Chapter 1 – Lecture notes

Ordinary Differential Equations (ODEs)

Generic Initial Value Problem (IVP):

- One independent variable x
- One dependant variable y, $y(x)$

Generic IVP:

$$y'(x) = F(x, y(x)) \qquad (1)$$

Eq. (1) is posed on an interval $I = [a,b]$, with initial condition $y(x_0) = y_0$, $x_0 \in I$.

Question: Can we solve (1) by Taylor Series?

Attempt:

$$y(x) = \overbrace{y(x_0)}^{y_0} + y'(x_0)(x-x_0) + \tfrac{1}{2} y''(x_0)(x-x_0)^2 + \dots \qquad (2)$$

$$y'(x) \overset{\text{Eq.(1)}}{=} F(x, y(x))$$

$$\Rightarrow y'(x_0) = F(x_0, y_0) \quad \checkmark$$

$$y''(x) = \frac{d}{dx} y'(x)$$

$$\overset{\text{Eq.(1)}}{=} \frac{d}{dx} F(x, y(x))$$

$$= \frac{\partial F}{\partial x} + \frac{\partial F}{\partial y}\frac{dy}{dx}$$

ODE

$$\Rightarrow y''(x_0) \overset{\text{Chain Rule}}{=} \frac{\partial F}{\partial x}(x_0, y_0) + \frac{\partial F}{\partial y}(x_0, y_0)\, F(x_0, y_0) \quad \checkmark$$

Iterate for $y'''(x_0)$ etc. CLOSED PROBLEM.

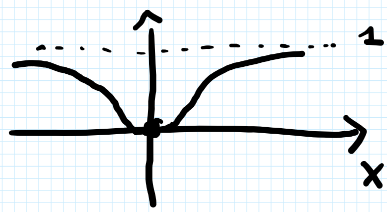
Problem: We don't know what is the radius of convergence of the Taylor series (2).

Even worse, the radius of convergence can be zero.

§ 1.1.1 — Pathological Example

$$y'(x) = F(x), \quad \text{where}$$

$$F(x) = \begin{cases} e^{-1/x^2}, & x \neq 0 \\ 0, & x = 0 \end{cases}.$$



By direct differentiation,

$$y^{(n)}(0) = 0, \quad n = 0, 1, 2, \ldots$$

So, by setting $x_0 = 0$ and $y_0 = 0$ corresponding to a particular initial condition, we get from (2) that

$$y(x) = 0 \quad (3)$$

is a solution of the IVP.

But we know that

$$y(x) = \begin{cases} x e^{-1/x^2} + \sqrt{\pi}\left(\operatorname{erf}\left(\tfrac{1}{x}\right) - 1\right) & x \neq 0 \\ 0, & x = 0 \end{cases} \quad (4)$$

The only way to avoid a contradiction between (3) and (4) is to conclude that the radius of convergence of the Taylor series is zero.

Conclusion: A general theory of ODEs can't be based on Taylor series.

We take a step back and define what it means for an IVP to be well posed (Def^{n} 1.1 in typed notes):

Given $y'(x) = F(x, y(x))$ on an interval I, with $x \in I$, the IVP is to find a differentiable solution $y(x)$ on a subinterval $J \subseteq I$ such that $y(x_0) = x_0$, for $x_0 \in J$ and $y_0 \in \mathbb{R}$. We say that the IVP is well posed if:

strong / classical solution (PDEs)

- A solution exists and is differentiable
- The solution is unique
- The solution depends continuously on the initial conditions.

Example: Ill-posed IVP:

$I = [-1, 1]$

$$y'(x) = \begin{cases} 0, & -1 \le x \le 0 \end{cases}$$

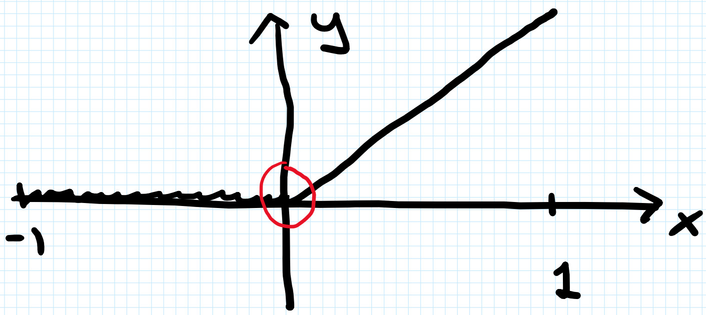
$$y'(x) = \begin{cases} 0, & -1 \le x \le 0 \\ 1, & 0 \le x \le 1 \end{cases}$$

IC: $y(0) = 0$, $x \in I$

Solution by integration:

$$y(x) = \begin{cases} 0, & -1 \le x \le 0 \\ x, & 0 \le x \le 1 \end{cases}$$

... constant of integration $= 0$ for continuity, to satisfy the IC



Discontinuity in first derivative at $x = 0$.

Solution fails to be differentiable at $x = 0$

Solution fails first criterion of well posedness.

<u>Remark</u>: This is a solution but it's not well posed.

A <u>weak</u> <u>solution</u>.

<u>Example</u>: $y'(x) = 3y^{2/3}$

Example of $y'(x) = F(\cancel{x}, y)$

$-1 \le x \le 1$, $I = [-1, 1]$

$-1 \le x \le 1, \quad I = [-1,1]$

$y(0) = 0 \quad \ldots \quad x_0 = 0, \; y_0 = 0.$

Separation of variables:

$$\frac{1}{3}\frac{1}{y^{2/3}}\frac{dy}{dx} = 1$$

$$\left\{ \frac{1}{3} y^{-2/3} \frac{dy}{dx} = \frac{d}{dx}\left(y^{1/3}\right) \right.$$

$$\Rightarrow \frac{d}{dx}\left(y^{1/3}\right) = 1$$

$$\Rightarrow y^{1/3} = x + C \overset{\text{I.C.}}{=} x$$

$$\Rightarrow \boxed{y = x^3}$$

Other solutions:

$$y(x) = 0 \quad \forall x \in I$$

$$y(x) = \begin{cases} 0, & -1 \le x \le x_1 \\ (x-x_1)^3, & x_1 \le x \le 1 \end{cases}$$

for any parameter $x_1 \in (-1,1)$.

Infinitely many solutions.

Well-posedness of ODEs (Chapter 1)

$$y'(x) = 3y^{2/3}, \quad y(0) = 0, \quad x \in \underbrace{[-1,1]}_{I}$$

Separation of variables:

$$y(x) = x^3$$

Another solution:

$$y(x) = 0$$

Or:

$$y(x) = \begin{cases} 0, & -1 \le x \le x_1 \\ (x-x_1)^3, & x_1 \le x \le 1 \end{cases}, \quad x_1 = \text{const. parameter},$$

where $x_1 \in (-1,1)$.

Hence, infinitely many solutions.

Hence, IVP is ill-posed.

To guarantee well-posedness, we need the Lipschitz condition on $F(x,y)$:

- Better than continuity (in the y-variable)
- Not as strong as C^∞

Defn (Defn 1.2) $F(x,y)$ satisfies the Lipschitz condition w.r.t. y in a region $D \in \mathbb{R}^2$ if there exists a $K > 0$ such that :

$$|F(x, y_1) - F(x, y_2)| \leq K |y_1 - y_2|$$

for all (x, y_1) and $(x, y_2) \in D$.

<u>Remark</u> : If $F(x,y)$ is C^∞ and D is closed, bounded and convex, then by the Mean Value Theorem,

$$K = \sup_{(x,y)\in D} \left| \frac{\partial F}{\partial y} \right| \quad \ldots \text{ Exercises \# 1}$$

Next step — re-express the IVP as an integral equation (§1.3).

$$\int_{x_0}^{x} \frac{dy}{dx} dx = \int_{x_0}^{x} F(x, y(x)) dx \,, \qquad y(x_0) = y_0$$

$$\Rightarrow \quad y(x) - \underbrace{y(x_0)}_{y_0} = \int_{x_0}^{x} F(x, y(x)) dx$$

$$\Rightarrow \quad \boxed{y(x) = y_0 + \int_{x_0}^{x} F(x, y(x)) dx}$$

Suggests an iterative method for solving the IVP :

$$y_0(x) = y_0 \quad \ldots \quad \text{i.e. the constant function } y_0$$

Picard

$$y_1(x) = y_0 + \int_{x_0}^{x} F(x, y_0) dx$$

$$y_2(x) = y_0 + \int_{x_0}^{x} F(x, y_1(x)) dx$$

Picard Iteration

$$y_2(x) = y_0 + \int_{x_0}^{x} F(x, y_1(x))\,dx$$

$$\vdots$$

$$y_{n+1}(x) = y_0 + \int_{x_0}^{x} F(x, y_n(x))\,dx$$

Motivation: Consider the IVP $y'(x) = k\,y(x)$, $\underbrace{y(0)}_{x_0} = \underbrace{1}_{y_0}$. i.e. $\frac{dy}{dx} = ky$. From ACM10060 we know the solution:

$$y(x) = e^{kx}.$$

Picard iteration: $F(x, y) = ky$ (i.e. no explicit x-dependence). Integral form of IVP:

$$y(x) = 1 + \int_0^x k\,y(s)\,ds.$$

$$y_0(x) = 1$$

$$y_1(x) = 1 + \int_0^x k \cdot 1\,ds = 1 + kx$$

$$y_2(x) = 1 + \int_0^x k \cdot y_1(s)\,ds$$

$$= 1 + \int_0^x k(1 + k \cdot s)\,ds$$

$$= 1 + kx + \frac{1}{2}k^2x^2$$

Guess the pattern:

$$y_n(x) = 1 + kx + \frac{1}{2}(kx)^2 + \cdots + \frac{1}{n!}(kx)^n .$$

Take the limit as $n \to \infty$:

$$\lim_{n\to\infty} y_n(x) = e^{kx} .$$

Hence:

$$\lim_{n\to\infty} y_n(x) = e^{kx} \quad \text{.... limit of Picard iteration}$$

$\|$ Limit of Picard iteration = Sln of IVP

$$y(x) = e^{kx} \quad \text{.... from ACM10060}$$

We would like to show that in the general case,

$$\underbrace{\lim_{n\to\infty} y_n(x)}_{\text{Limit of Picard iteration}} = \underbrace{y(x)}_{\text{Sln of IVP}}$$

We ~~do~~ prove this by proving the first form of Picard's Theorem.

Theorem 1.1 — First form of Picard's Theorem.

Let $F(x,y)$ be continuous for all $x \in [a,b]$ and all $y \in \mathbb{R}$, and let $F(x,y)$ satisfy a Lipschitz condition in the y-variable with Lipschitz constant K, say. Then the IVP $y'(x) = F(x, y(x))$ with $y(x_0) = y_0$ for some $x_0 \in [a,b]$ has a unique sl^{n} for all $x \in [a,b]$.

The proof has three steps:

EXAMINABLE

1. Show that the sequence of Picard iterates converges uniformly to $y(x)$ on $x \in [a,b]$.
2. Show that the limit function $y(x)$ satisfies the IVP.
3. Show that $y(x)$ is the unique solution.

We first of all need some background knowledge about pointwise and uniform convergence. (§ 1.4.1)

Def$^{\underline{n}}$ (Pointwise convergence) A series $\sum u_r(x)$ of continuous functions $u_r : I \to \mathbb{R}$ converges pointwise to a limit $u(x)$ if for each $x \in I$, and for a given $\epsilon > 0$ there exists N such that

$$\left| \sum_{r=0}^{n} u_r(x) - u(x) \right| < \epsilon \text{ whenever } n > N(x)$$

Pointwise convergence is not enough for our purposes.

E.g. $u_0(x) = 1$, $u_n(x) = x^n - x^{n-1}$, $x \in [0,1]$.

$$\sum_{r=0}^{n} u_r(x) = \begin{array}{l} u_n(x) \\ + u_{n-1}(x) \\ + \end{array} = \begin{array}{l} x^n - \cancel{x^{n-1}} \\ + \cancel{x^{n-1}} - \cancel{x^{n-2}} \\ \vdots \end{array}$$

$$\sum_{r=0} \qquad + u_{n-1}(x) + \cdots + u_0(x) \qquad + \cancel{x^{n-1}} - \cancel{x^{n-2}} \cdots + \cancel{x} - \cancel{x^0} + \cancel{x}$$

$$\overline{\qquad x^n \qquad}$$

$$\lim_{n\to\infty} \sum_{r=0}^{n} u_r(x) = \lim_{n\to\infty} x^n = \begin{cases} 1 & , x = 1 \\ 0, & 0 \le x < 1 \end{cases}$$

Limit function has a jump discontinuity at $x = 1$.

Def$^{\underline{n}}$: A series of continuous functions $\sum u_r(x)$ converges uniformly to $u(x)$ if, for a given $\epsilon > 0$ there exists N such that:

$$\left| \sum_{r=0}^{n} u_r(x) - u(x) \right| < \epsilon \quad \text{whenever } n > N$$

for all $x \in I$.

Some facts about uniform convergence:

- If the u_r's are continuous and $\sum u_r(x)$ converges uniformly to a limit function $u(x)$, the the limit function is continuous.
- Weierstrass M-test.

Theorem 1.2: Let M_n be numbers such that

$$|u_n(x)| \le M_n \qquad \forall x \in I, \quad n = 0, 1, 2, \ldots$$

$$\left(\sum^{\infty} M_n < \infty \right)$$

$|v_n(x)| \le M_n$

If the sum $\sum M_n$ converges $\left(\sum_{r=0}^{\infty} M_r < \infty \right)$, the sum $\sum v_r(x)$ converges uniformly to a limit function, for all $x \in I$.

We now apply the Weierstrass M-test to the Picard iterates.

$$u_0(x) = y_0$$

$$u_n(x) = y_n(x) - y_{n-1}(x).$$

$$\sum_{r=0}^{n} v_r(x) = \begin{array}{l} y_n - y_{n-1} \\ + y_{n-1} - y_{n-2} \\ + \ldots \\ + y_1 - y_0 \\ + y_0 \end{array} = y_n(x)$$

$$\Rightarrow y_n(x) = \sum_{r=0}^{n} v_r(x)$$

We have to try to find M_r's such that $|v_r(x)| \le M_r$

Look at $v_1(x)$: $\quad u_1(x) = y_1(x) - y_0$

$$= \int_{x_0}^{x} F(x, y_0)\,dx$$

$$|u_1(x)| = \left| \int^{x} F(x, y_0)\,dx \right|$$

$$|u_1(x)| = \left| \int_{x_0}^{x} F(x, y_0)\, dx \right|$$

Triangle inequality:

$$|a+b| \leq |a| + |b| \quad , \quad \forall\, a, b \in \mathbb{R}.$$

Extends to sums:

$$|x_1 + \cdots + x_n| \leq |x_1| + \cdots + |x_n| \quad \forall\, x_i \in \mathbb{R}$$

Extends to Riemann sums, and therefore to integrals:

$$\left| \int_\alpha^\beta f(x)\, dx \right| \leq \int_\alpha^\beta |f(x)|\, dx$$

for all $f : [\alpha, \beta] \to \mathbb{R}$ continuous.

TRIANGLE INEQUALITY (Δ) FOR INTEGRALS

Back to $u_1(x)$:

$$|u_1(x)| = | \quad \cdots \quad |$$

$$\leq \int_{x_0}^{x} |F(x, y_0)|\, dx \quad .$$

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$$y'(x) = F(x, y(x)), \quad y(x_0) = y_0. \quad (1)$$

Picard iteration:

$$y_{n+1}(x) = y_0 + \int_{x_0}^{x} F(x, y_n(x))\,dx$$

$$y_0(x) = y_0$$

Trick:

$$u_0(x) = y_0$$

$$u_n(x) = y_n(x) - y_{n-1}(x)$$

$$\boxed{\sum_{r=0}^{n} u_r(x) = y_n(x)}$$

Today: We apply the Weierstrass M-test to the $u_r(x)$'s.

Key result: Δ inequality:

$$\left| \int_\alpha^\beta g(x)\,dx \right| \leq \int_\alpha^\beta |g(x)|\,dx$$

true for all $g : [\alpha, \beta] \to \mathbb{R}$ continuous.

$$u_1(x) = y_1(x) - y_0(x)$$

$$= \int_{x_0}^{x} F(x, y_0)\, dx$$

$$|u_1(x)| \overset{\Delta}{\leq} \int_{x_0}^{x} |F(x, y_0)|\, dx$$

$$\leq M(x - x_0) \quad \ldots \text{ F is continuous, hence bounded.}$$

where we take $x > x_0$ WLOG.

Hence,

$$|u_1(x)| \leq M|b-a|.$$

$$u_2(x) = y_2(x) - y_1(x)$$

$$= \int_{x_0}^{x} F(x, y_1(x))\, dx - \int_{x_0}^{x} F(x, y_0(x))\, dx$$

$$= \int_{x_0}^{x} [F(x, y_1(x)) - F(x, y_0(x))]\, dx$$

$$|u_2(x)| \overset{\Delta}{\leq} \int_{x_0}^{x} |F(x, y_1(x)) - F(x, y_0(x))|\, dx$$

$$\overset{\text{Lipschitz}}{\leq} \int_{x_0}^{x} K\,|\overbrace{y_1(x) - y_0(x)}^{u_1(x)}|\, dx$$

$$\leq \int_{x_0}^{x} K M (x - x_0)\, dx$$

$$= \tfrac{1}{2} K M (x - x_0)^2.$$

Guess the pattern:

$$|u_n(x)| \leq \frac{1}{n!} M K^{n-1} (x-x_0)^n$$

$$\leq \frac{1}{n!} M K^{n-1} |b-a|^n = M_n$$

$$\sum_{n=0}^{\infty} M_n = \sum_{n=0}^{\infty} \frac{1}{n!} M K^{n-1} |b-a|^n$$

$$= \frac{M}{K} \sum_{n=0}^{\infty} \frac{1}{n!} K^n |b-a|^n$$

$$= \frac{M}{K} e^{K|b-a|} < \infty.$$

Hence, Weierstrass M-test applies:

$$y_n(x) = \sum_{r=0}^{n} u_r(x)$$

converges uniformly to a continuous limit function $y(x)$, on $[a,b]$.

STEP 1 OF PROOF COMPLETE.

IV

let

$$y(x) = \lim_{n\to\infty} y_n(x).$$

To show: $y(x)$ satisfies

$$y(x) = y_0 + \int_{x_0}^{x} F(x, y(x))\,dx$$

Proof: Look at

$$\left| y(x) - y_0 - \int_{x_0}^{x} F(x, y(x))\,dx \right|$$

$$= \left| y(x) - y_0 - \int_{x_0}^{x} F(x, y(x))\,dx - 0 \right|$$

$$= \left| y(x) - \cancel{y_0} - \int_{x_0}^{x} F(x, y(x))\,dx - \left(y_{n+1}(x) - \cancel{y_0} - \int_{x_0}^{x} F(x, y_n(x))\,dx \right) \right|$$

$$= \left| y(x) - y_{n+1}(x) - \int_{x_0}^{x} [F(x, y(x)) - F(x, y_n(x))]\,dx \right|$$

$$\overset{\Delta}{\leq} \left| y(x) - y_{n+1}(x) \right| + \int_{x_0}^{x} \left| F(x, y(x)) - F(x, y_n(x)) \right| dx$$

Since $y_{n+1}(x)$ and $y_n(x)$ converge to $y(x)$, given $\epsilon > 0$, there exists $N > 0$ such that

$$\left. \begin{aligned} |y_{n+1}(x) - y(x)| &< \epsilon \\ |y_n(x) - y(x)| &< \epsilon \end{aligned} \right\} \text{whenever } n > N.$$

Lipschitz: $\underline{V}$

$$\left| y(x) - y_0 - \int_{x_0}^{x} F(x, y(x))\,dx \right|$$

$$\leq |y(x) - y_{n+1}(x)| + \int_{x_0}^{x} |F(x,y(x)) - F(x, y_n(x))|\,dx$$

Lipschitz

$$\leq \underbrace{|y(x) - y_{n+1}(x)|}_{<\epsilon} + \int_{x_0}^{x} k\,\underbrace{|y(x) - y_n(x)|}_{<\epsilon}\,dx$$

Hence:

$$\left| y(x) - y_0 - \int_{x_0}^{x} (\dots) \right|$$

$$\leq \epsilon + \int_{x_0}^{x} k \cdot \epsilon\,dx \qquad \text{for } n > N.$$

$$\leq \epsilon + \epsilon k (x - x_0) \qquad (x > x_0)$$

$$\leq \epsilon + \epsilon k |b-a|$$

$$= \epsilon(1 + k|b-a|).$$

True for all $\epsilon > 0$. Hence:

$$\underbrace{\left| y(x) - y_0 - \int_{x_0}^{x} F(x, y(x))\,dx \right| = 0.}$$

Limit of Picard iteration solves the IVP.

STEP 2 COMPLETE.

Uniqueness :

$y(x)$ = solution of IVP, via Picard iteration.

Let $Y(x)$ be a second solution of the IVP.

$$y(x) = y_0 + \int_{x_0}^{x} f(x, y(x))\,dx$$

$$Y(x) = y_0 + \int_{x_0}^{x} f(x, Y(x))\,dx$$

$$Y(x) - y(x) = \int_{x_0}^{x} [F(x, Y(x)) - F(x, y(x))]\,dx$$

$$|Y(x) - y(x)| = | \;\dots\; |$$

$$\overset{\Delta}{\leq} \int_{x_0}^{x} |f(x, Y(x)) - f(x, y(x))|\,dx$$

$$\overset{\text{Lipschitz}}{\leq} \int_{x_0}^{x} K\,|\underbrace{Y(x) - y(x)}_{\text{bounded on } [a,b], \text{ by continuity}}|\,dx \quad (*)$$

$$\overset{\text{Continuity}}{\leq} \int_{x_0}^{x} K \cdot C\,dx$$

$$= K \cdot C\,|x - x_0| \qquad (x > x_0)$$

VII

$$|Y(x) - y(x)| \leq \int_{x_0}^{x} K \underbrace{|Y(x) - y(x)|}_{\leq K \cdot C (x - x_0)} dx$$

$$\leq \int_{x_0}^{x} K^2 C (x - x_0) dx$$

$$= \frac{1}{2} K^2 C (x - x_0)^2 .$$

Iterate :

$$|Y(x) - y(x)| \leq \frac{1}{n!} K^n C [x - x_0|^n$$

$$\leq \frac{1}{n!} \underline{K^n C |b - a|^n}$$

Take $n \to \infty$, RHS $\to 0$, hence :

$$|Y(x) - y(x)| \leq 0$$

$$\Rightarrow \quad Y(x) - y(x) = 0$$

$$\Rightarrow \quad Y(x) = y(x) .$$

<u>Uniqueness</u>

STEP 3 IS COMPLETE .

Continuous dependence on initial conditions

(§1.5)

Theorem 1.3 Let $y^{(\delta)}(x)$ be the solution of the IVP with initial condition

$$y^{(\delta)}(x_0) = y_0 + \delta .$$

Then

$$\left| y^{(\delta)}(x) - y(x) \right| \le |\delta| e^{K|x-x_0|}$$

Proof – via Picard iteration.

$$\left| y_0^{(\delta)}(x) - y_0(x) \right| = \left| y_0+\delta - y_0 \right|$$

$$= |\delta| .$$

$$\left| y_1^{(\delta)}(x) - y_1(x) \right| = \left| \cancel{y_0} + \delta + \int_{x_0}^{x} F(x, y_0+\delta)\,dx - \cancel{y_0} - \int_{x_0}^{x} F(x, y_0)\,dx \right|$$

$$\Rightarrow \left| y_1^{(\delta)}(x) - y_1(x) \right| = \left| \delta + \int_{x_0}^{x} \left[F(x, y_0+\delta) - F(x, y_0) \right] dx \right|$$

$$\overset{\Delta}{\le} |\delta| + \int_{x_0}^{x} \left| F(x, y_0+\delta) - F(x, y_0) \right| dx$$

$$\overset{\text{Lipschitz}}{\le} |\delta| + \int_{x_0}^{x} K|\delta| \, dx$$

$$= |\delta| \left(1 + K|x-x_0| \right) .$$

By a process similar to step #2 (uniqueness), we find:

$$\left|y_N^{(\delta)}(x) - y_N(x)\right| \le |\delta| \sum_{n=0}^{N} \frac{1}{n!}\left(K|x-x_0|\right)^n$$

$\forall x$

Take $N \to \infty$.

$$\left|y^{(\delta)}(x) - y(x)\right| \le |\delta| \, e^{K|b-a|}$$

Continuous dependence on initial conditions is shown. ∎

<u>Where we are at</u>:

- Solution of IVP exists and is continuous ☑
- Solution of IVP is unique ☑
- Continuous dependence on ICs ☑

Hence, well-posedness is shown ∎

These three proofs make up <u>item 1</u> on the list of examinable proofs.

§ 1.6 Picard's Theorem (Local version)

<u>Motivation</u>: ⟶

$$\frac{dy}{dx} = -y^2 \quad , \quad y(1) = 1, \quad \underbrace{x \in [-1,3]} .$$

$$\int \frac{dy}{y^2} = \int -dx$$

$$\boxed{y(x) = \frac{1}{x}} \leftarrow$$

Solution blows up at $x = 0$.

Solution is only valid on $x \in [0,3]$

Does this contradict the Lipschitz property?

$$\text{RHS} = F(y) = -y^2 .$$

$$|F(x, y_1) - F(x, y_2)| = |y_1^2 - y_2^2|$$

$$= \underbrace{|y_1 + y_2|}_{\text{☒}} \; \underbrace{|y_1 - y_2|}_{\text{☑}}$$

Not a constant

If we can keep the solution inside a domain D, then

$$K = \sup_{y_1, y_2 \in D} |y_1 + y_2|$$

plays the role of a "local" Lipschitz constant.

Local version of Picard's Theorem:

Let $F(x,y)$ be continuous for all $x \in [x_0-\alpha, x_0+\alpha]$ and $y \in [y_0-\beta, y_0+\beta]$, and let $F(x,y)$ satisfy a Lipschitz condition for all $y \in [y_0-\beta, y_0+\beta]$, with Lipschitz constant K, say. Further, let

$$M = \sup |F(x,y)|$$

over the domain. Then the IVP

$$y'(x) = F(x, y(x)) \quad \text{with} \quad y(x_0) = y_0$$

has a unique solution defined in the interval $|x-x_0| \leq \min(\alpha, \beta/M)$.

Proof — similar to before.

<u>Application</u> to $y'(x) = -y^2$, $x_0 = 1$,

$x = [-1, 3]$ $(\alpha = 2)$, $y_0 = y(x_0) = 1$.

Hence, $y \in [1-\beta, 1+\beta]$.

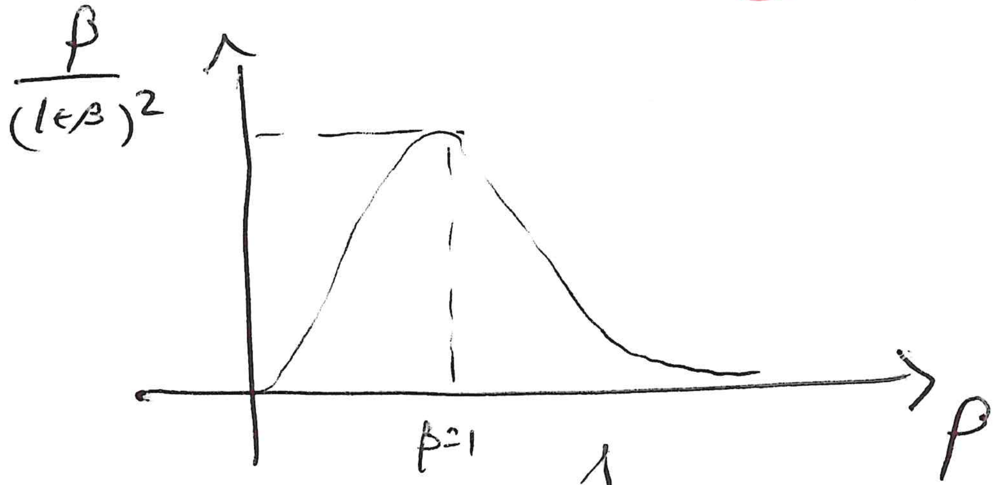
Max $y = (1+\beta)^2$ on interval of interest.

Theorem says:

$= 1/4\ (\max)$

$$|x-x_0| \le \min\left(2, \frac{\beta}{(1+\beta)^2}\right)$$

$\max(y^2)$



$x_0 = 1$

Hence, $|x-x_0| \le 1/4$.

$|x-1| \le 1/4$.

$$\Rightarrow \boxed{x \le 5/4, \quad x \ge 3/4}$$ <u>Picard</u>.

We know:

$$y(x) = 1/x, \quad 0 < x \le \infty.$$

<u>Direct calculation</u>

Sometimes we can extend the region of validity of the solution (beyond what Picard allows) but on a case-by-case basis.