Today: Finish Chapter 1 (well-posedness of ODEs)

A few loose ends: Looked at $y'(x) = F(x, y(x))$

- Systems of ODEs
- Higher-order ODEs

} § 1.7

A system of ODEs:

$$y_i'(x) = F_i(x, y_1(x), \ldots, y_n(x)), \quad i = 1, 2, \ldots, n$$

Or, in vector notation:

$$\underline{y}'(x) = \underline{F}(x, \underline{y}(x))$$

Definition (The vector IVP): Given the ODE $\underline{y}'(x) = \underline{F}(x, \underline{y}(x))$ on an interval $x \in I$, the IVP is to find a differentiable solution $\underline{y}(x)$ on a subinterval $J \subseteq I$ such that $\underline{y}(x_0) = \underline{y}_0$ for a given $x_0 \in J$ and $\underline{y}_0 \in \mathbb{R}^n$.

We need a Lipschitz condition for $\underline{y}'(x) = \underline{F}(x, \underline{y}(x))$.

There is more than one way to extend the modulus $|\cdot|$ to higher dimensions. For example:

$$\| \underline{y} \|_p = \left(\sum_{i=1}^{p} |y_i|^p \right)^{1/p} .$$

If we take the limit as $p \to \infty$, we get:

$$\| \underline{y} \|_\infty = \max_i |y_i| \quad *$$

However, in finite dimensions, all norms are

However, in finite dimensions, all norms are equivalent, e.g.

$$\|\underline{y}\|_2 \le \|\underline{y}\|_1 \le \sqrt{n}\,\|\underline{y}\|_2 \longleftarrow \underline{\text{Exercises 1}}$$

So we can just pick whichever norm suits our purposes.

Def$^{\underline{n}}$ (The Vector Lipschitz Condition) $\underline{F}(x, \underline{y})$ satisfies a Lipschitz condition w.r.t. $\underline{y}$ in a region $D \subset \mathbb{R}^{n+1}$ in the p-norm if there exists a constant $K > 0$ such that

$$\|\underline{F}(x, \underline{y}_1) - \underline{F}(x, \underline{y}_2)\|_p \le K\,\|\underline{y}_1 - \underline{y}_2\|_p$$

for all $(x, \underline{y}_1), (x, \underline{y}_2) \in D$.

Theorem 1.5 (Vector Picard's Theorem (second form))

Let $\underline{F}(x, \underline{y})$ be continuous for all $x \in [x_0 - \alpha, x_0 + \alpha]$ and $\|\underline{y} - \underline{y}_0\|_\infty < \beta$ and let $\underline{F}(x, \underline{y})$ satisfy the Lipschitz condition with respect to the ∞ norm with Lipschitz constant K, say. Further, let $M = \sup \|\underline{F}(x, \underline{y})\|$ over the domain. Then the vector IVP $\underline{y}'(x) = \underline{F}(x, \underline{y}(x))$ with $\underline{y}(x_0) = \underline{y}_0$ has a unique solution defined on the interval has a unique solution defined on the interval

$$|x-x_0| \le \min(\alpha, \beta/M).$$

§ 1.7.2 — Higher-order equations

E.g.

$$y^{(n)}(x) = F(x, y(x), y'(x), \dots, y^{(n-1)}(x))$$

Can be reduced to a system of first-order ODEs.

$$\begin{aligned} y_1(x) &= y(x) \\ y_2(x) &= y'(x) \\ &\vdots \\ y_n(x) &= y^{(n-1)}(x) \end{aligned} \longrightarrow \frac{dy_n}{dx} = \frac{dy^{(n)}}{dx} \overset{IVP}{=} F(\dots)$$

$$\frac{d}{dx}\begin{bmatrix} y_1(x) \\ \vdots \\ \vdots \\ y_{n-1}(x) \\ y_n(x) \end{bmatrix} = \begin{bmatrix} y_2(x) \\ \vdots \\ \vdots \\ y_n(x) \\ F(x, y_1(x), \dots, y_n(x)) \end{bmatrix}$$

.... n^{th}-order ODE can be reduced to a system of first-order ODEs, to which the previous version of Picard's Theorem can be applied. We will look into this more in Chapter 3.

Rest of the week: Comparison Theorems (Chapter 2)

- Help us to prove some of the results of Chapter 1 in a neater way
- Provide a very powerful tool to do a priori analysis of solutions of ODEs.

Our first result: Gronwall's Inequality

Lemma 2.1 Let $\sigma(x)$ be a differentiable function satisfying the differential inequality

$$\sigma'(x) \le c \cdot \sigma(x) \qquad x_0 \le x \le b, \qquad (1)$$

for some constant c. Then

$$\sigma(x) \le \sigma(x_0)\, e^{c(x-x_0)}, \qquad x_0 \le x \le b.$$

Remark $\quad \frac{d\sigma}{dx} \overset{\le}{=} c\sigma, \quad \int_{\sigma(x_0)}^{\sigma(x)} \frac{d\sigma}{\sigma} = \int_{x_0}^{x} c\, dx$

$$\Rightarrow \sigma(x) \overset{\le}{=} \sigma(x_0)\, e^{c\cdot(x-x_0)}$$

Proof: Start with the differential inequality (1) and multiply by e^{-cx} and re-arrange:

$$e^{-cx}\left[\frac{d\sigma}{dx} - c\,\sigma(x)\right] \le 0 \cdot e^{-cx} \qquad \text{I.F.} = e^{-c\cdot x}$$

We can do this because e^{-cx} maintains a positive sign, and hence the direction of the inequality is sign, and hence the direction of the inequality is preserved.

$$LHS = \frac{d}{dx}\left[\sigma(x) \cdot e^{-cx}\right]$$

$$RHS = 0$$

$$LHS \leq RHS$$

$$\therefore \quad \frac{d}{dx}\left[\sigma(x) e^{-cx}\right] \leq 0$$

Hence, $\sigma(x)e^{-cx}$ is non-increasing on $[x_0, b]$.

$$\Rightarrow \sigma(x) e^{-cx} \leq \sigma(x_0) e^{-cx_0} \quad , x \in [x_0, b]$$

$$\Rightarrow \boxed{\sigma(x) \leq \sigma(x_0) e^{c(x-x_0)} \quad , \quad x \in [x_0, b]}$$

First application of Gronwall's Inequality (§2.1.1)

Re-visit the IVP $y'(x) = F(x, y(x))$ with $y(x_0) = y_0$. Gronwall's Inequality gives alternative proofs for

- Uniqueness
- Continuous dependence on initial conditions,

without having to resort to Picard iteration.

without having to resort to Picard iteration.

Method : Let $y(x)$ solve the IVP with initial condition $y(x_0) = y_0$. Also, let $y^{(\delta)}(x)$ solve the IVP with initial condition $y^{(\delta)}(x_0) = y_0 + \delta$.

Let

$$\sigma(x) = \left[y(x) - y^{(\delta)}(x) \right]^2 .$$

$$\frac{d\sigma}{dx} = 2\left[y(x) - y^{(\delta)}(x) \right]\left[y'(x) - y^{(\delta)\prime}(x) \right]$$

$$\overset{IVP}{=} 2\left[y(x) - y^{(\delta)}(x) \right]\left[F(x, y(x)) - F(x, y^{(\delta)}(x)) \right]$$

$$\leq 2\left| y(x) - y^{(\delta)}(x) \right| \left| F(x, y(x)) - F(x, y^{(\delta)}(x)) \right|$$

$$\overset{\text{Lipschitz}}{\leq} 2\left| y(x) - y^{(\delta)}(x) \right| K \left| y(x) - y^{(\delta)}(x) \right|$$

$$= 2K \left| y(x) - y^{(\delta)}(x) \right|^2$$

$$= 2k\, \sigma(x) .$$

Hence:

$$\frac{d\sigma}{dx} \leq 2K\, \sigma(x)$$

By Gronwall:

$$e^{2k(x-x_0)}$$

By Gronwall:

$$\sigma(x) \le \sigma(x_0)\, e^{2k(x-x_0)}.$$

But $\sigma(x_0) = |\delta|^2 = |\, y(x) - y^{(\delta)}(x)|^2$

Take square roots

$$|y(x) - y^{(\delta)}(x)| \le |\delta|\, e^{k(x-x_0)}$$

Two results for the price of one:

i) Continuous dependence on initial conditions:

$$\lim_{\delta \to 0} |y(x) - y^{(\delta)}(x)| = 0$$

ii) Uniqueness: If $\delta = 0$, then

$$|y(x) - y^{(\delta)}(x)| = 0 \quad \forall\, x$$

$$\Rightarrow \quad y(x) = y^{(\delta)}(x) \qquad \text{UNIQUENESS}$$

Comparison Theorems (Ch. 2)

Today: General comparison theorem

Let $F(x,y)$ satisfy a Lipschitz condition for x and y, for $x \in [a,b]$. Suppose that $y(x)$ satisfies $y'(x) = F(x, y(x))$. Also, suppose that $z(x)$ satisfies

$$\boxed{z'(x) \leq F(x, z(x))},$$

for all $x \in [a,b]$. Suppose that $y(x)$ and $z(x)$ have the same initial condition:

$$y(x_0) = z(x_0), \quad \text{some } x_0 \in [a,b].$$

Then:

$$\boxed{z(x) \leq y(x)}, \text{ for all } x \geq x_0 .$$

Proof: $\sigma(x) = z(x) - y(x)$.

$$\sigma'(x) = z'(x) - y'(x)$$

$$\leq F(x, z(x)) - F(x, y(x))$$

Lipschitz condition on $F(x,z(x)) - F(x,y(x))$

$$-K|z(x) - y(x)| \leq F(x, z(x)) - F(x, y(x)) \leq K|\overbrace{z(x) - y(x)}^{\sigma(x)}|$$

Put the three highlighted results together:

$$\sigma'(x) \leq K|\sigma(x)| \ .$$

If $\sigma(x) \leq 0$, then $z(x) \leq y(x)$ and <u>we are done</u> (Case 1). Look at Case 2, where $\sigma(x) > 0$.

Therefore, $\sigma'(x) \leq K|\sigma(x)|$ becomes $\sigma'(x) \leq K \cdot \sigma(x)$.

We apply Gronwall's Inequality:

$$\sigma(x) \leq \sigma(x_0)\, e^{K(x-x_0)}, \qquad x \geq x_0 .$$

But $\sigma(x_0) = 0$ (same initial conditions), therefore $\sigma(x) \leq 0$, therefore $z(x) - y(x) \leq 0$, therefore:

$$z(x) \leq y(x), \qquad x \geq x_0 . \qquad \blacksquare$$

<u>Corollary</u>: Let

$$y'(x) = F(x, y(x))$$
$$z'(x) = G(x, z(x))$$

with the same initial conditions $y(x_0) = z(x_0)$. Suppose that F and G have a Lipschitz condition on the second variable. If

$$G(x,y) \leq F(x,y) \quad \text{for all } x \text{ and } y$$

then $z(x) \leq y(x)$, for all $x \geq x_0$.

<u>Example</u>: We can use comparison theorems to do

... solutions of ODEs.

Example : [illegible]

a priori analysis of solutions of ODEs.

$$y'(x) = \underbrace{\cosh(x + y(x)) + (y(x))^2}_{\boxed{\geq 1 + (y(x))^2}}.$$

$$y(0) = 0$$

The solution blows up: $y(x) \to \infty$ as $x \to x_*$, where $0 < x_* \leq \pi/2$.

To show this, we use the comparison theorems.

We know that $\cosh(z) \geq 1$, so

$$\boxed{\cosh(x + y(x)) \geq 1}.$$

Let $F(x,y) = \cosh(x+y) + y^2$,

$G(x,y) = 1 + y^2$.

$$F(x,y) \geq G(x,y).$$

Introduce a second ODE:

$$z'(x) = G(x, z(x)) = 1 + (z(x))^2$$

$$z(0) = 0$$

By comparison theorems, since $G(x,\cdot) \leq F(x,\cdot)$,

$z(x) \leq y(x)$, for all $x \geq 0$.

ODE for z:

$$\frac{dz}{} = 1 + z^2 \qquad z(0) = 0.$$

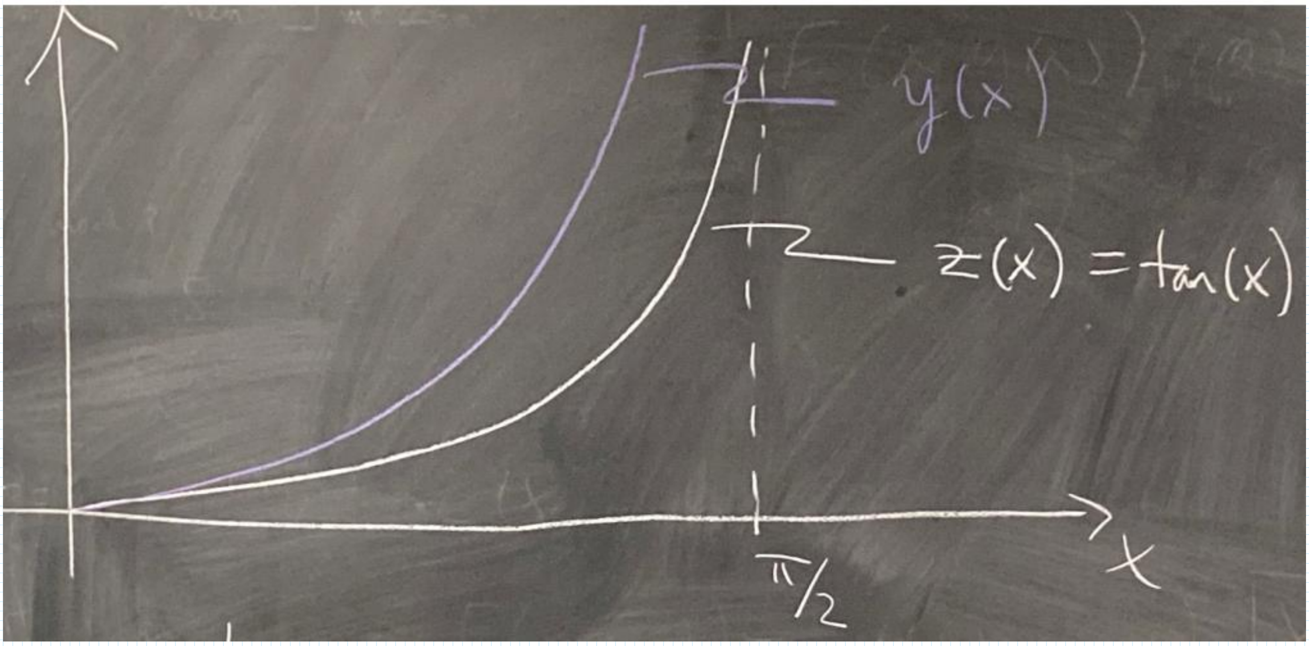
$$\frac{dz}{dx} = 1 + z^2 \quad , \quad z(0) = 0.$$

Solution: $z(x) = \tan(x)$.

Since $\tan(x) \to \infty$ as $x \uparrow \pi/2$, and

$$\underbrace{\tan(x)}_{z(x)} \leq y(x)$$

it follows that there exists $x_* \in (0, \pi/2)$ such that $y(x) \to \infty$ as $x \uparrow x_*$.



That's all about comparison theorems... for now! (Week 12)

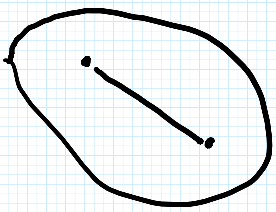
Rest of today's lecture – Exercises #1.

Q1. Prove the statement in Chapter 1 of the lecture notes: If $F(x,y)$ is continuously differentiable

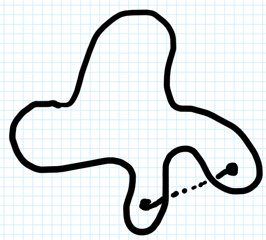
Q1. Prove the statement in chapter [illegible] of the lecture notes: If $F(x,y)$ is continuously differentiable and $(x,y) \ni D$ is bounded and convex, then F satisfies a Lipschitz condition with respect to y, with

$$K = \sup_{(x,y)\in D} \left|\frac{\partial F}{\partial y}\right| .$$

First, we look at the definition of a convex set. A set D is convex if the line segment joining any two points in D is contained entirely in D.



CONVEX SET



NOT CONVEX

In symbols, a set D is convex, if, given any two points $\underline{x}$ and $\underline{y}$ in D, the line segment

$$\underline{x}(t) = \underline{x} \cdot t + \underline{y}(1-t), \qquad t \in [0,1]$$

is contained entirely in D, for all $t \in [0,1]$.

<u>Solution to the question</u>: Let $\underline{x}_1 = (x_1, y_1)$ and $\underline{x}_2 = (x_2, y_2)$ in D. Because D is convex, the

line segment

$$L: \quad \underline{x}(t) = \underline{x}_1(1-t) + \underline{x}_2 t \qquad t \in [0,1]$$

is contained entirely in D. Introduce the function $g(t)$ of a single variable:

$$g(t) = F(\underline{x}(t)).$$

$$\frac{dg}{dt} = \underline{\nabla} F \cdot \frac{d\underline{x}}{dt} \quad \dots \quad \begin{cases} \text{Chain rule,} \\ F \text{ is continuously} \\ \text{differentiable} \end{cases}$$

$$= \underline{\nabla} F \cdot (\underline{x}_2 - \underline{x}_1) \quad \dots \quad \frac{d\underline{x}}{dt} \text{ along } L$$

Since g is continuously differentiable, by the Mean Value Theorem, there exists a $t_* \in (0,1)$ such that

$$\left.\frac{dg}{dt}\right|_{t_*} = \frac{g(1) - g(0)}{1} = F(\underline{x}_2) - F(\underline{x}_1)$$

$$|F(\underline{x}_2) - F(\underline{x}_1)| = \left| \left.\frac{dg}{dt}\right|_{t_*} \right|$$

$\underline{x}_1 = (x_1, y_1)$
$\underline{x}_2 = (x_2, y_2)$

$$= |\underline{\nabla} F(\underline{x}(t_*)) \cdot (\underline{x}_2 - \underline{x}_1)|$$

Take $x_1 = x_2 = x$. Hence, $\underline{x}_2 - \underline{x}_1 = (y_2 - y_1)\underline{j}$

$|F(\underline{x}_2) - F(\underline{x}_1)|$ becomes:

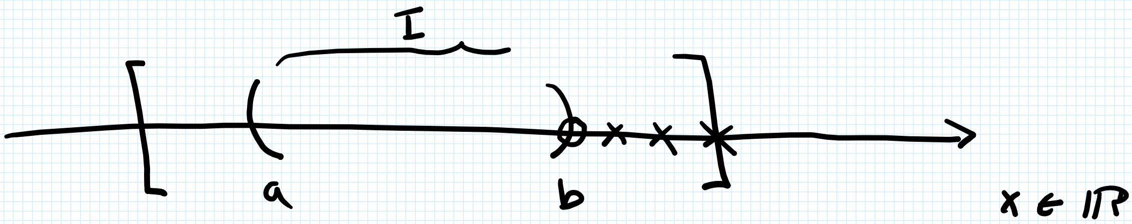
$$\left| F(x,y_2) - F(x,y_1) \right| = \left| \frac{\partial F}{\partial y}\bigg|_{\underline{x}(t^*)} (y_2 - y_1) \right|$$

$$= \left| \frac{\partial F}{\partial y}\bigg|_{\underline{x}(t^*)} \right| |y_2 - y_1|$$

$$\leq \underbrace{\sup_{(x,y)\in D}}_{K} \left| \frac{\partial F}{\partial y} \right| |y_2 - y_1|$$

$$\therefore \quad K = \sup_{(x,y)\in D} \left| \frac{\partial F}{\partial y} \right| \quad \blacksquare$$

where "sup" means "least upper bound".



$$\sup_{x\in I} x = b.$$

Every bounded set of real numbers has a least upper bound — Completeness Axiom.

I

Week 2, Lecture 3 30/01/2026

Today

- Wrap up assignment 1
- Linear ODEs

Question 1: All finite-dimensional norms are equivalent. That is, if I have two norms $\|\cdot\|_\beta$ and $\|\cdot\|_\alpha$, then there exist positive constants c_1 and c_2 such that

$$c_1\|\underline{x}\|_\beta \leq \|\underline{x}\|_\alpha \leq c_2\|\underline{x}\|_\beta,$$

for all $\underline{x} \in \mathbb{R}^n$.

Using the Cauchy-Schwarz inequality, show that the L^1 and L^2 norms are equivalent, such that:

$$\|\underline{x}\|_2 \leq \|\underline{x}\|_1 \leq \sqrt{n}\|\underline{x}\|_2.$$

Solution: We have

$$\|\underline{x}\|_1 = \sum_{i=1}^{n} |x_i|$$

Also, take $\underline{x} = (x_1, \ldots, x_n)$

Take $\underline{y} = (|x_1|, \ldots, |x_n|)$, $\underline{a} = (1, 1, \ldots, 1)$.

Take

$$\underline{a} \cdot \underline{y} = \underbrace{|x_1| + \cdots |x_n|}_{\|\underline{x}\|_1}$$

$$\underset{C.S.}{\leq} \|\underline{a}\|_2 \|\underline{x}\|_2$$

Hence:

$$\|\underline{x}\|_1 \leq \|\underline{a}\|_2 \|\underline{x}\|_2$$

$$= \sqrt{1^2 + \dots + 1^2}\, \|\underline{x}\|_2$$

$$= \sqrt{n}\, \|\underline{x}\|_2$$

$$\therefore \quad \boxed{\|\underline{x}\|_1 \leq \sqrt{n}\, \|\underline{x}\|_2} . \qquad (1)$$

II

Take

$$\left[\, |x_1| + \dots + |x_n| \,\right]^2$$

$$= |x_1|^2 + \dots + |x_n|^2 + 2|x_1||x_2| + \dots$$

$$\geq |x_1|^2 + \dots + |x_n|^2$$

$$= \|\underline{x}\|_2^2 .$$

Take square roots:

$$\boxed{\underbrace{|x_1| + \dots + |x_n|}_{= \|\underline{x}\|_1} \geq \|\underline{x}\|_2} \qquad (2)$$

Combine (1) and (2):

$$\|\underline{x}\|_2 \leq \|\underline{x}\|_1 \leq \sqrt{n}\, \|\underline{x}\|_2 .$$

Question 3 — Look at it yourselves.

Question 4 — " .

We will move on to <u>Chapter 3</u>, linear homogeneous ODEs.

Standard form:

$$p_n(x)\,y^{(n)}(x) + p_{n-1}(x)\,y^{(n-1)}(x) + \cdots + p_0(x)\,y(x) = 0, \quad x \in [a,b] \qquad (1)$$

Remark:

- RHS $= 0 \Rightarrow$ homogeneous case
- For the homogeneous case, $y(x) = 0$ solves (1) <u>with zero initial conditions</u>.
- We have to assume
  $$p_n(x) \neq 0 \quad \forall x \in [a,b].$$

With $p_n(x)$ different from zero, we can compute the Lipschitz constant in the ∞ norm:

$$K = (n-1) + \sum_{i=0}^{n-1} \max_{x\in[a,b]} \left| \frac{p_i(x)}{p_n(x)} \right| .$$

(Exercise 2)

Hence, Equation (1) is <u>well posed</u>.

Equation (1) is a linear ODE: $y(x)$ and its derivatives appear raised to the zeroth and first powers only.

<u>Clarify</u>: By "solution" I mean a $y(x)$ that solves (1), without specifying initial conditions. By "solution of the IVP" I mean a $y(x)$ that solves (1) <u>with</u> <u>specified initial conditions</u>.

Linear independence of solutions (§3.2)

<u>Linear operator</u>:

$$L = p_n(x)\frac{d^n}{dx^n} + p_{n-1}(x)\frac{d^{n-1}}{dx^{n-1}} + \cdots + p_0(x)$$

Hence, (1) can be re-written as:

$$L[y] = 0.$$

If u and v are solutions of (1), then:

$$L[u+v] \overset{\text{linearity}}{=} \cancel{L[u]} + \cancel{L[v]}$$
$$= 0$$

$\Rightarrow$ $u+v$ is a solution.

Also,
$$L[c\cdot u] \overset{\text{linearity}}{=} c\cdot L[u] = 0.$$

$\Rightarrow$ $c \cdot u$ is a solution, for all V
c constant in $\mathbb{R}$.

So: Solutions are closed
- under addition
- under scalar multiplication

Hence, the set of solutions of (1) is a vector space over $\mathbb{R}$ (solution space).

We should try to find the dimension of the solution space. Answer: n.

We have to build up to this!

We start with the definition of linear dependence:

Def^{n} The set $\{v_1(x), .., v_r(x)\}$ of solutions to (1) are linearly dependent if there exist constants $c_1, \dots, c_r$ not all zero such that:

$$c_1 v_1(x) + \dots + c_r v_r(x) = 0 \quad \forall x \in \underbrace{I}_{[a,b]}$$

Conversely, the set $\{v_1(x), .., v_r(x)\}$ is linearly independent if it is not linearly dependent.

Proof strategy:

- We show that the dimension of the solution space is at most n.
- We show that the dimension of the solution space is exactly n.

<u>Theorem 3.1</u>: The dimension of the solution space of (1) is at most n.

<u>Proof</u>: Suppose we have $(n+1)$ solutions of (1):

$$u_1(x), \dots, u_n(x), u_{n+1}(x).$$

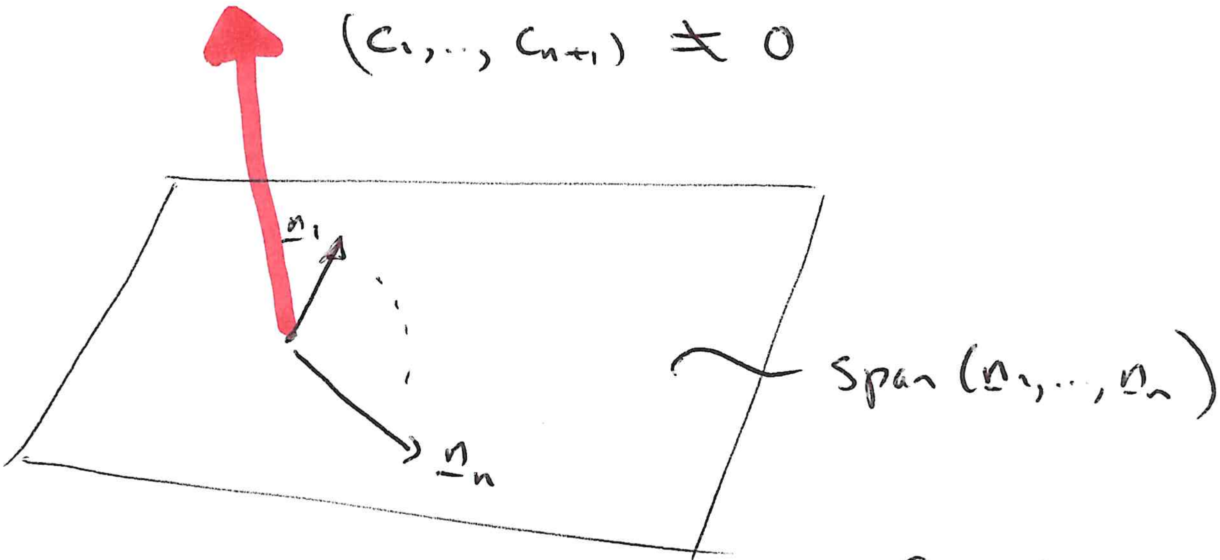
Consider the problem of finding constants $c_1, \dots, c_n, c_{n+1}$ such that, for some $x_0 \in [a,b]$

$$n\left\{\begin{aligned}
c_1 u_1(x_0) + \dots + c_n u_n(x_0) + c_{n+1} u_{n+1}(x_0) &= 0 \\
c_1 u_1'(x_0) + \dots + c_n u_n'(x_0) + c_{n+1} u_{n+1}'(x_0) &= 0 \\
&\vdots \\
c_1 u_1^{(n-1)}(x_0) + \dots + c_n u_n^{(n-1)}(x_0) + c_{n+1} u_{n+1}^{(n-1)}(x_0) &= 0
\end{aligned}\right.$$

In other words, we seek a vector $(c_1, \dots, c_{n+1})$ such that:

$$\left.\begin{aligned}
(c_1, \dots, c_{n+1}) \cdot \underline{n}_1 &= 0, & \underline{n}_1 &= (u_1(x_0), \dots, u_{n+1}(x_0)) \\
(c_1, \dots, c_{n+1}) \cdot \underline{n}_2 &= 0, & \underline{n}_2 &= (u_1'(x_0), \dots, u_{n+1}'(x_0)) \\
&\vdots \\
(c_1, \dots, c_{n+1}) \cdot \underline{n}_n &= 0, & \underline{n}_n &= (u_1^{(n-1)}(x_0), \dots, u_{n+1}^{(n-1)}(x_0))
\end{aligned}\right\}$$

A non-zero vector $(c_1, \dots, c_{n+1})$ can always be found satisfying these conditions



Hence, $(c_1, \dots, c_{n+1}) \neq 0$ can be found.

Introduce

$$V(x) = c_1 V_1(x) + \dots + c_{n+1} V_{n+1}(x)$$

$V(x_0) = 0$, by definition.

Also,

$$\left.\begin{aligned} V'(x_0) &= 0 \\ &\vdots \\ V^{(n-1)}(x_0) &= 0 \end{aligned}\right\} \text{by definition of the } c_i\text{'s}$$

Finally,

$L[V] = 0$, by linearity.

(2)

Equation (2) is an IVP:

$$L[v] = 0$$

$$v(x_0) = 0, \dots, v^{(n-1)}(x_0) = 0.$$

Solution: $v(x) = 0$ (unique solution).

i.e. there exist constants $c_1, \dots, c_{n+1}$ not all zero such that

$$\underbrace{c_1 u_1(x) + \dots + c_{n+1} u_{n+1}(x)}_{v(x)} = 0 \quad \forall x \in [a,b]$$

Hence, $\{u_1(x), \dots, u_{n+1}(x)\}$ is a linearly dependent set.

$\Longrightarrow$ at most n linearly independent solutions of (1).

$\blacksquare$