta_b u'(b) v'(b) - \alpha_b u'(b) v(b)$$

$$\Rightarrow 0 \neq \underbrace{-u'(b)}_{\neq 0}\left[\alpha_b v(b) + \beta_b v'(b)\right]$$

Hence, $\alpha_b v(b) + \beta_b v'(b) \neq 0$, so v does not satisfy the RHBC.

However, we still have the candidate solution:

$$y(x) = -u(x)\int_b^x \frac{\cancel{r(s)}\, v(s)}{p_2(s)W(s)}\,ds + v(x)\int_a^x \frac{u(s)r(s)}{p_2(s)W(s)}\,ds$$

$$y(x) = -u(x)\cancel{\int_b \frac{r(s)\,v(s)}{p_2(s)W(s)}\,ds} + v(x)\int_a \frac{u(s)\,r(s)}{p_2(s)W(s)}\,ds$$

We require:

$$0 = \alpha_a \underline{y(a)} + \beta_a y'(a)$$

$$0 = \alpha_b y(b) + \beta_b y'(b)$$

$\underline{x=a}$:

$$\Big[\underbrace{\cancel{\alpha_a u(a) + \beta_a u'(a)}}_{=0}\Big]\int_a^b \frac{v(s)r(s)}{p_2(s)W(s)}\,ds = 0$$

LHBC ok.

$\underline{x=b}$:

$$\Big[\underbrace{\alpha_b v(b) + \beta_b v'(b)}_{\neq 0}\Big]\int_a^b \frac{v(s)r(s)}{p_2(s)W(s)}\,ds = 0$$

So, for the BVP to have a solution:

$$\int_a^b \frac{v(s)r(s)}{p_2(s)W(s)}\,ds = 0 \qquad \text{SOLVABILITY CONDITION}$$

$\underline{\text{Theorem 6.2}}$: Let $L[y]=r$, $x\in[a,b]$, with homogeneous Robin BCs at $x=a$ and $x=b$. Suppose that $L[u]=0$ and $L[v]=0$, and that u satisfies the LHBC and the RHBC, and that u and v are lin. independent. Then, either:

$$\int_a^b \frac{v(s)r(s)}{p_2(s)W(s)}\,ds \neq 0 \qquad \text{BVP has no solutions}$$

OR

$$\int_a^b \frac{v(s)r(s)}{p_2(s)W(s)}\,ds \neq 0, \qquad \text{BVP has infinitely many solutions.}$$

- Called the BVP alternative.

- Called the BVP alternative.
- Infinite solutions:

$$y(x) = -u(x) \int_b^x \frac{v(s)\cdot\lambda \quad r(s)\,ds}{p_2(s)\,W(u,\lambda v)\,ds} \quad , \quad \lambda \in \mathbb{R}.$$

$$+ v(x)\cdot\lambda \int_a^x \frac{u(s)r(s)}{p_2(s)\,W(u,\lambda v)}\,ds$$

<u>Comparison with linear algebra</u>:

$Ax = b, \quad A \in \mathbb{R}^{n\times n}, \quad x, b \in \mathbb{R}^n, \quad x$ unknown.

- A invertible $\Rightarrow$ $x = A^{-1}b$, unique solution
- A not invertible:
  - b is in the image of A: there exists a $b_0 \in \mathbb{R}^n$ such that $Ab_0 = b$

    Solution: $x = \lambda x_0 + b$, $\quad x_0 \in \ker(A)$

    $\Rightarrow$ infinitely many solutions
  - b is not in the image of A: no <u>solution</u>

Worked example: § 6.3.3

$y'' + y = r, \quad x \in [0, \pi]$

BCs: $y(0) = 0, \quad y(\pi) = 0$

i) $r(x) = 1$

$u(x) = \sin(x)$, satisfies both BCs

$v(x) = \cos(x)$

$y_{PI} = 1.$

$y = A\sin(x) + B\cos(x) + \overbrace{1}^{y_{PI}}$ ATTEMPTED SOLUTION

<u>$x=0$</u>: $A\cdot 0 + B\cdot 1 + 1 = 0 \Rightarrow B = -1$

$x = \pi$: $A \cdot 0 - B + 1 = 0 \Rightarrow B = +1$

Inconsistent $\Rightarrow$ no solution of the BVP

ii) $r(x) = \sin(2x)$

$$y_{PI} = -\frac{1}{3}\sin(2x)$$

$$y(x) = A\sin(x) + B\cos(x) - \overbrace{\frac{1}{3}\sin(2x)}^{y_{PI}}$$ ATTEMPTED SOLUTION

$x = 0$: $A \cdot 0 + B \cdot 1 + 0 = 0 \Rightarrow B = 0$

$x = \pi$: $A \cdot 0 + B \cdot (-1) + 0 = 0 \Rightarrow B = 0$

$$\Rightarrow y(x) = \underbrace{A\sin(x)}_{\in \ker(L)} - \underbrace{\frac{1}{3}\sin(2x)}_{\in \operatorname{Im}(L)},$$

valid for all A, hence infinitely many solutions.

Check the BVP alternative:

i) $r(s) = 1$, $W(s) = \begin{vmatrix} \sin(x) & \cos(x) \\ \cos(x) & -\sin(x) \end{vmatrix} = -1$

$$\int_0^{\pi} \frac{\overbrace{\sin(s)}^{u(s)} \cdot 1}{(-1)}\, ds = -\cos(s)\Big|_0^{\pi} = 2 \neq 0$$

$\Rightarrow$ NO SL^{N} ☑

ii) $r(s) = \sin(2s)$

$$\int_0^{\pi} \frac{\overbrace{\sin(s)}^{u}\overbrace{\sin(2s)}^{r}}{(-1)}\, ds = -\frac{2}{3}\sin^3(s)\Big|_0^{\pi} = 0$$

$\Rightarrow$ Infinitely many Sl^{ns} ☑

Chapter 7 — Fredholm Integral Equations (FIEs)

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Motivation: Solution of a BVP can be converted into a FIE.

$$\frac{d}{dx}\left(p_2 \frac{dy}{dx}\right) + p_0 y = r$$

$$\Rightarrow \quad \underbrace{\frac{d}{dx}\left(p_2 \frac{dy}{dx}\right)}_{\text{NEW ODE}} = \underbrace{r - p_0 y}_{\text{NEW RHS}}$$

New ODE: $\quad \frac{d}{dx}\left(p_2 \frac{dy}{dx}\right) = 0$

$$\Rightarrow \quad v(x) = 1, \quad u(x) = P(x) = \int^{x} \frac{1}{p_2(x)}\, ds$$

$$W = \begin{vmatrix} u & v \\ u' & v' \end{vmatrix} = -\frac{1}{p_2(x)}$$

$$G(x,s) = \begin{cases} -P(s), & a \le s \le x \le b \\ -P(x) & a \le x \le s \le b \end{cases}$$

$$y(x) = \int_a^b G(x,s)\,[\overbrace{r(s) - p_0(s)\,y(s)}^{\text{NEW RHS}}]\,ds$$

$$= F(x) + \int_a^b \underbrace{\left(-G(x,s)\,p_0(s)\right)}_{K(x,s)} y(s)\,ds$$

This has the form of a generic FIE:

$$y(x) = F(x) + \lambda \int_a^b K(x,s)\,y(s)\,ds$$

FIEs are important in the analysis of ODEs:

FIEs are important in the analysis of ODEs:

- Easier to study the FIE than the ODE
- A nice example of a reproducing kernel Hilbert Space (these come up a lot in Data Science)

Separable kernels (§ 7.2)

$$K(x,s) = \sum_{j=0}^{n} u_j(x) v_j(s) \quad , \quad n \text{ finite}$$

Example: $K(x,s) = \sin(x+s) = \sin x \cos s + \sin s \cos x$

FIE:

$$y(x) = F(x) + \lambda \int_a^b \left(\sum_{j=1}^{n} u_j(x) v_j(s) \right) y(s) ds$$

$$= F(x) + \lambda \sum_{j=1}^{n} u_j(x) \underbrace{\int_a^b v_j(s) y(s) ds}_{= c_j}$$

$$\Rightarrow y(x) = F(x) + \lambda \sum_{j=1}^{n} u_j(x) c_j$$

Multiply both sides by $v_i(x)$ and integrate:

$$\underbrace{\int_a^b v_i(x) y(x) dx}_{c_i} = \underbrace{\int_a^b v_i(x) F(x) dx}_{b_i} + \lambda \sum_{j=1}^{n} c_j \underbrace{\int_a^b v_i(x) u_j(x) dx}_{A_{ij}}$$

$$\Rightarrow c_i = b_i + \lambda \sum_{j=1}^{n} A_{ij} c_j$$

In matrix form:

$$\underline{c} = \underline{b} + \lambda A \underline{c}$$

$$\Rightarrow (\mathbb{I} - \lambda A)\underline{c} = \underline{b}$$

<u>Homogeneous case</u>: $F(x) \Rightarrow \underline{b} = 0$

$$\Rightarrow (\mathbb{I} - \lambda A)\underline{c} = 0$$

Non-trivial sl$^{\underline{n}}$ $\Rightarrow$ $\det(\mathbb{I} - \lambda A) = 0$ Characteristic eq$^{\underline{n}}$

<u>Caution</u>:

- Roots of the characteristic eq$^{\underline{n}}$ are the eigenvalues of A^{-1}.
- Roots of the characteristic eq$^{\underline{n}}$ are the <u>eigenvalues</u> of the FIE.

Week 7, Lecture 1.

MB did § 7.3

— Separable kernels —

Worked example