Linear ODEs – Inhomogeneous Case (Chapter 4)

$$\underbrace{\left[p_n(x)\frac{d^n}{dx^n} + p_{n-1}(x)\frac{d^{n-1}}{dx^{n-1}} + \cdots + p_0(x)\right]}_{L} y(x) = r(x), \quad x \in [a,b]$$

where $p_n(x) \neq 0$, $x \in [a,b]$

We focus here on the second-order problem:

$$p_2(x)y''(x) + p_1(x)y'(x) + p_0(x)\,y(x) = r(x). \qquad (1)$$

Solution method: Variation of parameters. Idea: look at two linearly independent solutions of the corresponding homogeneous problem. Call them $u(x)$ and $v(x)$. Thus:

$$L[u]=0, \qquad L[v] = 0.$$

In the homogeneous case, a general solution of $L[y]=0$ would be

$$y(x) = \alpha\, u(x) + \beta\, v(x) \quad (2)$$

Homogeneous Case

where α and β are constant. We propose a solution to the inhomogeneous case by looking at (2), and promoting α and β to be functions of x:

$$y(x) = \alpha(x)\, u(x) + \beta(x) v(x) \quad (3)$$

Inhomogeneous Case

There are conditions on $\alpha(x)$ and $\beta(x)$ such that (3) satisfies the ODE.

Sub (3) into (1):

Product Rule Twice (Leibniz Rule)

$$p_2(x)\left[\alpha(x)u''(x) + 2\alpha'(x)u'(x) + \alpha''(x)u(x)\right]$$

$$p_2(x)\left[\cancel{\alpha(x)u''(x)} + 2\alpha'(x)u'(x) + \cancel{\alpha(x)u(x)}\right]$$
$$+ \; p_1(x)\left[\cancel{\alpha(x)u'(x)} + \cancel{\alpha'(x)u(x)}\right] + p_0(x)\,\cancel{\alpha(x)u(x)}$$
$$+ \; p_2(x)\left[\cancel{\beta(x)v''(x)} + 2\beta'(x)v'(x) + \cancel{\beta''(x)v(x)}\right]$$
$$+ \; p_1(x)\left[\cancel{\beta(x)v'(x)} + \cancel{\beta'(x)v(x)}\right] + p_0(x)\,\cancel{\beta(x)v(x)} = r(x)$$

Terms multiplying $p_1(x)$:

$$\alpha'(x)u(x) + \beta'(x)v(x) = 0$$

differentiate

$$\Rightarrow \quad \alpha'(x)u'(x) + \alpha''(x)u(x) + \beta'(x)v'(x) + \beta''(x)v(x) = 0$$

Hence: $p_2(x)\left[\alpha'(x)u'(x) + \beta'(x)v'(x)\right] = r(x)$

Hence, $\alpha(x)$ and $\beta(x)$ are uniquely determined, through:

$$\alpha'(x)u(x) + \beta'(x)v(x) = 0$$

$$\alpha'(x)u'(x) + \beta'(x)v'(x) = \frac{r(x)}{p_2(x)}.$$

Linear system in $\alpha'(x)$ and $\beta'(x)$:

$$\begin{pmatrix} u(x) & v(x) \\ u'(x) & v'(x) \end{pmatrix}\begin{pmatrix} \alpha'(x) \\ \beta'(x) \end{pmatrix} = \begin{pmatrix} 0 \\ \frac{r(x)}{p_2(x)} \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} \alpha'(x) \\ \cdots \end{pmatrix} = \frac{1}{W(\ldots)}\begin{pmatrix} v'(x) & -v(x) \\ \cdots & \cdots \end{pmatrix}\begin{pmatrix} 0 \\ \frac{r(x)}{\cdots} \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} \alpha'(x) \\ \beta'(x) \end{pmatrix} = \frac{1}{W(x)} \begin{pmatrix} \cdots & \cdots \\ -u'(x) & u(x) \end{pmatrix} \begin{pmatrix} \cdots \\ \frac{r(x)}{p_2(x)} \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} \boxed{\alpha'(x)} \\ \beta'(x) \end{pmatrix} = \frac{1}{W(x)} \begin{pmatrix} \boxed{-v(x)\,r(x)\,/\,p_2(x)} \\ +u(x)\,r(x)\,/\,p_2(x) \end{pmatrix}$$

$$\Rightarrow \begin{cases} \alpha(x) = \alpha_0 - \displaystyle\int_{x_\alpha}^{x} \frac{v(s)\,r(s)}{p_2(s)W(s)}\,ds \\[2ex] \beta(x) = \beta_0 + \displaystyle\int_{x_\beta}^{x} \frac{u(s)\,r(s)}{p_2(s)W(s)}\,ds \end{cases}$$

Choose reference points x_α and x_β such that $\alpha_0 = 0$ and $\beta_0 = 0$. Hence:

$$\alpha(x) = -\int_{x_\alpha}^{x} \frac{v(s)\,r(s)}{p_2(s)W(s)}\,ds, \quad \text{same for } \beta(x).$$

Hence, the general sl^{n} of (1) is:

$$\left.\begin{aligned} y(x) = {} & A\,u(x) + B\,v(x) \\ & - u(x) \int_{x_\alpha}^{x} \frac{v(s)\,r(s)}{p_2(s)W(s)}\,ds \\ & + v(x) \int_{x_\beta}^{x} \frac{u(s)\,r(s)}{p_2(s)W(s)}\,ds \end{aligned}\right\} \quad (4)$$

The IVP: Suppose that we are solving (1) with

$$y(x_0) = 0, \quad y'(x_0) = 0.$$

The solution of this IVP is called the particular integral (PI) of (1). From (4), we see that the PI ($= y_p(x)$) is:

$$y_p(x) = -u(x)\int_{x_0}^{x} \frac{v(s)r(s)}{p_2(s)W(s)}\,ds + v(x)\int_{x_0}^{x} \frac{u(s)r(s)}{p_2(s)W(s)}\,ds$$

Check:

$$y_p(x_0) = -u(x_0)\int_{x_0}^{x_0} (\ldots)\,ds + \cdots = 0 \quad \boxed{\checkmark}$$

$$y_p'(x) = -u'(x)\int_{x_0}^{x} \frac{v(s)r(s)}{p_2(s)W(s)}\,ds - \cancel{u(x)\frac{v(x)r(x)}{p_2(x)W(x)}}$$

$$+ v'(x)\int_{x_0}^{x} \frac{u(s)r(s)}{p_2(s)W(s)}\,ds + \cancel{\frac{v(x)u(x)r(x)}{p_2(x)W(x)}}$$

$$\Rightarrow y_p'(x) = -u'(x_0)\int_{x_0}^{x_0} (\ldots)\,ds + \cdots = 0 \quad \boxed{\checkmark}$$

Building up to a general IVP:

$$L[y] = r(x)$$

$$y(x_0) = y_0, \quad y'(x_0) = y_0'$$

Solution:

$$\boxed{y(x) = \underbrace{A\,u(x) + B\,v(x)}_{\substack{\text{Solves homogeneous} \\ \text{problem, takes care} \\ \text{of initial conditions}}} + \underbrace{y_p(x)}_{\substack{\text{Solves inhomogeneous} \\ \text{problem, with zero} \\ \text{initial conditions}}}}$$

Check:

$L[y](x) = r(x)$, automatically.

So it remains to check the initial conditions (ICs):

$$y(x_0) = A u(x_0) + B v(x_0) + \cancel{y_p(x_0)} = y_0$$

$$y'(x_0) = A u'(x_0) + B v'(x_0) + \cancel{y_p'(x_0)} = y_0'$$

Solve for A and B:

$$\underbrace{\begin{pmatrix} u(x_0) & v(x_0) \\ v'(x_0) & v'(x_0) \end{pmatrix}}\begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} y_0 \\ y_0' \end{pmatrix}$$

$\det(..) = W(x_0) \neq 0$, since

$u(x)$ and $v(x)$ are linearly independent.

Hence, A and B can be found, and the PI can be constructed via method of variation of parameters, so the general IVP can be solved. ∎

Example:

$$L[y] = y''(x) + 2y'(x) + 2y(x) = r(x),$$

with $y(0) = 1$ and $y'(0) = 0$.

Linearly independent solutions of $L[y]=0$ are:

$$u(x) = e^{-x} \cos(x)$$

$$v(x) = e^{-x} \sin(x)$$

Wronskian:

$$W(x) = \begin{vmatrix} u(x) & v(x) \\ u'(x) & v'(x) \end{vmatrix}$$

$$= \begin{vmatrix} e^{-x} \cos x & e^{-x} \sin x \\ -e^{-x} \sin x - e^{-x} \cos x & e^{-x} \cos x - e^{-x} \sin x \end{vmatrix} \begin{array}{l} \leftarrow 1^{st} \text{ Row} \\ \leftarrow 2^{nd} \text{ Row} \end{array}$$

$$= \begin{vmatrix} e^{-x} \cos x & e^{-x} \sin x \\ -e^{-x} \sin x & e^{-x} \cos x \end{vmatrix}$$

$$e^{-2x} \cos^2 x + e^{-2x} \sin^2 x = e^{-2x}.$$

$$\therefore \quad W(x) = e^{-2x}$$

Check:

$$\frac{dW}{dx} = -\left[\frac{p_1(x)}{p_2(x)}\right] W = -\frac{2}{1} W(x)$$

$$\Rightarrow \frac{dW}{dx} = -2W(x) \Rightarrow W(x) = W(0) e^{-2x}.$$

$$W(u) = \begin{vmatrix} 1 & 0 \\ -1 & 1 \end{vmatrix} = 1 \quad ☑$$

Particular integral:

$$y_p(x) = -u(x)\int_0^x \frac{v(s)\,r(s)}{1\cdot W(s)}\,ds + v(x)\int_0^x \frac{u(s)\,r(s)}{1\cdot W(s)}\,ds$$

$$= -e^{-x}\cos(x)\int_0^x e^{-s}\sin(s)\,r(s)\,e^{2s}\,ds$$

$$+ e^{x}\sin(x)\int_0^x e^{-s}\cos(s)\,r(s)\,e^{2s}\,ds$$

$$= \int_0^x e^{(s-x)}\left[\underbrace{\sin(x)\cos(s) - \sin(s)\cos(x)}\right] r(s)\,ds$$

$$= \int_0^x e^{s-x}\sin(x-s)\,r(s)\,ds$$

$$\Rightarrow \boxed{y_p(x) = \int_0^x e^{s-x}\sin(x-s)\,r(s)\,ds}$$

"convolution with kernel function"

OR $$y_p(x) = \int_0^x K(x,s)\,r(s)\,ds$$

Jump straight back into example on Method of Variation of parameters.

$$y_p(x) = \int_0^x \underbrace{e^{s-x} \sin}(x-s)\, r(s)\, ds$$
$$= \int_0^x K(x,s)\, r(s)\, ds$$
} Particular Integral

ODE: $y''(x) + 2y'(x) + 2y(x) = r(x)$

ICs: $y(0) = 1, \quad y'(0) = 0.$

Particular integral satisfies ODE but has zero ICs.

Solution with correct ICs:

$$y(x) = Au(x) + Bv(x) + y_p(x)$$
$$= Ae^{-x}\cos(x) + \underline{Be^{-x}\sin(x)} + \underline{y_p(x)}$$

Given:

$y(0) = 1$

But $y(0) = A$. Hence, $A = 1$.

$$y'(x) = \cancel{Ae^{-x}(-\sin(x))} - Ae^{-x}\cos(x)$$
$$+ Be^{-x}(\cos(x)) \cancel{- Be^{-x}\sin(x)} + \cancel{y_p'(x)}$$

$$y'(0) = -A + B \overset{\text{I.C.}}{=} 0 \Rightarrow B = A \Rightarrow B = 1.$$

Solution: $y(x) = Au(x) + Bv(x) + y_p(x)$

Hence:

$$\boxed{y(x) = e^{-x}\cos(x) + e^{-x}\sin(x) + y_p(x)}$$

Moving on, we look at:

- Green's Functions
- Volterra Integral Equations

Context: Chapter 1:

$$y(x) = y_0 + \int_{x_0}^{x} F(s, y(s))\, ds,$$

solves $\frac{dy}{dx} = F(x, y(x))$, $y(x_0) = y_0$.

Example – linear ODEs:

$$y_p(x) = \int_0^x K(x,s)\, r(s)\, ds$$

Second equation is an example of a solution of an integral equation.

Green's Functions – Solutions of ODEs (in integral form) via convolution.

Generic second-order LINEAR ODE:

$$\underbrace{p_2(x)y''(x) + p_1(x)y'(x) + p_0(x)y(x)}_{L[y]} = r(x) \qquad (1)$$

subject to $y(x_0) = y_0$, $y'(x_0) = y_0'$

Partial integral $y_p(x)$, satisfying the ODE (1)

with zero initial conditions:

with zero initial conditions :

$$y_p(x) = -u(x)\int_{x_0}^{x} \frac{v(s)\,r(s)}{P_2(s)W(s)}\,ds + v(x)\int_{x_0}^{x} \frac{u(s)\,r(s)}{P_2(s)W(s)}\,ds$$

where $u(x)$ and $v(x)$ are linearly independent solutions of the homogeneous version of (1), with $W(x) \neq 0$.

Re-write :

$$y_p(x) = \int_{x_0}^{x} \underbrace{\left[\frac{-u(x)v(s) + v(x)u(s)}{p_2(s)W(s)}\right]}_{G(x,\,s)} r(s)\,ds$$

What's a convolution? It's this integral here.

So $y_p(x)$ is a convolution of $r(\cdot)$ with the Green's Function.

Definition :

$$G(x,s) = \frac{-u(x)v(s) + v(x)u(s)}{p_2(s)W(s)} \qquad (2)$$

is the Green's Function for the IVP (1).

Theorem 4.1 : $G(x,s)$ satisfies the following properties :

1. Defined for $x_0 \le s \le x$
2. $L_x[G] = 0$
3. $G(x, x^-) = 0$

4. $$\frac{\partial G}{\partial x}(x,x^-) = \frac{-u'(x)v(x) + v'(x)u(x)}{p_2(x)W(x)} = \frac{1}{p_2(x)}$$

Property 1:

$$y_p(x) = \int_{x_0}^{x} G(x,s)f(s)\,ds \quad , \qquad x \geq x_0$$

Hence, $x_0 \leq s \leq x$. ☑

Property 2:

$L_x[G]$ means

$$p_2(x)\frac{\partial^2 G(x,s)}{\partial x^2} + p_1(x)\frac{\partial G(x,s)}{\partial x} + p_0(x)\,G(x,s).$$

But, in the x-variable, G is a linear combination of $u(x)$ and $v(x)$. Thus,

$$L_x[G] = 0.$$ ☑

Property 3:

$G(x,x^-)$ means $\lim_{s \uparrow x} G(x,s)$ (one-sided limit).

$$G(x,s) = \frac{-u(x)v(s) + v(x)u(s)}{p_2(s)W(s)}$$

Take $s \uparrow x$:

$$\lim_{s\uparrow x} G(x,s) = \frac{-u(x)v(x) + v(x)u(x)}{p_2(x)W(x)} = 0$$ ☑.

Property 4 :

$$\frac{\partial G}{\partial x}(x,s) = \frac{-u'(x)v(s) + v'(x)u(s)}{p_2(s)W(s)}$$

$$\lim_{s \uparrow x} \frac{\partial G}{\partial x}(x,s) = \frac{-u'(x)v(x) + v'(x)u(x)}{p_2(x)W(x)} = \frac{W(x)}{p_2(x)W(x)}$$

$$= \frac{1}{p_2(x)}$$

$$= \frac{\partial G}{\partial x}(x,x^-) .$$

<u>Adjoint Operator</u> Context : Self-adjoint problems are nice because the Green's function is symmetric, $G(x,s) = G(s,x)$. §4.3.1

Generic linear operator :

$$L[y] = p_2(x)y''(x) + p_1(x)y'(x) + p_0(x)y(x)$$

<u>Define</u>

$$M[y] = \big(p_2(x)y(x)\big)'' - \big(p_1(x)y(x)\big)' + p_0(x)y(x)$$

The M-operator satisfies :

$$\int u\big(L[v]\big)dx = \int v\,\big(M[u]\big) + \text{BOUNDARY TERMS}$$

Definition : M is the adjoint operator of L.

The idea is much clearer if we use inner-product notation :

$$\langle u, v \rangle = \int uv \, dx$$

$$\langle u, L[v] \rangle = \langle Lv, u \rangle \quad \ldots \text{ inner product is symmetric}$$

$$= \langle v, M[u] \rangle \quad \ldots \text{ adjoint.}$$

$$= \langle M[u], v \rangle \quad \ldots \text{ symmetry}$$

Notation : $L^{+} \equiv M$.

Def^n : A differential operator is called self-adjoint if $L^{+} = L$.

Question : Can we put conditions on p_1 and p_2 to make L self-adjoint ?

$$L[y] = p_2 y'' + p_1 y' + p_0 y$$

$$M[y] = (p_2 y)'' - (p_1 y)' + p_0 y.$$

$$= p_2 y'' + 2p_2' y' + y\, p_2'' - p_1 y' - y\, p_1' + p_0$$

Require : $\boxed{2p_2' - p_1 = p_1 \Rightarrow p_2' = p_1}$

Also : $\boxed{\cancel{p_0} = p_2'' - p_1' + \cancel{p_0}}$

If the first condition is satisfied, then the second condition is satisfied automatically.

Result : If $L = L^+$, then $p_2' = p_1$

$$L[y] = p_2 y'' + p_1 y' + p_0 y$$

$$= p_2 y'' + p_2' y' + p_0 y$$

$$\Rightarrow L[y] = \frac{d}{dx}\left(p_2 \frac{dy}{dx}\right) + p_0 y$$

Apply Abel's Lemma to a self-adjoint problem:

$$\frac{dW}{dx} = -\frac{p_1(x)}{p_2(x)} W(x)$$

$$= -\frac{p_2'(x)}{p_2(x)} W(x) \quad \ldots \text{self-adjoint problem}$$

$$\Rightarrow \frac{1}{W}\frac{dW}{dx} = -\frac{1}{p_2}\frac{dp_2}{dx}$$

$$\Rightarrow \int_{x_0}^{x} \frac{d}{dx} \log W(x)\, dx = -\int_{x_0}^{x} \frac{d}{dx} \log p_2(x)\, dx$$

$$\Rightarrow \log \frac{W(x)}{\quad} = -\log \frac{p_2(x)}{\quad}$$

$$\Rightarrow \quad \log \frac{W(x)}{W(x_0)} = -\log \frac{p_2(x)}{p_2(x_0)}$$

$$= \log \frac{p_2(x_0)}{p_2(x)}$$

exponentiate

$$\Rightarrow \quad \frac{W(x)}{W(x_0)} = \frac{p_2(x_0)}{p_2(x)}$$

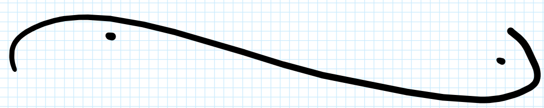
$$\Rightarrow \quad W(x)\,p_2(x) = W(x_0)\,p_2(x_0) = \text{CONSTANT}.$$

$$\Rightarrow \quad \boxed{W(x)\,p_2(x) = \text{CONSTANT}} \qquad L \text{ self-adjoint}$$

Back to Green's Functions:

$$G(x,s) = \frac{-u'(x)\,v(s) + v'(x)\,u(s)}{\underbrace{p_2(s)\,W(s)}_{\text{CONSTANT}}}$$

Very simple form for self-adjoint problems.



Week 4, Lecture 3
13/02/2026

Today: Plan
- Self adjoint ODEs
- Volterra Integral Equation
- Exercises # 3

} Ch. 4

(1) $p_2(x)\, y''(x) + p_1(x)\, y'(x) + p_0(x)\, y(x) = \phi(x)$

To make (1) self-adjoint, we require:

(2) $p_1 = p_2'$

If this is true, then (1) can be re-written as:

$$p_2(x) y''(x) + p_2'(x) y'(x) + p_0(x) y(x) = r(x)$$

$$\Rightarrow \frac{d}{dx}\left(p_2(x)\frac{dy}{dx}\right) + p_0(x)\, y(x) = r(x).$$

Today: Every equation of type (1) can be brought to self-adjoint form using an integrating factor $I(x)$ $\longrightarrow$

$$\underline{I p_2} y'' + \underline{I p_1} y' + I p_0 y = I r .$$

$$(I p_1) = (I p_2)'$$

$$\Rightarrow \quad I' p_2 + I p_2' = I p_1$$

$$\Rightarrow \quad I' = \frac{I (p_1 - p_2')}{p_2}$$

$$\Rightarrow \quad \frac{1}{I} \frac{dI}{dx} = \frac{p_1}{p_2} - \frac{1}{p_2} \frac{dp_2}{dx}$$

$$\Rightarrow \quad \frac{d}{dx} \log I = \frac{p_1}{p_2} - \frac{d}{dx} \log p_2$$

$$\Rightarrow \quad \log I = \int_{x_0}^{x} \frac{p_1}{p_2} dx - \log p_2 + \text{Const.}$$

$$\Rightarrow \quad I = e^{(\cdots\cdots)}$$

$$= \text{Const.} \frac{e^{\int_{x_0}^{x} \frac{p_1}{p_2} dx}}{p_2(x)} .$$

Hence, $\boxed{I(x) = \frac{1}{p_2(x)} e^{\int_{x_0}^{x} \frac{p_1(s)}{p_2(s)} ds}}$

Hence, there is no longer any loss of generality in just focusing on self-adjoint problems. $\longrightarrow$

§ 4.4 — Writing the IVP as an integral equation

Look at § 4.4.1 EXAMINABLE

Start with a self-adjoint linear second-order ODE:

$$\frac{d}{dx}\left(p_2 \frac{dy}{dx}\right) + p_0 y = r .$$

$$\Longrightarrow \quad \underbrace{\frac{d}{dx}\left(p_2 \frac{dy}{dx}\right)}_{\text{"NEW ODE"}} = \underbrace{r - p_0 y}_{\text{"NEW RHS"}}$$

Method of variation of parameters.

Corresponding homogeneous problem for new ODE:

$$\frac{d}{dx}\left(p_2 \frac{dy}{dx}\right) = 0 .$$

$$y(x) = \text{Const.} \quad \Longrightarrow \quad v(x) = 1 .$$

$$p_2 \frac{dy}{dx} = \text{Const.}$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{\text{Const.}}{p_2(x)}$$

$$\Rightarrow \quad y(x) = \text{Const.} \int^x \frac{ds}{p_2(s)}$$

$$u(x) = \underbrace{\int^x \frac{ds}{p_2(s)}}_{P(s)} .$$

IV

Wronskian:

$$W(x) = \begin{vmatrix} u & v \\ u' & v' \end{vmatrix} \qquad v = 1.$$

$$= -vu' = -\frac{1}{p_2(x)}.$$

Particular integral:

$$y_p(x) = \int_{x_0}^{x} \frac{[-u(x)v(s) + v(x)u(s)]}{p_2(s)W(s)} [r(s) - p_0(s)y(s)]$$

OR

$$y_p(x) = \int_{x_0}^{x} \frac{[-u(x)v(s) + v(x)u(s)]}{\underbrace{p_2(s)W(s)}_{=-1}} RHS(s)\, ds$$

$$= \int_{x_0}^{x} [+u(x)v(s) - v(x)u(s)]\, RHS(s)\, ds$$

$$= \int_{x_0}^{x} [P(x)\cdot 1 - 1\cdot P(s)]\, RHS(s)\, ds$$

$$= P(x) \int_{x_0}^{x} RHS(s)\, ds - \int_{x_0}^{x} P(s)\, RHS(s)\, ds.$$

General solution:

$$y(x) = A\,P(x) + B + y_p(x)$$

$$\Rightarrow y(x) = A\,P(x) + B + P(x)\int_{x_0}^{x} [r(s) - p_0(s)\,y(s)]\,ds - \int_{x_0}^{x} P(s)\,[r(s) - p_0(s)\,y(s)]\,ds.$$

Final form:

$$y(x) = A\,P(x) + B + \int_{x_0}^{x} [P(x) - P(s)][r(s) - p_0(s)\,y(s)]\,ds.$$

$$\Rightarrow y(x) = \underbrace{A\,P(x) + B + \int_{x_0}^{x} [P(x) - P(s)]\,r(s)\,ds}_{F(x)} + \int_{x_0}^{x} \underbrace{[P(s) - P(x)]\,p_0(s)}_{K(x,s)}\,y(s)\,ds$$

$\longrightarrow$

Same form as the Volterra Integral Equation:

$$y(x) = F(x) + \int_{x_0}^{x} K(x,s)\, y(s)\, ds \, .$$

Compare with Fredholm Integral Equation:

$$y(x) = F(x) + \int_{a}^{b} K(x,s)\, y(s)\, ds$$

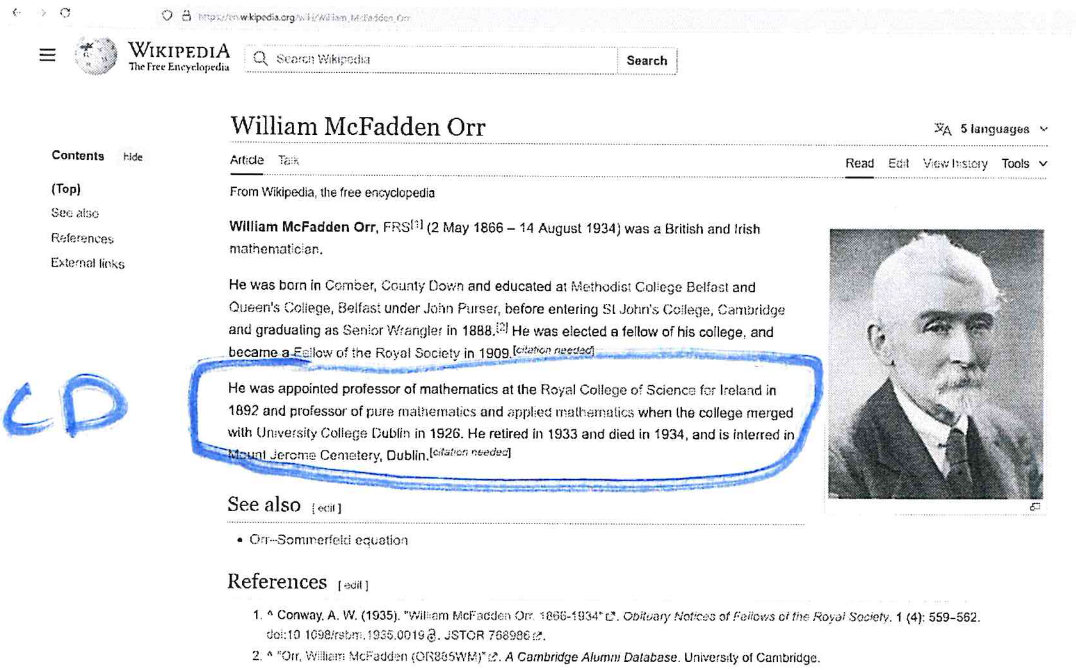
Exercises # 3 Orr — Sommerfeld Equation:

$$ik(z-c)(\partial_z^2 - k^2)\Phi = \frac{1}{Re}(\partial_z^2 - k^2)^2 \Phi \quad (1)$$

- k, Re are real positive parameters
- c is a complex number (eigenvalue)
- z is the the independent variable, $z \in [0,1]$.

Boundary conditions:

$$\Phi = \Phi' = 0 \text{ @ } z = 0,\ z = 1 \quad (2)$$



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From Wikipedia, the free encyclopedia

William McFadden Orr, FRS[1] (2 May 1866 – 14 August 1934) was a British and Irish mathematician.

He was born in Comber, County Down and educated at Methodist College Belfast and Queen's College, Belfast under John Purser, before entering St John's College, Cambridge and graduating as Senior Wrangler in 1888.[2] He was elected a fellow of his college, and became a Fellow of the Royal Society in 1909.[citation needed]

He was appointed professor of mathematics at the Royal College of Science for Ireland in 1892 and professor of pure mathematics and applied mathematics when the college merged with University College Dublin in 1926. He retired in 1933 and died in 1934, and is interred in Mount Jerome Cemetery, Dublin.[citation needed]

UCD

See also [edit]

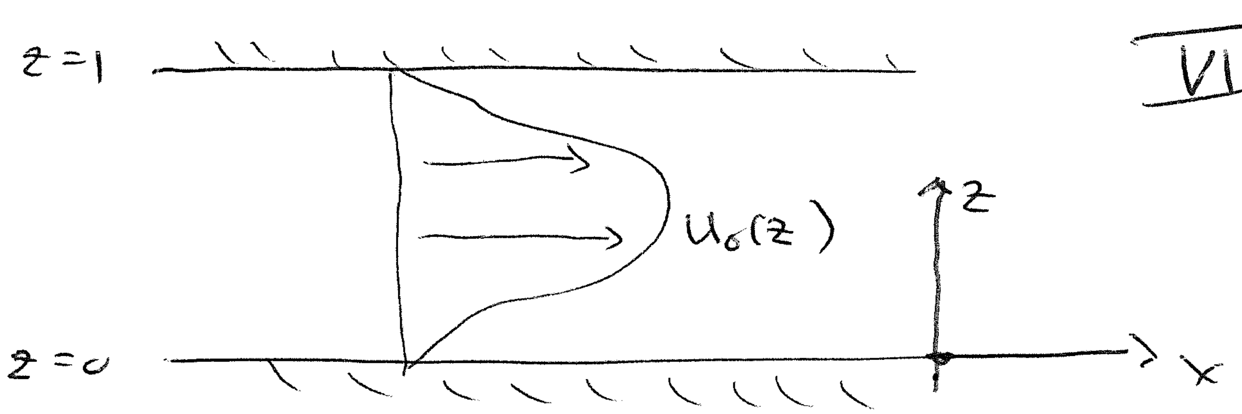
- Orr–Sommerfeld equation

References [edit]

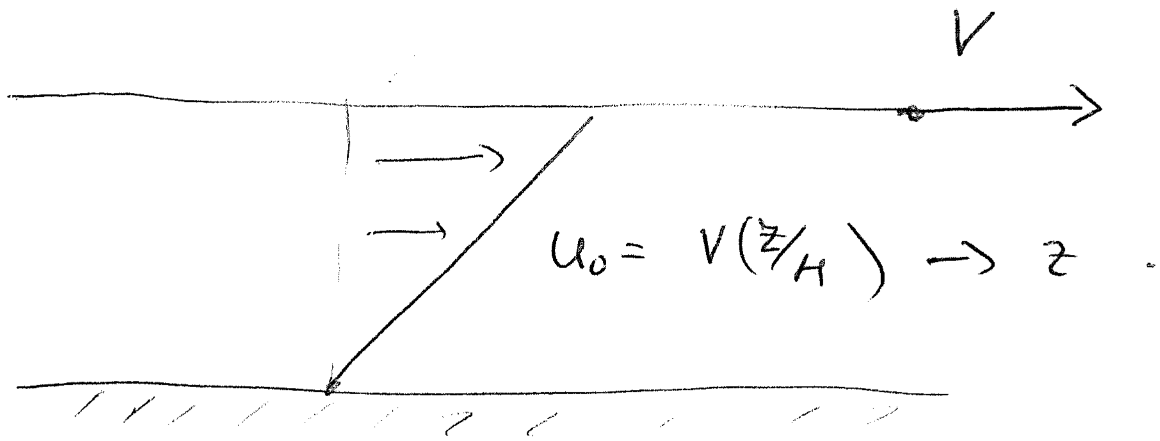
1. ^ Conway, A. W. (1935). "William McFadden Orr. 1866-1934". Obituary Notices of Fellows of the Royal Society. 1 (4): 559–562. doi:10.1098/rsbm.1935.0019. JSTOR 768986.
2. ^ "Orr, William McFadden (OR885WM)". A Cambridge Alumni Database. University of Cambridge.

Figure 2: Screenshot of the Wikipedia page on William McFadden Orr, professor of mathematics at the Royal College of Science for Ireland (1892) and professor of pure mathematics and applied mathematics when that college merged with University College Dublin in 1926.

VII



$U_0(z) = z$ (Couette Flow)



Couette Flow satisfies the Navier-Stokes Equations.

$$\underline{u}(x,z,t) = \Big(U_0(z) + \delta u(x,z,t),\ 0,\ \delta w(x,z,t) \Big)$$

- Plug into Navier-Stokes Equations
- Neglect non-linear terms
- Introduce a streamfunction $\psi(x,z,t)$
- Normal-mode decomposition:

$$\psi(x,z,t) = \Phi(z)\, e^{ik(x-ct)}$$

- Then, Orr-Sommerfeld Equation is obtained.
- if $\mathrm{Im}(c) > 0$, then base flow $U_0(z)$ is LINEARLY UNSTABLE.

VII)

1. Let $\boxed{v = (\partial_z^2 - k^2)\Phi_1}$.

$$ik(z-c)\underbrace{(\partial_z^2-k^2)\Phi_1}_{v} = \frac{1}{Re}(\partial_z^2-k^2)\underbrace{(\partial_z^2-k^2)\Phi_1}_{v}$$

$$\Rightarrow ik(z-c)v = \frac{1}{Re}(\partial_z^2-k^2)v \quad \mathbf{(3)}$$

Second-order homogeneous problem:

$v=0$ is a solution.

Hence: $(\partial_z^2-k^2)\Phi_1 = 0$.

Hence, $\Phi_1 = \begin{cases} \cosh(kz) & ☑ \\ \sinh(kz) & ☑ \end{cases}$

are solutions of the OS equation.

2. Rescaling: $\tilde{z} = z - c - ik/Re$.

Hence, (3) becomes:

$$\left[\frac{d^2}{dz^2} - k^2 - \underline{ikRe}\,(z-c)\right]v = 0.$$

$$\Rightarrow \left\{\frac{d^2}{dz^2} - ikRe\Big[\underbrace{z-c-\frac{ik}{Re}}_{\tilde{z}}\Big]\right\}v = 0.$$

$$\frac{d}{dz} = \frac{d}{d\tilde{z}}$$

Hence, (3) becomes:

$$\left(\frac{d^2}{d\tilde{z}^2} - ik\,Re\,\underline{\tilde{z}} \right) v = 0. \qquad \textbf{(4)}$$

Second transformation:

$$\xi = \lambda \tilde{z} \Rightarrow \frac{\xi}{\lambda} = \tilde{z}.$$

$$\frac{d}{d\tilde{z}} = \frac{d\xi}{d\tilde{z}} \frac{d}{d\xi} = \lambda \frac{d}{d\xi}$$

$$\Rightarrow \frac{d^2}{d\tilde{z}^2} = \lambda^2 \frac{d^2}{d\xi^2}.$$

Hence, under rescaling, (4) becomes:

$$\left(\lambda^2 \frac{d^2}{d\xi^2} - ik\,Re \frac{\xi}{\lambda} \right) v = 0.$$

$$\Rightarrow \left(\cancel{\lambda^3} \frac{d^2}{d\xi^2} - \cancel{ik\,Re}\, \xi \right) v = 0.$$

Choose $\lambda^3 = ik\,Re$.

$$= e^{i\pi/2} k\,Re$$

$$\Rightarrow \lambda = \left[e^{i\pi/2} k\,Re \right]^{1/3}$$

$$\Rightarrow \boxed{\lambda = e^{i\pi/6} (k\,Re)^{1/3}}.$$

$$\left(\frac{d^2}{d\xi^2} - \xi\right) v = 0.$$

Airy's ODE.

$$v = Ai(\xi), \quad Bi(\xi), \quad \blacksquare.$$

Question (3):

$$v = (\partial_z^2 - k^2)\Psi.$$

$$\Rightarrow (\partial_z^2 - k^2)\Psi = \begin{cases} Ai(\xi) \\ Bi(\xi) \end{cases} \qquad (5)$$

Next time: use method of variation of parameters to solve (5).