Bessel's Equation (§5.6)

$$x^2 y'' + x y' + (x^2 - \nu^2) y = 0, \qquad \nu \in \mathbb{C} \qquad (1)$$

For applications, see the graded assignment :

* $\nabla^2 \phi = 0$ in cylindrical polar coordinates, gives Bessel's Eq$^{\underline{n}}$ in in the radial direction

* Hence, applications in :
  - Heat transfer
  - Electromagnetism
  - Fluid Mechanics (potential flow)

Re-write (1) as:

$$y'' + \underbrace{\frac{1}{x}}_{p(x)} y'(x) + \underbrace{\left(1 - \frac{\nu^2}{x^2}\right)}_{q(x)} y = 0 .$$

$$\left.\begin{aligned} x\,p(x) &= 1 \\ x^2 q(x) &= x^2 - \nu^2 \end{aligned}\right\} \text{ polynomials, hence analytic @ } x = 0.$$

Hence, $x = 0$ is a regular singular point.

We therefore propose a Frobenius series solution of Eq$^{\underline{n}}$ (1) :

$$y(x) = x^{\alpha} \left(\sum_{n=0}^{\infty} a_n x^n \right) \qquad (2)$$

$$= \sum_{n=0}^{\infty} a_n x^{n+\alpha}$$

$y'' \to (n+\alpha)(n+\alpha-1) \times x^{n+\alpha-2}$

Sub (2) back into (1) :

$$\sum_{n=0}^{\infty} (n+\alpha)(n+\alpha-1)\, x^{n+\cancel{\alpha}-2} \cdot x^2 \cdot a_n$$

x^{α}

$$+ \sum_{n=0}^{\infty} (n+\alpha)\, x^{n+\cancel{\alpha}-1} \cdot x \cdot a_n$$

$$+ \sum_{n=0}^{\infty} a_n x^{n+\cancel{\alpha}} \cdot x^2 \; - \; \nu^2 \sum_{n=0}^{\infty} a_n x^{n+\cancel{\alpha}} \; = \; 0.$$

Hence:

$$\sum_{n=0}^{\infty} a_n (n+\alpha)(n+\alpha-1)\, x^n \; + \; \sum_{n=0}^{\infty} a_n (n+\alpha)\, x^n$$

$$\boxed{+ \sum_{n=0}^{\infty} a_n x^{n+2}} \; - \; \nu^2 \sum_{n=0}^{\infty} a_n x^n \; = 0$$

DUMMY VARIABLE

Re-index:

$$p = n-2$$

$$\Rightarrow p+2 = n \quad , \quad p_{start} = n_{start} - 2 = -2 \; \checkmark$$

Hence:

$$\sum_{p=-2}^{\infty} a_{p+2}\, (p+2+\alpha)(p+1+\alpha)\, x^{p+2}$$

$$+ \sum_{p=-2}^{\infty} a_{p+2}\, (p+2+\alpha)\, x^{p+2}$$

$$+ \boxed{\sum_{p=0}^{\infty} a_p\, x^{p+2}} \; - \; \nu^2 \sum_{p=-2}^{\infty} a_{p+2}\, x^{p+2} \; = \; 0$$

Hence:

$$C_{-2}\, x^{-2+2} + C_{-1}\, x^{-1+2} + \sum_{p=0}^{\infty} x^{p} \cdot C_p = 0,$$

to conclude $C_{-2} = 0$, $C_{-1} = 0$, $C_p = 0 \;\; \forall\, p \geq 0$.

i.e. we set coefficients of powers of x to zero.

<u>$p=-2$</u>

$$a_0\left[\alpha(\alpha-1)+\alpha-\nu^2\right]=0.$$

$$F(s)=s(s-1)+s-\nu^2, \quad s\in\mathbb{C}.$$

$$\Rightarrow \boxed{a_0\cdot F(\alpha)=0}$$ Indicial Equation (3)

<u>$p=-1$</u>

$$a_1\left[\alpha(\alpha+1)+(\alpha+1)-\nu^2\right]=0.$$

$$\Rightarrow a_1 F(\alpha+1)=0.$$

<u>$p>-1$</u>

$$a_{p+2}(p+2+\alpha)(p+1+\alpha)+a_{p+2}(p+2+\alpha)+a_p-\nu^2 a_{p+2}=0.$$

$$\Rightarrow a_{p+2}\Big[\underbrace{\overset{s}{(p+2+\alpha)}(p+1+\alpha)+(p+2+\alpha)-\nu^2}_{F(\alpha+p+2)}\Big]=-a_p$$

$$\Rightarrow \boxed{a_{p+2}F(\alpha+p+2)=-a_p}$$ RECURRENCE RELATION (4)

$$a_0 \; F(\alpha) = 0.$$

We demand that $a_0 \neq 0$. Therefore

$$F(\alpha) = 0.$$

But
$$
\begin{aligned}
F(\alpha) &= \alpha(\alpha-1) + \alpha - v^2 \\
&= \alpha^2 - \cancel{\alpha} + \cancel{\alpha} - v^2 \\
&= \alpha^2 - v^2
\end{aligned}
$$

$$F(\alpha) = 0 \Rightarrow \alpha = \pm v .$$

Two roots: $\boxed{\alpha_1 = v}$, $\alpha_2 = -v$.

Let $\alpha = v$, and move to $\underline{p = 1}$:

$$a_1 \; F(\overset{=\alpha}{\widetilde{v}} + 1) = 0.$$

$$\Rightarrow a_1 \left[(v+1)(v) + v + 1 - v^2 \right] = 0$$

$$\Rightarrow a_1 \left[\cancel{v^2} + v + v + 1 - \cancel{v^2} \right] = 0$$

$$\Rightarrow a_1 \left[2v + 1 \right] = 0 \quad (5)$$

Look at non-special case first,

$$\alpha_1 - \alpha_2 \neq \text{integer}$$

$$\Rightarrow v - (-v) \neq \text{integer}$$

$$\Rightarrow 2v \neq \text{integer}$$

$\Rightarrow \quad 2\nu \neq -1$

$\Rightarrow \quad 2\nu+1 \neq 0$

Hence, the only way to satisfy (5) is for $a_1 = 0$.

Hence by the recurrence relation (4) (steps of two),

$a_3 = a_5 = a_7 = \dots = 0.$

Recurrence relation: $\quad p+2 = n$

$$a_n F(\underset{\underset{\underset{p+2+\alpha}{\uparrow}}{n+\alpha}}{\overset{}{\underset{\uparrow}{n+\nu}}}) = -a_{n-2}$$

$$\Rightarrow a_n\left[(\nu+n)(\nu+n-1) + n+\nu - \nu^2\right] = -a_{n-2}$$

$$\Rightarrow a_n\left[\cancel{\nu^2} + \nu n - \cancel{\nu} + \nu n + n^2 - \cancel{n} + \cancel{\nu} + \cancel{n} - \cancel{\nu^2}\right] = -a_{n-2}$$

$$\Rightarrow a_n\left[2\nu n + n^2\right] = -a_{n-2}$$

$$\Rightarrow a_n \cdot n(2\nu+n) = -a_{n-2}$$

$$\Rightarrow \boxed{a_n = -\frac{a_{n-2}}{n(n+2\nu)}}$$

Only even terms in the power series survive: $n = 2\ell$

$$a_{2\ell} = -\frac{a_{2\ell-2}}{}$$

$$2\ell \cdot 2(\ell+\nu)$$

$$\Rightarrow a_{2\ell} = -\frac{a_{2(\ell-1)}}{2^2\,\ell\,(\ell+\nu)}$$

$$= -\frac{1}{2^2\,\ell\,(\ell+\nu)}\,a_{2(\ell-1)}$$

$$= -\frac{1}{2^2\,\ell\,(\ell+\nu)}\,\frac{a_{2(\ell-2)}}{2^2(\ell-1)}\,\frac{(-1)}{(\ell-1+\nu)}$$

$$= \frac{(-1)(-1)(-1)\qquad a_{2(\ell-3)}}{2^2\cdot 2^2\cdot 2^2\;\ell\,(\ell-1)(\ell-2)\;(\ell+\nu)(\ell-1+\nu)(\ell-2+\nu)}$$

Guess pattern:

$$a_{2\ell} = \frac{(-1)^p\qquad a_{2(\ell-p)}}{\underbrace{2^2\cdots 2^2}_{p\text{ times}}\;\boxed{\ell(\ell-1)\cdots(\ell-p+1)}\,(\ell+\nu)\cdots(\ell-p+1+\nu)}$$

keep going until $\ell = p$:

$$a_{2\ell} = \frac{(-1)^\ell}{2^{2\ell}\,\ell!}\,\frac{a_0}{(\ell+\nu)\cdots(1+\nu)}$$

Hence, the recurrence relation is solved.

Since a_0 is free (but non-zero), we set

$$a_0 = \frac{1}{2^\nu}.$$

Solution: $\qquad n = 2\ell$

$$y(x) = x^\nu \sum_{n\text{ even}}^{\infty} a_n x^n$$

$$y(x) = \quad x \quad \text{n even}$$

$$= \underbrace{\left(\frac{x}{2}\right)^{\nu} \sum_{\ell=0}^{\infty} \frac{(-1)^{\ell}}{2^{2\ell}\,\ell!} \cdot \frac{1}{(1+\nu)\cdots(\ell+\nu)} \cdot \left(\frac{x}{2}\right)^{2\ell}}_{J_\nu(x)}$$

Second solution : $\alpha_2 = -\nu$, where :

$$\left\{ \begin{array}{ll} \alpha_1, \alpha_2 \neq 0 & (\text{i.e. } \nu \neq 0) \\ \alpha_1 - \alpha_2 \neq \text{Integer} & (2\nu \neq \text{Integer}) \end{array} \right\} \text{ GENERAL CASE}$$

p=-2 : $a_0 \underbrace{\cancel{F(-\nu)}}_{=0} = 0$. ☒ $\nu \to -\nu$

p=-1 : $a_1 \underbrace{(1-2\nu)}_{\neq 0,\ \text{GENERAL CASE}} = 0$.

$\Rightarrow a_1 = 0$.

p > -1 : $a_n F(-\nu+n) = -a_{n-2}$

Becomes $a_n = -\dfrac{a_{n-2}}{n(n-2\nu)}$

Since 2ν is not an integer, this recurrence relation is valid for all $n \in \{0,1,2,\dots\}$.

This gives:

$$J_{-\nu}(x) = \left(\frac{x}{2}\right)^{-\nu} \sum^{\infty} \frac{(-1)^{\ell}}{\ell!\,(\ \ \cdots\ \ )} \left(\frac{x}{2}\right)^{\ell}$$

$$J_{-\nu}(x) = \left(\frac{x}{2}\right)^{-\nu} \sum_{l=0} \frac{(-1)^l}{l!\,(l-\nu)\cdots(1-\nu)} \left(\frac{x}{2}\right)^{2l}$$

So our two linearly independent solutions in the non-special case are:

$$J_\nu(x) \text{ and } J_{-\nu}(x).$$

Wednesday : $\nu = 0$ $\begin{cases} \text{Frobenius Series S1?} \\ y_2(x) = \left(\frac{\partial y_\alpha}{\partial \alpha}\right)_{\alpha=\alpha_1} \end{cases}$

Bessel's ODE:

$$y'' + \frac{1}{x}y' + \left(1 - \frac{\nu^2}{x^2}\right)y = 0 \qquad (1)$$

$$y(x) = x^{\alpha}\left(\sum_{n=0}^{\infty} a_n x^n\right)$$

General value of ν:

Indicial Equation: $a_0 F(\alpha) = 0, \quad a_0 \neq 0$

$$\Rightarrow F(\alpha) = 0, \quad F(\alpha) = \alpha^2 - \nu^2$$

Hence, $\alpha = \pm\nu$, for a general value of ν.

This gave us $J_\nu(x)$ and $J_{-\nu}(x)$ as solutions of (1).

Today: Take $\nu = 0$ (§5.6.2).

$p = -2$: $a_0 \cdot F(\alpha) = 0$

$$\Rightarrow a_0 \cdot (\alpha^2 - \cancel{\nu^2}) = 0$$

$$\Rightarrow a_0 \cdot \alpha^2 = 0, \quad a_0 \neq 0 \Rightarrow \boxed{\alpha = 0}$$

$p = -1$: $a_1 F(\alpha+1) = 0$

$$\Rightarrow a_1 \cdot \left[(\alpha+1)^2 - \cancel{\nu^2}\right] = 0$$

$$\Rightarrow a_1 \cdot 1 = 0 \quad \Rightarrow \quad a_1 = 0$$

$n > 1$: $a_n F(n + \cancel{\nu}) = -a_{n-2}$

$$\Rightarrow a_n n^2 = -a_{n-2}$$

$$\Rightarrow a_n = -\frac{a_{n-2}}{n^2} \qquad \text{RECURRENCE REL}^{\underline{N}}$$

Hence, $a_3 = a_5 = \cdots \quad 0$

Let $n \to 2n$ in the recurrence relation:

$$a_{2n} = -\frac{a_{2n-2}}{2^2 n^2} .$$

Same procedure as in previous lecture:

$$a_{2n} = \frac{(-1)^n a_0}{2^{2n} (n!)^2} ,$$

giving

$$J_0(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(n!)^2} \left(\frac{x}{2}\right)^{2n}$$

We can't get a second solution using this method.

We use Frobenius's Method instead.

$$y_\alpha(x) = x^\alpha \sum_{n=0}^{\infty} a_n(\alpha) x^n$$

We don't impose the indicial equation, but we do impose the recurrence relation:

$$a_0 F(\alpha) = ?$$

$$a_1 F(\alpha+1) = 0 \qquad F(s) = s^2 .$$

$$a_n = -\frac{a_{n-2}}{F(n+\alpha)}$$

So again, $a_3 = a_5 = \cdots \quad 0$, so

$$y_\alpha(x) = x^\alpha \sum_{n=0}^{\infty} a_{2n}(\alpha) x^{2n} .$$

From last week,

$$y_2(x) = \left(\frac{\partial y_\alpha(x)}{\partial \alpha} \right)_{\alpha = \alpha_1 = 0}$$

We need to calculate $\dfrac{d\, a_{2n}(\alpha)}{d\alpha}$.

From the recurrence relation,

$$a_{2n} = -\frac{a_{2n-2}}{(2n+\alpha)^2} \quad \cdots\cdots \quad = -\frac{a_{2n-2}}{F(2n+\alpha)}$$

Hence,

$$a_{2n}(\alpha) = \frac{(-1)^n \, a_0}{(2n+\alpha)^2 \left(2(n-1)+\alpha\right)^2 \cdots (2+\alpha)^2} \quad \text{- set } a_0 = 1$$

$$a_{2n}(\alpha) = \frac{(-1)^n}{\mathcal{F}(\alpha)}$$

$$\Rightarrow \frac{d}{d\alpha} a_{2n}(\alpha) = (-1)^n \left(-\frac{1}{\mathcal{F}^2} \frac{d\mathcal{F}}{d\alpha} \right)$$

$$= \frac{-(-1)^n}{\mathcal{F}} \left(\frac{1}{\mathcal{F}} \frac{d\mathcal{F}}{d\alpha} \right)$$

$$= \frac{-(-1)^n}{\mathcal{F}} \frac{d \log \mathcal{F}}{d\alpha} \quad (2)$$

But

$$\mathcal{F} = (2n+\alpha)^2 \left(2(n-1)+\alpha\right)^2 \cdots (2+\alpha)^2$$

$$= \prod_{j=1}^{n} (2j+\alpha)^2$$

$j=1$

$$\Rightarrow \log \mathcal{J} = \log\left[\prod_{j=1}^{n} (2j+\alpha)^2\right]$$

$$= \sum_{j=1}^{n} \log (2j+\alpha)^2$$

$$= 2 \sum_{j=1}^{n} \log(2j+\alpha)$$

$$\Rightarrow \frac{d}{d\alpha} \log \mathcal{J} = 2 \sum_{j=1}^{n} \frac{d}{d\alpha} \log(2j+\alpha)$$

$$= \sum_{j=1}^{n} \frac{2}{2j+\alpha} \qquad (3)$$

Back to (2):

$$\frac{d}{d\alpha} a_{2n}(\alpha) = -\overbrace{\left[\frac{(-1)^n}{\mathcal{J}(\alpha)}\right]}^{a_{2n}(\alpha)} \frac{d}{d\alpha} \log \mathcal{J}(\alpha)$$

$$\overset{(3)}{=} -a_{2n}(\alpha) \sum_{j=1}^{n} \frac{\cancel{2}}{\cancel{2}j+\cancel{\alpha}}$$

Hence:

$$\left[\frac{d}{d\alpha} a_{2n}(\alpha)\right]_{\alpha=0} = -a_{2n}(0) \sum_{j=1}^{n} \frac{1}{j}$$

$$= -\boxed{a_{2n}(0)} \left(1 + \frac{1}{2} + \cdots + \frac{1}{n}\right)$$

Back to:

$$y_2(x) = \left(\frac{\partial y_\alpha(x)}{\partial \alpha}\right)_{\alpha=0}$$

$$= \frac{\partial}{\partial\alpha}\left(x^\alpha \sum_{n=0}^{\infty} a_{2n}(\alpha)\, x^{2n}\right)_{\alpha=0}$$

$$\frac{\partial}{\partial \alpha} x^\alpha = \frac{\partial}{\partial \alpha} e^{\log x^\alpha}$$

$$= \frac{\partial}{\partial \alpha} e^{\alpha \log x}$$

$$= \log x \; \underbrace{e^{\alpha \log x}}$$

$$= x^\alpha \log x$$

$$\left\{ \begin{aligned} &\frac{\partial}{\partial \alpha} e^{\alpha \cdot c} \\ &= c \, e^{\alpha \cdot c} \end{aligned} \right.$$

$$y_2(x) = \left(\frac{\partial y_\alpha(x)}{\partial \alpha} \right)_{\alpha = 0}$$

$$= \frac{\partial}{\partial \alpha} \left(x^\alpha \sum_{n=0}^{\infty} a_{2n}(\alpha) \, x^{2n} \right)_{\alpha = 0}$$

$$= \left(x^\alpha \log x \sum_{n=0}^{\infty} a_{2n}(\alpha) \, x^{2n} + x^\alpha \sum_{n=0}^{\infty} \frac{d a_{2n}(\alpha)}{d\alpha} x^{2n} \right)_{\alpha = 0}$$

$$= \cancel{x^\alpha}^{=1} \log x \sum_{n=0}^{\infty} a_{2n}(0) \, x^{2n}$$

$$+ \cancel{x^\alpha}^{=1} \sum_{n=0}^{\infty} \frac{d a_{2n}(0)}{d\alpha} x^{2n}$$

$$= \log x \; J_0(x)$$

$$+ \sum_{n=0}^{\infty} \left[- a_{2n}(0) \right] \left(1 + \frac{1}{2} + \cdots + \frac{1}{n} \right) x^{2n}$$

$$\ldots J(\ldots) \ldots \sum^{\infty} \frac{(-1)^{n-1}}{\ldots} \left(1 + \frac{1}{2} + \cdots + \frac{1}{n} \right) x^{2n}$$

$$= \log x \, J_0(x) + \sum_{n=0}^{\infty} \frac{(-1)^{n-1}}{2^{2n}(n!)^2}\left(1+\tfrac{1}{2}+\dots+\tfrac{1}{n}\right)x^{2n}$$

$$= Y_0(x)$$

Second solution:

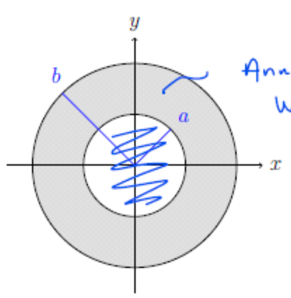
$$Y_0(x) = \log x \, J_0(x) + \sum_{n=0}^{\infty} \frac{(-1)^{n-1}}{2^{2n}(n!)^2}\left(1+\dots+\tfrac{1}{n}\right)x^{2n}$$

Note: $J_0(x)$ is defined for $x \to 0$,
but $Y_0(x)$ is not.

This is a general property of Bessel functions of various orders.

Remark (filled in after lecture) $Y_0(x)$ comes into play in applied problems when $x=0$ is not a possibility, e.g. in an annulus – see e.g. graded assignment:

Applied Analysis | Graded Assignment



Annulus.
We avoid $(x,y)=0$
$(\rho=0)$.

$\rho=0$, $\cancel{Y_0}$

$\rho>0$ Y_0

Figure 1: Setup for Question 4

4. **Physical Application of Bessel Functions**

A very long hollow cylinder of inner radius a and outer radius b (whose cross section ios indicated in Figure 1 is made of conducting material of diffusivity κ. If the inner and outer surfaces are kept at temperature zero while the initial temperature is a given function $f(\rho)$, where ρ is the radial distance from the axis, show that the temperature at any later time t is given by:

$$u(\rho,t) = \sum_{m=1}^{\infty} A_m e^{-\kappa\lambda_m^2 t} u_0(\lambda_m \rho), \qquad (6)$$

Bessel Equation

where

$$u(\rho,t)=\sum_{m=1}^{\infty}A_m e^{-\kappa\lambda_m^2 t}u_0(\lambda_m\rho), \tag{6}$$

Bessel Equation

where

$$\underline{u_0(\lambda_m\rho)}=Y_0(\lambda_m a)J_0(\lambda_m\rho)-J_0(\lambda_m a)\underline{Y_0(\lambda_m\rho)}. \tag{7}$$

allowed, since ρ is never zero

Here also, $f(\rho)$ has been expanded in terms of u_0:

$$f(\rho)=\sum_{m=1}^{\infty}A_m u_0(\lambda_m\rho),$$

where

$$A_m=\frac{\int_a^b \rho f(\rho)u_0(\lambda_m\rho)\mathrm{d}\rho}{\int_a^b \rho[u_0(\lambda_m\rho)]^2\mathrm{d}\rho}. \tag{8}$$

Key information:

- The heat equation:

$$\boxed{\frac{\partial u}{\partial t}=\kappa\left(\frac{\partial^2 u}{\partial\rho^2}+\frac{1}{\rho}\frac{\partial u}{\partial\rho}\right), \qquad t>0.}$$

- Fixed-temperature boundary conditions: $u(a,t)=0$, $u(b,t)=0$.
- Bounded solution $|u(\rho,t)|<M$, for all $t\geq 0$.

Exercises #4, Question 4

$$xy''+(c-x)y'-ay=0 \tag{1}$$

Classify $x=0$:

$$y''+\underbrace{\left(\frac{c}{x}-1\right)}_{p(x)}y'-\underbrace{\frac{a}{x}}_{q(x)}y=0$$

$$\left.\begin{aligned} x\,p(x) &= c-x \\ x^2 q(x) &= -ax \end{aligned}\right\}$$ polynomials, these trivially have a Taylor series centred @ $x=0$.

Hence, $x=0$ is a regular singular point.

Frobenius series solution:

$$y(x)=x^{\alpha}\sum_{n=0}^{\infty}a_n x^n=\sum_{n=0}^{\infty}a_n x^{n+\alpha}$$

Sub in to ODE (1)

$$\underline{x}y''+cy'-xy'-ay=0$$

Hence:

$n+\cancel{\alpha}-2+1$

Hence:

$$\sum_{n=0}^{\infty} a_n (n+\alpha)(n+\alpha-1)\, x^{n+\cancel{\alpha}-2+1} + c\sum_{n=0}^{\infty} a_n (n+\alpha)\, x^{n+\cancel{\alpha}-1} - \sum_{n=0}^{\infty} a_n (n+\alpha)\, x^{n+\cancel{\alpha}\underbrace{\cancel{-1}+\cancel{1}}_{\text{Cancel}}} - a\sum_{n=0}^{\infty} a_n x^{n+\cancel{\alpha}} = 0.$$

x^n

Re-index: $p = n-1$, $\begin{cases} n = p+1 \\ p_{start} = n_{start} - 1 = 0-1 = -1 \end{cases}$

Hence:

$$\sum_{p=-1}^{\infty} a_{p+1}(p+1+\alpha)(p+\alpha)\, x^p + c\sum_{p=-1}^{\infty} a_{p+1}(p+1+\alpha)\, x^p \underbrace{- \sum_{p=0}^{\infty} a_p (p+\alpha) x^p - a\sum_{p=0}^{\infty} a_p x^p}_{\text{Dummy-variable trick}} = 0$$

$$\sum_{p=-1}^{\infty} C_p x^p = 0 \Rightarrow C_p = 0$$

Set coefficients of powers of x to zero.

$p=-1$: $\underline{x^{-1}}$ $\quad a_0\left[\alpha(\alpha-1) + c\,\alpha\right] = 0.$

$$F(s) = s(s-1) + c\cdot s$$

Indicial Equation: $a_0 \cdot F(\alpha) = 0$, $a_0 \neq 0$

$$\Rightarrow F(\alpha) = 0$$

$$\alpha(\alpha-1) + c\cdot\alpha = 0$$

$$\Rightarrow \begin{cases} \alpha = 0 \\ \alpha = 1-c \end{cases}$$

Distinct roots (unless $c=1$).

Week 6, Lecture 3

Exercises # 4, question 4.

Refer to Wednesday lecture/ handout.

$$\sum_{p=-1}^{\infty} a_{p+1}(p+1+\alpha)(p+\alpha)x^p$$

x^p

$$+\ c\sum_{p=-1}^{\infty} a_{p+1}(p+1+\alpha)x^p$$

$$-\sum_{p=0}^{\infty} a_p(p+\alpha)x^p - a\sum_{p=0}^{\infty} a_p x^p = 0.$$

$\underline{p=-1}$: x^{-1} $\quad a_0\left[\underbrace{\alpha(\alpha-1)+c\alpha}\right] = 0.$

$$F(s) = s(s-1)+c\cdot s$$

Hence: $\quad a_0 F(\alpha) = 0, \quad a_0 \neq 0.$

$\Rightarrow \quad F(\alpha) = 0.$

$\Rightarrow \quad \begin{cases} \alpha = 0 \\ \alpha = 1-c \end{cases}$ Distinct roots, unless $c = 1$.

$\underline{\alpha = 0}$:

$$a_{p+1}\left[(p+1)p + c(p+1)\right] = a_p\left[p+a\right]$$

RECURRENCE REL^{N}.

Re-index again : $n = p+1$.

$$\Rightarrow a_n \left[n(n-1) + c \cdot n \right] = \underset{n-1}{a_{\cancel{p}}} \left[n-1+a \right]$$

$$\Rightarrow a_n n \left[n-1+c \right] = a_{n-1} \left[n-1+a \right].$$

$$\Rightarrow a_n = \frac{\underline{a_{n-1}} \, (n-1+a)}{n(n-1+c)} \qquad \boxed{n \geq 1}$$

Avoid : $c + n - 1 = 0$.

$$\Rightarrow c = 1-n, \quad n = 1, \ldots$$

$$\Rightarrow c = 0, -1, -2, \ldots$$

Solve the recurrence relation:

$$a_n = a_{n-1} \frac{(n-1+a)}{n(n-1+c)}$$

$$= \frac{\underline{a_{n-2}} \, (n-2+a)}{n-1 \, (n-2+c)} \cdot \frac{(n-1+a)}{n(n-1+c)}$$

$$= \frac{a_{n-p} \, (n-p+a)}{(n-p+1) \, (n-\underline{p+c})} \cdots\cdots \frac{(n-1+a)}{n(n-1+c)}.$$

All the way ~~down~~ up to $\underline{p = n}$:

$$\Rightarrow a_n = a_0 \frac{(a) \cdots (n-1+a)}{1 \cdots n \quad c \cdots (n-1+c)}$$

$$= a_0 \frac{a \cdots (n-1+a)}{n! \quad c \cdots (n-1+c)}$$

Rising factorial / Pochammer symbol — Abramowitz and Stegun.

$$(a)_n = a(a+1)\cdots(a+n-1) = \prod_{k=0}^{n-1}(a+k)$$

$$\Rightarrow \quad a_n = a_0 \frac{(a)_n}{n!\,(c)_n}$$

Hence:

$$y_1(x) = \underbrace{a_0 \sum_{n=0}^{\infty} \frac{(a)_n}{n!\,(c)_n} x^n}_{{}_1F_1(a;c;x)}$$

Confluent hypergeometric function, $c \notin 0 \cup -\mathbb{N}$.

Second solution: $\alpha = 1-c$.

$$a_{p+1}\left[(\overbrace{p+1}^{n}\underbrace{+1-c}_{\alpha})(\overbrace{p+}^{n}\underbrace{1-c}_{\alpha}) + c(\overbrace{p+1}^{n}+\underbrace{1-c}_{\alpha})\right]$$

$$= a_p\left[(\overbrace{p+}^{n}\underline{1-c}) + a\right]$$

Re-index, $n = p+1$

$$a_n\left[(n+1-c)\underline{(n-c)} + \underline{c}(n+1-c)\right]$$

$$= a_{n-1}\left[n-c+a\right]$$

$$\Rightarrow \quad a_n(n+1-c)(\underline{n}) = a_{n-1}(n-c+a)$$

$$\Rightarrow \quad a_n = \frac{a_{n-1}\,(n-c+a)}{n\,(n+1-c)}$$

$$= \frac{a_{n-1}\,[\,n-1+(a-c+1)\,]}{n\,[\,n-1+(2-c)\,]}$$

$$= \frac{a_0\,(a-c+1)_n}{n!\,(2-c)_n}$$

Hence, $2-c \notin 0 \cup -\mathbb{N}$.

Second solution:

$$y_2(x) = \frac{a_0 x^{1-c}}{x^a} \sum_{n=0}^{\infty} \frac{(a-c+1)_n}{n!\,(2-c)_n}\, x^n$$

$$= a_0\, x^{1-c}\, {}_1F_1\left(a-c+1;\, 2-c;\, x\right)$$