Chapter 3 – Linear homogeneous ODEs

$$\left.\begin{aligned} &\underbrace{\left[p_n(x)\frac{d^n}{dx^n} + p_{n-1}(x)\frac{d^{n-1}}{dx^{n-1}} + \cdots + p_0(x)\right]}_{L} y(x) = 0, \quad x \in [a,b] \\ &\text{where} \quad p_n(x) \neq 0 \quad \forall\, x \in [a,b] \end{aligned}\right\} \quad (1)$$

or $L[y] = 0$.

Last time: We showed there are at most n linearly independent solutions of (1).

Today: Show there are exactly n ... of (1).

Def^n: The Wronskian of a set of r $(r-1)$-times continuously differentiable functions $\{v_1(x), \dots, v_r(x)\}$ is defined as the following r-dimensional determinant:

$$W(x) = \begin{vmatrix} v_1(x) & v_2(x) & \cdots & v_r(x) \\ v_1'(x) & v_2'(x) & \cdots & v_r'(x) \\ \vdots & & & \\ v_1^{(r-1)}(x) & v_2^{(r-1)}(x) & \cdots & v_r^{(r-1)}(x) \end{vmatrix}$$

$\leftarrow \quad r \quad \rightarrow$

Theorem: If $\{v_1(x), \cdots, v_r(x)\}$ are linearly dependent, then their Wronskian is zero identically.

Proof: See Theorem 3.2, page 21.

Caution :

If $\{u_1(x), \dots, u_r(x)\}$ lin. dependent $\implies$ $W(x) = 0$.

Implication does not go the other way around in general :

$\{u_1(x), \dots, u_r(x)\}$ lin dependent $\cancel{\Leftarrow}$ $W(x) = 0$

In general

Counter-example :

$$u_1(x) = \begin{cases} 0 & x \le 0 \\ x^2, & x \ge 0 \end{cases} \qquad u_2(x) = \begin{cases} x^2 & x \le 0 \\ 0 & x \ge 0 \end{cases}$$

$$W(x) = \begin{cases} \begin{vmatrix} 0 & x^2 \\ 0 & 2x \end{vmatrix} = 0, & x \le 0 \\ \begin{vmatrix} x^2 & 0 \\ 2x & 0 \end{vmatrix} = 0 & x \ge 0 \end{cases}$$

$\implies$ $W(x) = 0$ identically.

Are $u_1(x)$ and $u_2(x)$ linearly dependent ?

$$c_1 u_1(x) + c_2 u_2(x) = 0 \quad \forall \; x .$$

Pick particular values of x:

$x = -1$: $0 \cdot c_1 + c_2 \cdot 1 = 0 \implies c_2 = 0$.

$x = 1$: $1 \cdot c_1 + c_2 \cdot 0 = 0 \implies c_1 = 0$.

$x = 1$: $1 \cdot c_1 + c_2 \cdot 0 = 0 \Rightarrow c_1 = 0$.

$$c_1 u_1(x) + c_2 u_2(x) = 0 \quad \forall x \Rightarrow c_1, c_2 = 0.$$

Therefore, $u_1(x)$ and $u_2(x)$ are linearly independent.

Example:

$W(x) = 0$, u_1 and u_2 are —"— ———— .

In general

$$W(x) = 0 \cancel{\Rightarrow} u_1 \text{ and } u_2 \text{ are linearly dependent.}$$

The next step: Prove Abel's identity.

Lemma 3.3 (Abel's Identity) Suppose that $\{u_1(x), \dots, u_n(x)\}$ satisfy the ODE (1.) Then the Wronskian satisfies:

$$W(x) = W(x_0)\, e^{-\int_{x_0}^{x} \frac{p_{n-1}(s)}{p_n(s)}\, ds}$$

for any $x_0 \in [a, b]$.

To prove Abel's Identity, we first need some facts about determinants.

$$A = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ a_{21} & \cdots & a_{2n} \\ \vdots & & \\ a_{n1} & \cdots & a_{nn} \end{bmatrix}$$

$$\det(A) = \sum_{\tau \in S_n} \operatorname{sign}(\tau) \prod_{i=1}^{n} a_{i\tau(i)}$$

$$= \sum_{\sigma \in S_n} \operatorname{sign}(\sigma) \prod_{i=1}^{n} a_{\sigma(i)\, i}$$

where τ and σ are permutations of $\{1, \dots, n\}$ and therefore live in the permutation group S_n and where

$$\operatorname{Sign}(\tau) = \begin{cases} 1 & \text{if } \tau \text{ is an even permutation of } \{1,2,\dots,n\} \\ -1 & \text{if } \tau \text{ is an odd permutation } \dots \\ 0 & \text{otherwise.} \end{cases}$$

The same for σ.

$(2\,1\ 3\ 4 \dots n) \to (1\ 2 \dots n)$ 1 transposition $\Rightarrow$ odd permutation

$(2\,3\,1\ 4 \dots n) \longrightarrow (2\,1\,3\,4 \dots n)$ 2 transpositions

$(1\,2 \dots n) \Rightarrow$ even permutation

A more compact way of writing determinant:

Eq. (2)

$$\boxed{\det(A) = \epsilon_{i_1 i_2 \dots i_n} a_{1 i_1} a_{2 i_2} \cdots a_{n i_n}}$$

SUMMATION CONVENTION

Repeated indices get summed

where $\epsilon_{i_1 i_2 \dots i_n}$ is the Lévi-Cevita symbol:

$$\epsilon_{i_1 i_2 \dots i_n} = \begin{cases} 1 & \text{if } i_1 i_2 \dots i_n \text{ is an even permutation of } \{1, \dots, n\} \\ -1 & \text{if } i_1 i_2 \dots i_n \text{ is an odd permutation } \dots \\ 0 & \text{otherwise} \end{cases}$$

$\quad\Big\lfloor\; 0 \quad$ otherwise

$\epsilon_{i_1 i_2 \cdots i_n} = 0$ if there are repeated entries in the list $\{i_1, i_2, \ldots, i_n\}$, e.g.

$$\epsilon_{1134\cdots n} = 0 .$$

Proof of Abel's Identity: Suppose that the matrix A (previously introduced) depends on a single parameter, x. That is, each entry a_{ij} depends on x. Then,

$$\frac{d}{dx}\det(A) \overset{(2)}{=} \frac{d}{dx}\, \epsilon_{i_1 \cdots i_n} a_{1i_1} a_{2i_2} \cdots a_{ni_n}$$

$$= \epsilon_{i_1 \cdots i_n} \frac{da_{1i_1}}{dx} a_{2i_2} \cdots a_{ni_n} \quad \leftarrow \text{first row}$$

$$+ \cdots + \epsilon_{i_1 \cdots i_n} a_{1i_1} \cdots a_{n-1\, i_{n-1}} \frac{da_{ni_n}}{dx}$$

$$= \sum_{i=1}^{n} \det\left[A_i(x)\right] \qquad (3)$$

where $A_i(x)$ is the matrix A again, but with the i^{th} row of A replaced with the derivative of the i^{th} row of A.

$$\frac{d}{dx} W(x) \overset{(3)}{=} \begin{vmatrix} u_1'(x) & \cdots & u_n'(x) \\ u_1'(x) & \cdots & u_n'(x) \\ \vdots & & \\ \cdots & & \cdots \end{vmatrix} + \cdots$$

$$\begin{vmatrix} \vdots & & \\ v_1^{(n-1)}(x) & \dots, & v_n^{(n-1)}(x) \end{vmatrix} \qquad \cdot$$

$$\begin{vmatrix} v_1(x) & \dots & v_n(x) \\ v_1'(x) & \dots & v_n'(x) \\ \vdots & & \\ v_1^{(n-2)}(x) & \dots & v_n^{(n-2)}(x) \\ v_1^{(n)}(x) & \dots & v_n^{(n)}(x) \end{vmatrix} \qquad (4)$$

All but the last determinant vanishes.

$$p_n(x)\, v_1^{(n)}(x) + p_{n-1}(x)\, v_1^{(n)}(x) + \dots + p_0(x)\, v_1(x) = 0$$

$$\Rightarrow v_1^{(n)}(x) + \frac{1}{p_n(x)}\left[p_{n-2}(x)\, v_1^{(n-2)}(x) + \dots + p_0(x)\, v_1(x) \right]$$

$$= -\frac{p_{n-1}(x)\, v_1^{(n-1)}(x)}{p_n(x)}$$

← Replace last row of (4) with terms like this.

If the highlighted term and I add it to $v_1^{(n)}(x)$ in the last row of (4), it doesn't change the determinant. Reason: the highlighted term is linearly dependent on all the other entries in the first column of (4).

We do a similar thing for $v_2^{(n)}(x)$, $v_3^{(n)}(x)$, ... in the last row of (4), meaning that (4) can be re-written as:

$$W'(x) = \begin{vmatrix} u_1(x) & \cdots & u_n(x) \\ \vdots & & \\ u_1^{(n-2)}(x) & \cdots & u_n^{(n-2)}(x) \\ -\frac{p_{n-1}(x)}{p_n(x)} u_1(x) & \cdots & -\frac{p_{n-1}(x)}{p_n(x)} u_n(x) \end{vmatrix}$$

$$= -\frac{p_{n-1}(x)}{p_n(x)} W(x)$$

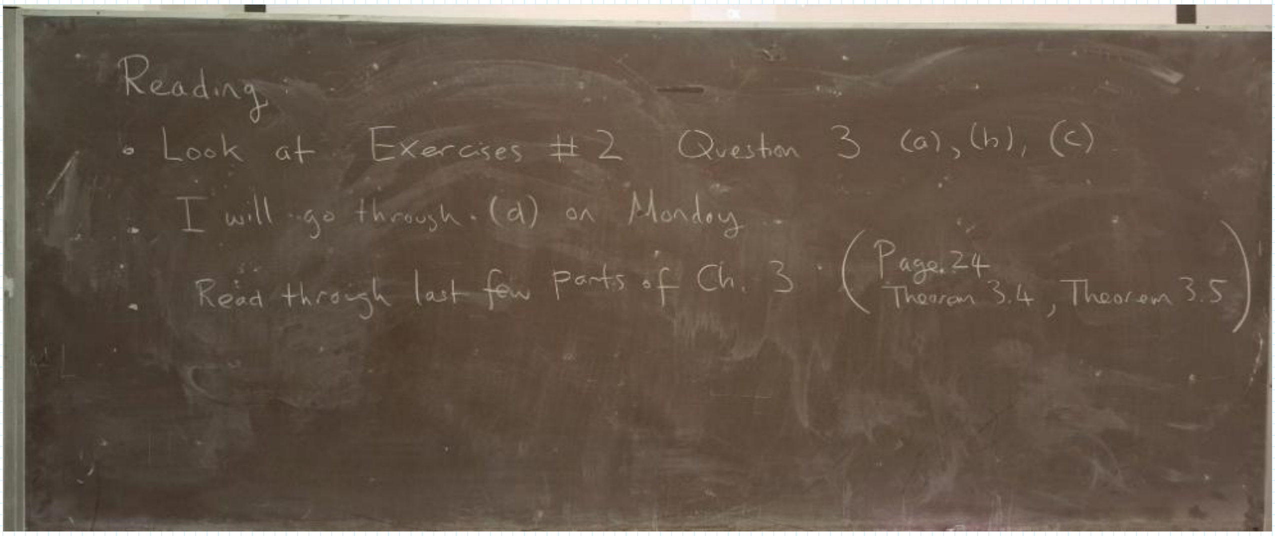
$$\Rightarrow \frac{dW}{dx} = -\frac{p_{n-1}(x)}{p_n(x)} W(x)$$

$$\Longrightarrow W(x) = W(x_0)\, e^{-\int_{x_0}^{x} \frac{p_{n-1}(s)}{p_n(s)} ds},$$

for any $x_0 \in [a,b]$. $\blacksquare$

Exercises 2, Question 1 and Question 2 done on board.

Plan for the rest of Week 3:



Below, I copy in some rough notes to help you go through the last parts of Chapter 3.

Last few steps:

VII

Theorem 3.4 If $u_1(x) \dots u_n(x)$ are solutions of $L[u]=0$ and $W(x)=0$, then $u_1(x) \dots, u_n(x)$ are linearly dependent.

Proof: Since $W(x)=0$ in general, then $W(x_0)=0$ for some particular x_0, hence there is a non-zero solution of:

$$\begin{pmatrix} u_1(x_0) & \cdots & u_n(x_0) \\ u_1'(x_0) & \cdots & u_n'(x_0) \\ \vdots & & \\ u_1^{(n-1)}(x_0) & \cdots & u_n^{(n-1)}(x_0) \end{pmatrix} \underbrace{\begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix}}_{\text{non-zero}} = 0$$

Let $v(x) = c_1 u_1(x) + \dots + c_n u_n(x)$

Then:
$$\left.\begin{aligned} &L[v]=0, \text{ and} \\ &v(x_0)=0 \\ &\vdots \\ &v^{(n-1)}(x_0)=0 \end{aligned}\right\} \text{IVP}$$

hence $v(x)=0$ identically (uniqueness of solutions of IVPs), hence $v(x)=0$ identically, so $u_1(x) \dots, u_n(x)$ ARE LINEARLY DEPENDENT

∎

VIII

To wrap up (at last!):

Theorem 3.5: There are n linearly independent solutions of Equation (*).

Proof: Choose solutions $\{u_1(x), \dots, u_n(x)\}$ with

$$u_i^{(j-1)}(x) = \delta_{i,j-1}$$

where $\delta_{i,j-1} = \begin{cases} 1 & \text{if } i = j-1 \\ 0 & \text{otherwise} \end{cases}$

Then: $W(x_0) = 1$, so $\{u_1(x), \dots, u_n(x)\}$ are linearly independent. ∎