We first wrap up our study of the Orr–Sommerfeld Equation before moving on to new topics.

Exercises #3, starting with Q3.

Q3.

$$V = (\partial_z^2 - k^2)\Psi$$

$V = 0$ ✓, $V = Ai(\xi), Bi(\xi)$

P_2

$$\underbrace{(\partial_z^2 - k^2)}_{V}\Psi = Ai(\xi), \quad Bi(\xi).$$

Method of variation of parameters: $\Psi_1 = \cosh(kz)$, $\Psi_2(kz)$.

$$W = \begin{vmatrix} \Psi_1 & \Psi_2 \\ \Psi_1' & \Psi_2' \end{vmatrix} = \begin{vmatrix} \cosh(kz) & \sinh(kz) \\ k\sinh(kz) & k\cosh(kz) \end{vmatrix}$$

$$= k\left[\cosh^2(kz) - \sinh^2(kz)\right] = k$$

Also, $P_2 = 1$

Hence, $P_2 W = k$ (= Const.)

Method of Variation of Parameters:

$$\Psi = \int_0^z \frac{\left[-\Psi_1(z)\Psi_2(z') + \Psi_1(z')\Psi_2(z)\right]}{k \ (= P_2 W)} Ai(\xi(z'))\,dz'$$

$$= -\frac{\cosh(kz)}{k}\int_0^z \sinh(kz')\,Ai(\xi(z'))\,dz'$$

$$+ \frac{\sinh(kz)}{k}\int_0^z \cosh(kz')\,Ai(\xi(z'))\,dz'$$

Hence:

$$\Psi = \frac{1}{k}\int_0^z \left[\sinh(kz)\cosh(kz') - \cosh(kz)\sinh(kz')\right] Ai(\xi(z'))\,dz'$$

$$\boxed{\chi_1(z) = \frac{1}{k}\int_0^z \sinh[k(z-z')]\,Ai(\xi(z'))\,dz'}$$

Also,

$$\chi_2(z) = \frac{1}{k}\int_0^z \sinh[k(z-z')]\,Bi(\xi(z'))\,dz'$$

Fill in for ξ:

$$\xi = (ikRe)^{1/3}\left(z - c - \frac{ik}{Re}\right)$$

Hence,

$$\chi_1(z) = \frac{1}{k}\int_0^z \sinh(k(z-z'))\,Ai\left((ikRe)^{1/3}\left(z' - c - \frac{ik}{Re}\right)\right)dz'$$

For linearly ind. sl$^{\underline{ns}}$s of OS eq$^{\underline{n}}$. General sl$^{\underline{n}}$:

$$\Psi(z) = A\,\Psi_1(z) + B\,\Psi_2(z) + \underline{C}\,\chi_1(z) + D\,\chi_2(z)$$

Q.4. Boundary conditions:

$\Psi, \Psi' = 0$ @ $z=0$ and $z=1$.

Notice: $\chi_i(0) = 0$, $\chi_i'(0) = 0$.

$z=0$: $\quad \underline{A\cdot 1} + B\cdot 0 + C\cdot 0 + D\cdot 0 = \underline{0}$

$\quad A\cdot 0 + \underline{k\cdot B}\cdot 1 + C\cdot 0 + D\cdot 0 = 0$.

$z=1$. $\quad A\cancel{\Psi_1(1)} + B\cancel{\Psi_2(1)} + C\chi_1(1) + D\chi_2(1) = 0$

$\quad A\cancel{\Psi_1'}(1) + B\cancel{\Psi_2'}(1) + C\chi_1'(1) + D\chi_2'(1) = 0$.

$$\begin{pmatrix} \chi_1(1) & \chi_2(1) \\ \chi_1'(1) & \chi_2'(1) \end{pmatrix}\begin{pmatrix} C \\ D \end{pmatrix} = 0.$$

Non-trivial sl$^{\text{n}}$ if and only if $\det(\ldots) = 0$.

$$\Rightarrow \quad \chi_1(1)\chi_2'(1) - \chi_2(1)\chi_1'(1) = 0$$

$$\Rightarrow \quad F(k, Re, c) = 0$$

$$\Rightarrow \quad c = G(k, Re)$$ DISPERSION RELATION

The function G is complex-valued, so c is a complex wave speed.

For Couette flow, $c_i < 0$, meaning Couette flow is linearly stable. Remark: For Poiseuille flow, $U_0(z) \propto z(1-z)$, the flow is linearly stable for $Re < 5772$:

$$c_i \leq 0 \quad \forall k, \quad Re \leq 5772,$$

$$c_i > 0 \text{ for some } k, \quad Re > 5772.$$

The Orr-Sommerfeld equation is one part of the description of Turbulence.

We were taught mathematical physics in first year honours by William McFadden Orr. He was a Belfast Presbyterian who had won the Royal Studentship in 1887 and was later a senior wrangler in the Cambridge Tripos. He was a great gentleman but very stern in every sense. He would sit with the pocket watch out waiting for 9 o'clock to start the lecture on the dot. If you came in after 9 a.m. you got a 'late' on the roll. He had come to U.C.D. from being professor in the College of Science. He was heard to say that research in the College of Science consisted of solving a quadratic equation which had not been solved before. We learned little from him for most of the time was spent criticizing the bad treatment of Newton's laws. We did not know very well what the laws were but we knew that Ernest Mach was the man to be respected.

J. R. Timoney

When will we see the next Irish Nobel Prize in Literature laureate?

Message 1 of 38

From onaraigh
To lettersed@irish-times.ie
Date 2024-10-10 16:10

Sir,

Daniel Mulhall asks when will we see the next Irish Nobel Prize in Literature laureate (Irish Times, 10th October). Given that three out of four of the Irish Nobel Lauretes in Literature were from a Protestant background and lived at a time when the Irish Protestant population was higher in relative terms, an answer (only partly facetious) would be when the Protestant population begins to increase again. The same might be said for Nobel Prizes in Science - two out of two to scientists from a Protestant background (one a devout Methodist). The Methodist (Ernest Walton) was a son of a clergyman - a commonality shared with several great Irish scientists of the 19th Century - George Gabriel Stokes, George Francis FitzGerald, John Joly, and Henrietta Beaufort, to name but a few. It would be wonderful to know what magical creative ingredient was stored in those damp old 19th Century manses, parsonages, and vicarages, and to see how much if any remains today.

Yours, etc.

Lennon Ó Náraigh

School of Mathematics and Statistics

University College Dublin

Belfield

Dublin 4

Chapter 5 – Constructing solutions of ODEs via power series.

- New notation:

$$y''(z) + p(z)\,y'(z) + q(z)\,y(z) = 0. \qquad (1)$$

Classification of the point $z=0$. Why? We will develop a power-series sl^n of (1), centred at $z=0$.

If we want to develop a power-series sl^n of (1) centred at z_0, then we would classify the point z_0 instead.

<u>Clarification</u>: Strictly speaking, we will develop a Frobenius-series sl^n of (1), centred at $z=0$:

$$y(z) = z^{\alpha}\left(\sum_{n=0}^{\infty} a_n z^n\right) \qquad \text{FROBENIUS SERIES}$$

Back to the classification:

- $z=0$ is an ordinary point if $p(z)$ and $q(z)$ are analytic (have a Taylor-series expansion) at $z=0$. ☑

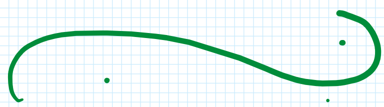
- $z=0$ is a singular point if $p(z)$ and $q(z)$ are not analytic at $z=0$.
  - Regular singular point if $z\,p(z)$ and $z^2 q(z)$ are analytic at $z=0$. ☑

– Otherwise, $z=0$ is an irregular singular point. $\boxed{\times}$

Example:

$$y''(z) + \underbrace{\left(\frac{3}{z} + \frac{2}{z^3}\right)}_{p(z)} y'(z) + 0 \cdot y(z) = 0.$$

$z\,p(z) = 3 + \frac{2}{z^2}$, does not have a Taylor series centred at zero, so $z=0$ is an irregular singular point.



Chapter 5

$$y''(z) + p(z)y'(z) + q(z)y(z) = 0 \qquad (1)$$

$$\left.\begin{array}{l} zp(z) \\ z^2 q(z) \end{array}\right\} \text{analytic}$$

- $z=0$ is a ordinary point if $p(z)$ and $q(z)$ are analytic at $z=0$.
- $z=0$ is a regular singular point if
  $\left.\begin{array}{l} zp(z) \\ z^2 q(z) \end{array}\right\}$ are analytic at $z=0$.

In these cases, a solution of (1) is available as a Frobenius series:

$$y(z) = z^{\alpha}\left(\sum_{n=0}^{\infty} a_n z^n\right)$$

Theorem 5.1 If $p(z)$ and $q(z)$ are analytic in the disc $|z|<R$, then there exist two linearly independent $s|^{n}s$ $y_1(z)$ and $y_2(z)$ of Eq^{n} (1) such that:

- $y_1(z)$ and $y_2(z)$ are analytic in $|z|<R$ (and possibly a larger disc)
- $y_1(0) \neq 0$, if $y_2(0)=0$ and $y_2'(0) \neq 0$.

Plan for this chapter: We don't prove Theorem 5.1.

Instead, we look at examples that make the theorem look very plausible.

look very plausible.

Example : $p(z)$ and $q(z)$ constant and equal to p_0 and q_0.

$$y''(z) + p_0 y'(z) + q_0\, y(z) = 0.$$

TEMPLATE ORDINARY POINT @ $z=0$

$$y(z) = e^{\alpha z}.$$

$$\boxed{\alpha^2 + p_0\alpha + q_0 = 0}.$$ characteristic equation

Case 1 : Distinct roots α_1 and α_2.

$$y_1(z) = e^{\alpha_1 z} \quad \ldots \quad y_1(0) \neq 0$$

$$y_2(z) = e^{\alpha_2 z} - e^{\alpha_1 z}. \quad \begin{cases} y_2(0) = 0 \\ y_2'(0) = \alpha_2 - \alpha_1 \neq 0 \end{cases}$$

Case 2 : Equal roots α_1 and $\alpha_2 = \alpha_1$

$$y_1(z) = e^{\alpha_1 z}$$

$$y_2(z) = z e^{\alpha_1 z}.$$

New ODE (§ 5.3) — Euler's Linear Equation

$$y''(z) + \frac{p_0}{z} y'(z) + \frac{q_0}{z^2} y(z) = 0$$

TEMPLATE REG. SINGULAR POINT @ $z=0$

Trial sl^n: $y(z) = z^\alpha$.

$$\alpha(\alpha-1)\cancel{z^{\alpha-2}} + p_0\,\alpha\, \cancel{z^{\alpha-1}\cdot\frac{1}{z}} + q_0\, \cancel{z^\alpha \cdot \frac{1}{z^2}} = 0.$$

$$\Rightarrow \boxed{\alpha(\alpha-1) + p_0\alpha + q_0 = 0}$$ Characteristic Equation.

$$\alpha^2 + \alpha(p_0 - 1) + q_0 = 0.$$

$$\alpha = \frac{(1-p_0) \pm \sqrt{(p_0-1)^2 - 4q_0}}{}$$

Distinct roots: $(p_0-1)^2 \neq 4q_0$.

$$y_1(z) = z^{\alpha_1}$$

$$y_2(z) = z^{\alpha_2} - z^{\alpha_1}$$

Repeated roots: $(p_0-1)^2 = 4q_0$, $\alpha_1 = \frac{1-p_0}{2}$ $(=\alpha_2)$

$$\boxed{y_1(z) = z^{\alpha_1}}$$

This is a Frobenius series: $y(z) = z^{\alpha} \underbrace{\left(\sum_{n=0}^{\infty} a_n z^n\right)}_{=1 \text{ (Euler)}}$

<u>Second solution</u>: $y_2(z) = y_1(z) \cdot u(z)$

Sub into Euler's ODE:

$$\cancel{y_1''(z)u(z)} + 2y_1'(z)u'(z) + y_1(z)u''(z)$$
$$+ \cancel{\frac{p_0}{z} y_1'(z)u(z)} + \frac{p_0}{z} y_1(z)u'(z) = 0$$
$$+ \cancel{\frac{q_0}{z^2} y(z)u(z)}$$

$$u''(z) + u'(z)\left[\frac{2y_1'(z)}{y_1(z)} + \frac{p_0}{z}\right] = 0$$

$$y_1(z) = z^{\alpha_1}, \quad y_1'(z) = \alpha_1 z^{\alpha_1 - 1}$$

$$\Rightarrow \frac{y_1'(z)}{y_1(z)} = \frac{\alpha_1 z^{\alpha_1-1}}{z^{\alpha_1}} = \frac{\alpha_1}{z}$$

$$\Rightarrow \quad u''(z) + u'(z)\left[\frac{2\alpha_1}{z} + \frac{p_0}{z}\right] = 0$$

$$\Rightarrow \quad u''(z) + u'(z)\,(2\alpha + p_0) \quad = 0$$

$$\Rightarrow \quad u''(z) + \frac{u'(z)}{z}(2\alpha_1 + p_0) = 0$$

Let $V(z) = u'(z)$.

Thus: $V'(z) + V(z)\dfrac{2\alpha_1 + p_0}{z} = 0$.

Repeated roots: $\alpha_1 = \dfrac{1-p_0}{2} \qquad (= \alpha_2)$

$$\Rightarrow \quad 2\alpha_1 = 1 - p_0$$

$$\Rightarrow \quad 2\alpha_1 + p_0 = 1.$$

ODE: $V'(z) + \dfrac{V(z)}{z} = 0$.

Hence: $\dfrac{dV}{dz} = -\dfrac{V}{z}$.

$$\Rightarrow \quad \frac{1}{V}\frac{dV}{dz} = -\frac{1}{z}.$$

$$\Rightarrow \quad \frac{d}{dz}\log V = -\frac{d}{dz}\log z.$$

$$\Rightarrow \quad \log V = -\log z = \log z^{-1}.$$

$$\Rightarrow \quad V = \frac{1}{z}. \qquad \boxed{\log z = \ln z}$$

But $V = u'$.

Hence:

$$\frac{du}{dz} = \frac{1}{z} \quad \Rightarrow \quad u = \log z.$$

Hence $y_2(z) = z^{\alpha_1} \cdot \log z$

Remark: since $z\,p(z)$ and $z^2 q(z)$ are analytic at

$z = 0$ for a regular singular point, we have:

$$\left.\begin{aligned} z\,p(z) &\simeq p_0 \\ z^2 q(z) &\simeq q_0 \end{aligned}\right\} \Rightarrow \begin{aligned} p(z) &\simeq \frac{p_0}{z} \\ q(z) &\simeq \frac{q_0}{z^2} \end{aligned}$$

Hence, in the neighborhood of $z = 0$ (a regular singular point), the ODE behaves like Euler's ODE, so the solution will look like:

Distinct roots
$$\begin{cases} y_1(z) = z^{\alpha_1} + \text{Higher powers of } z \\ y_2(z) = z^{\alpha_2} - z^{\alpha_1} + \text{Higher powers of } z \end{cases}$$

Equal roots
$$\begin{cases} y_1(z) = z^{\alpha_1} + \text{Higher powers of } z \\ y_2(z) = z^{\alpha_1} \cdot \log(z) + \text{Higher powers of } z \end{cases}$$

Rest of chapter – finding the higher powers of z.

Back to Eq^n (1):

$$y''(z) + p(z)\,y'(z) + q(z)\,y(z) = 0.$$

Assume: $z = 0$ is a regular singular point (w.l.o.g.)

$$z\,p(z) = \sum_{n=0}^{\infty} p_n z^n$$

$$z^2 q(z) = \sum_{n=0}^{\infty} q_n z^n$$

Ordinary points are included here by taking $\begin{cases} p_0 = 0 \\ q_0, q_1 = 0 \end{cases}$.

Trial solution of Equation (1):

$$y(z) = z^{\alpha}\left(\sum_{n=0}^{\infty} a_n z^n\right), \qquad (2) \qquad z^{n+\alpha}$$

Where the form of the trial solution is motivated by Euler's Equation.

To fix the solution, we demand: $\boxed{a_0 \neq 0}$

$L[y] = 0$:

$$L[y] = \sum_{n=0}^{\infty} (n+\alpha)(n+\alpha-1)\, a_n z^{n+\alpha-2} + \left(\frac{1}{z} \sum_{n=0}^{\infty} p_n z^n\right)\left(\sum_{n=0}^{\infty} (n+\alpha)\, a_n z^{n+\alpha-1}\right) + \left(\frac{1}{z^2} \sum_{n=0}^{\infty} q_n z^n\right)\left(\sum_{n=0}^{\infty} a_n z^{n+\alpha}\right) = 0.$$

Or :

$$\cancel{\frac{z^\alpha}{z^2}} \sum_{n=0}^{\infty} (n+\alpha)(n+\alpha-1)\, a_n z^n + \cancel{\frac{z^\alpha}{z^2}} \left(\sum_{n=0}^{\infty} p_n z^n\right)\left(\sum_{n=0}^{\infty} (n+\alpha)\, a_n z^n\right) + \cancel{\frac{z^\alpha}{z^2}} \left(\sum_{n=0}^{\infty} q_n z^n\right)\left(\sum_{n=0}^{\infty} a_n z^n\right) = 0$$

Cauchy product :

$$\left(\sum_{n=0}^{\infty} \overset{= \text{"}a_n\text{"}}{\beta_n} z^n\right)\left(\sum_{n=0}^{\infty} \overset{= \text{"}p_n\text{"},\ \text{"}q_n\text{"}}{\gamma_n} z^n\right) = \sum_{n=0}^{\infty} z^n \left(\sum_{j=0}^{n} \beta_j \gamma_{n-j}\right)$$

Hence, ODE becomes:

$$\sum_{n=0}^{\infty} a_n (n+\alpha)(n+\alpha-1) z^n + \sum_{n=0}^{\infty} z^n \left(\sum_{j=0}^{n} a_j (j+\alpha)\, p_{n-j}\right) + \sum z^n \left(\sum_{j=0}^{n} a_j\, q_{n-j}\right) = 0$$

$$+ \sum_{n=0}^{\infty} z^n \left(\sum_{j=0}^{n} a_j \, q_{n-j} \right)$$

Hence:

$$\sum_{n=0}^{\infty} a_n (n+\alpha)(n+\alpha-1) z^n$$

$$= - \sum_{n=0}^{\infty} z^n \left\{ \sum_{j=0}^{n} a_j \left[(j+\alpha) p_{n-j} + q_{n-j} \right] \right\}$$

$$= - \sum_{n=0}^{\infty} z^n \left\{ \sum_{j=0}^{n-1} a_j \left[(j+\alpha) p_{n-j} + q_{n-j} \right] + \left[a_n (n+\alpha) \cdot p_0 + q_0 \right] \right\}$$

Re-group:

$$\Rightarrow \sum_{n=0}^{\infty} a_n \left\{ (n+\alpha)(n+\alpha-1) + (n+\alpha) p_0 + q_0 \right\} z^n$$

$$= - \sum_{n=0}^{\infty} z^n \left\{ \sum_{j=0}^{n-1} a_j \left[(j+\alpha) p_{n-j} + q_{n-j} \right] \right\}$$

Equate coefficients of a_0:

z^0 ... contribution on RHS only.

$$= \quad a_0 \left[\underbrace{\alpha \cdot (\alpha-1) + \alpha \cdot p_0 + q_0}_{F(\alpha)} \right] = 0.$$

$$a_0 \cdot F(\alpha) = 0.$$

Demand: $a_0 \neq 0$

$$\Rightarrow \boxed{F(\alpha) = 0} \quad \text{Indicial Equation.}$$

Week 5, Lecture 3 20/02/2026

$$y''(z) + p(z)y'(z) + q(z)y(z) = 0 \quad \textbf{(1)}$$

$z = 0$ is a reg. singular point (or an ordinary point):

- $z\,p(z)$ has a Taylor series centred @ $z = 0$
- $z^2 q(z)$ ——"——

Frobenius series solution for (1):

$$y(z) = z^\alpha \left(\sum_{n=0}^{\infty} a_n z^n \right) \quad \textbf{(2)}$$

Sub (2) into (1) and equate coefficients of z^n:

$$z^0: \quad a_0 \left[\underbrace{\alpha(\alpha-1) + \alpha p_0 + q_0}_{F(\alpha)} \right] = 0$$

$$\underbrace{z^n,\ n \geq 1: \quad a_n \left[\underbrace{(n+\alpha)(n+\alpha-1) + (n+\alpha)p_0 + q_0}_{F(n+\alpha)} \right] = -\sum_{j=0}^{n-1} a_j \left[(j+\alpha)p_{n-j} + q_{n-j} \right]}_{\text{RECURRENCE RELATION}}$$

Identify:

$$F(s) = s(s-1) + sp_0 + q_0, \quad s \in \mathbb{C}$$

We have:

II

z^0: $$a_0 F(\alpha) = 0.$$

We require $a_0 \neq 0$

$$\Rightarrow \boxed{F(\alpha) = 0} \quad \text{INDICIAL EQUATION}$$

The indicial equation has two roots α_1 and α_2, possibly complex. Hence $F(s)$ can be re-written as:

$$\boxed{F(s) = (s-\alpha_1)(s-\alpha_2)}$$

Today, we first of all look at what happens when α_1 and α_2 are distinct and what's more, they do not differ from an integer:

$$\alpha_1 - \alpha_2 \notin \mathbb{Z}.$$

$$F(n+\alpha) = (n+\underline{\underline{\alpha-\alpha_1}})(n+\alpha-\alpha_2)$$

But the indicial equation tells us that we can no longer have α arbitrary, rather it is fixed as $\alpha = \alpha_1$ or $\alpha = \alpha_2$.

Hence

$$F(n+\alpha_1) = n(n+\alpha_1-\alpha_2)$$

NOT AN INTEGER

$$F(n+\alpha_2) = n(n+\alpha_2-\alpha_1)$$

Sub $F(n+\alpha_1)$, $F(n+\alpha_2)$ back into the recurrence relation, e.g. $(\alpha=\alpha_1)$

(3)
$$\boxed{a_n = -\frac{1}{n(n+\alpha_1-\alpha_2)} \sum_{j=0}^{n-1} a_j \left[(j+\alpha_1) p_{n-j} + q_{n-j} \right]}$$

So, once a_0 is fixed (as an initial condition), we get $a_1, a_2, \dots$ from the recurrence relation.

Two linearly independent solutions in this way:

- $\alpha = \alpha_1$
- $\alpha = \alpha_2$

Exceptional Cases:

- Equal Roots
- Roots differing by an integer.

Equal roots: Then, the different recurrence relations for α_1 and α_2 in Equation (3) collapse into one recurrence relation:

$$a_n = -\frac{1}{n^2}\sum_{j=0}^{n-1} a_j\left[(j+\overset{=\alpha_2}{\alpha_1})p_{n-j} + q_{n-j}\right]$$

So, we can only generate one solution with this method.

There has to be a way to find a second linearly independent solution!

Roots differing by an integer:

$$\alpha_1 = \alpha_2 + N, \quad N \in \{1, 2, \dots\}$$

(convention: α_1 is the bigger of the two roots)

$$F(n+\alpha_1) = n(n+N) \quad \text{never zero}$$

$$\Rightarrow a_n = -\frac{1}{n(n+N)}\sum_{j=0}^{n-1} a_j\left[(j+\alpha_1)p_{n-j} + q_{n-j}\right]$$

OK.

But look at

$$F(n+\alpha_2) = n(n-N))$$

$$\Rightarrow a_n = -\frac{1}{n(n-N)} \sum_{j=0}^{n-1} a_j [\cdots]$$

Recurrence relation fails at $n = N$.
So we fail to generate a second solution by a Frobenius Series.

Frobenius's Method (§5.5)

- Use a Frobenius Series (with $F(\alpha_1)=0$) to obtain a first solution.

- Second solution: Let
$$Y(z) = z^{\alpha} \sum_{n=0}^{\infty} a_n z^n$$
be a series, but not necessarily a sl$^{\underline{n}}$ of the ODE 1.

We can certainly sub $Y(z)$ into the ODE, but we won't have

$$L[Y] = 0.$$

For $Y(z)$, we demand:

- $F(\alpha) \neq 0$ (α arbitrary),
  so actually, $Y(z) \equiv Y_\alpha(z)$.

- We pick the coefficients a_j to satisfy:

$$a_n = -\frac{1}{F(n+\alpha)} \sum_{j=0}^{n-1} a_j \left[(j+\alpha) p_{n-j} + q_{n-j} \right]$$

$a_n(\alpha)$

We sub $Y_\alpha(z)$ into the ODE:

$$\frac{d^2}{dz^2} Y_\alpha(z) + p(z) \frac{d}{dz} Y_\alpha(z) + q(z) Y_\alpha$$

$$= a_0 F(\alpha) z^{\alpha-2}$$

$$= a_0 (\alpha - \alpha_1)(\alpha - \alpha_2) z^{\alpha-2}$$

(Check!)

We differentiate both sides with respect to α, using e.g.

$$\frac{\partial}{\partial \alpha} p(z) \frac{d}{dz} Y_\alpha(z)$$

$$= p(z) \frac{d}{dz} \frac{\partial Y_\alpha(z)}{\partial \alpha}$$

$$\frac{d^2}{dz^2}\frac{\partial Y_\alpha}{\partial \alpha} + p(z)\frac{d}{dz}\frac{\partial Y_\alpha}{\partial \alpha} + q(z)\frac{\partial Y_\alpha}{\partial \alpha}$$

$$= \frac{\partial}{\partial \alpha}\left[a_0 \, \underline{(\alpha-\alpha_1)}(\alpha-\alpha_2)\, z^{\alpha-2}\right]$$

$$= a_0 \cdot 1 \cdot (\alpha-\alpha_2) z^{\alpha-2}$$

$$+ \; a_0 (\alpha-\alpha_1) \cdot 1 \cdot z^{\alpha-2}$$

$$+ \; a_0 (\alpha-\alpha_1)(\alpha-\alpha_2) \frac{\partial}{\partial \alpha} z^{\alpha-2}$$

Use:

$$\frac{\partial}{\partial \alpha} z^{\alpha-2} = \frac{\partial}{\partial \alpha} e^{(\alpha-2)\log z}$$

$$= \log z \; e^{(\alpha-2)\log z}$$

$$= z^{\alpha-2} \log z$$

Hence:

$$\frac{d^2}{dz^2}\frac{\partial Y_\alpha}{\partial \alpha} + p(z)\frac{d}{dz}\frac{\partial Y_\alpha}{\partial \alpha} + q(z)\frac{\partial Y_\alpha}{\partial \alpha}$$

$$= a_0 \cancel{(\alpha-\alpha_2)} z^{\alpha-2} + \cancel{a_0(\alpha-\alpha_1)} z^{\alpha-2} + a_0\cancel{(\alpha-\alpha_1)}(\alpha-\alpha_2) z^{\alpha-2}\log z = 0$$

Repeated roots, $\alpha_1 = \alpha_2 \in \mathbb{C}$

Take limit as $\alpha \to \alpha_1$ on both sides:

Hence,

$$\left(\frac{\partial Y_\alpha(z)}{\partial \alpha}\right)_{\alpha=\alpha_1} = y_2(z)$$

is the second solution of the ODE, where:

$$Y_\alpha(z) = z^\alpha \sum_{n=0}^{\infty} a_n(\alpha) z^n.$$

Other exceptional case, when $\alpha_1 = \alpha_2 + N$ where $N \in \{1, 2, \dots\}$. By a similar procedure,

$$\left\{\frac{\partial}{\partial \alpha}\left[(\alpha-\alpha_2) Y_\alpha(z)\right]\right\}_{\alpha=\alpha_2} = y_2(z).$$