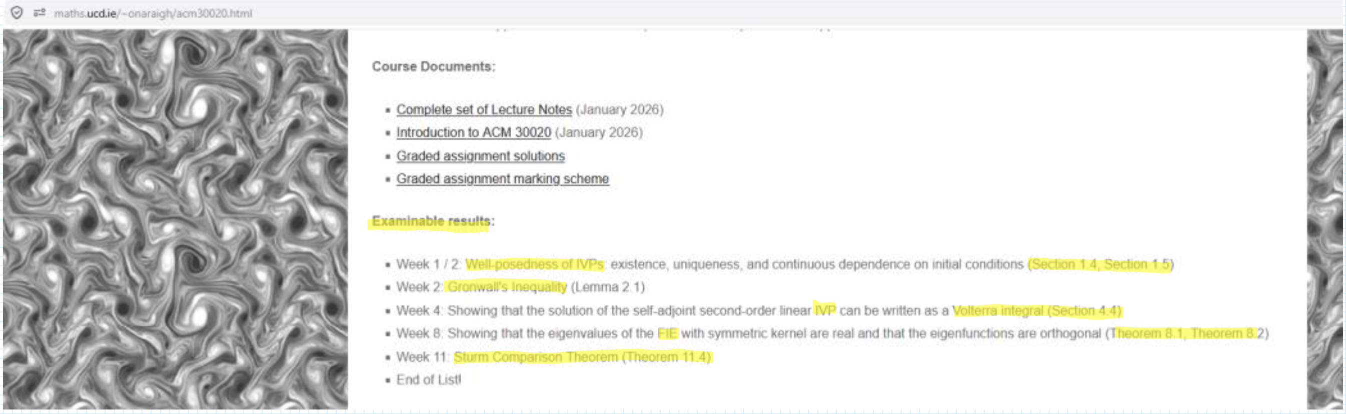
Week 12, Lecture 1

20 April 2026 12:40

§11.6 Sturm Comparison Theorem

$$\left.\begin{aligned} &L[u](x) = -\lambda r(x)u(x), \qquad x\in(a,b)\\ &\text{BCs:}\\ &\qquad \alpha_a u(a)+\beta_a u'(a)=0\\ &\qquad \alpha_b u(b)+\beta_b u'(b)=0\\ &\text{where:}\\ &L[u](x)=\frac{d}{dx}\left[p(x)\frac{du}{dx}\right]+q(x)u(x)\end{aligned}\right\}\quad (1)$$

and where $p(x)>0$, $r(x)>0$.



maths.ucd.ie/~onaraigh/acm30020.html

Course Documents:

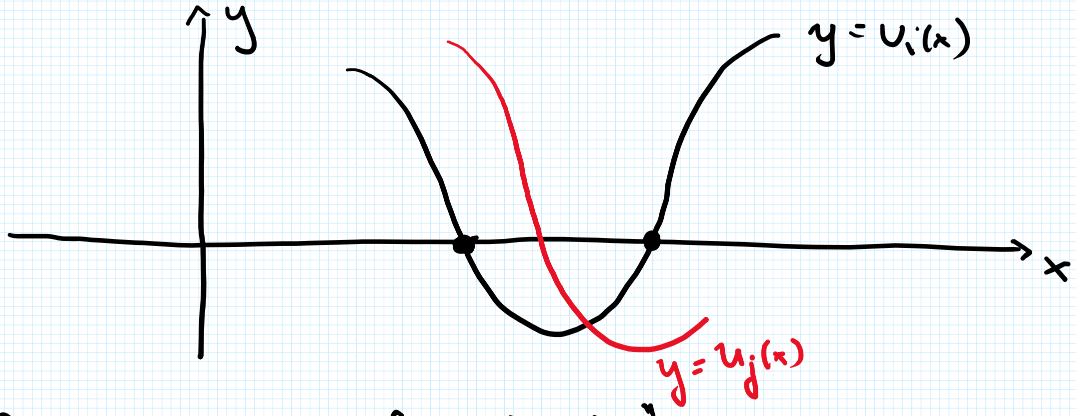
- Complete set of Lecture Notes (January 2026)
- Introduction to ACM 30020 (January 2026)
- Graded assignment solutions
- Graded assignment marking scheme

Examinable results:

- Week 1 / 2: Well-posedness of IVPs: existence, uniqueness, and continuous dependence on initial conditions (Section 1.4, Section 1.5)
- Week 2: Gronwall's Inequality (Lemma 2.1)
- Week 4: Showing that the solution of the self-adjoint second-order linear IVP can be written as a Volterra integral (Section 4.4)
- Week 8: Showing that the eigenvalues of the FIE with symmetric kernel are real and that the eigenfunctions are orthogonal (Theorem 8.1, Theorem 8.2)
- Week 11: Sturm Comparison Theorem (Theorem 11.4)
- End of List!

Theorem 11.4 Let u_i and u_j be two eigenfunctions of (1) with eigenvalues λ_i and λ_j, where $\lambda_j > \lambda_i$. Then, between any two zeros of u_i there exists a zero of u_j.

<u>Sketch</u>:



$y = u_j(x)$

" u_j oscillates faster than u_i, for $\lambda_j > \lambda_i$ " .

<u>Proof</u>: For simplicity and without any loss of generality, we re-label:

- eigenvalues λ_1 and λ_2, $\lambda_2 > \lambda_1$
- eigenfunctions u_1 and u_2

Eigenvalue problems:

$$L[u_1](x) = -\lambda_1 r(x) u_1(x) \qquad \times\ u_2$$

$$L[u_2](x) = -\lambda_2 r(x) u_2(x) \qquad \times\ u_1$$

Subtract:

$$u_2\, L[u_1](x) - u_1\, L[u_2](x) = -\lambda_1 r(x) u_1(x) u_2(x) + \lambda_2 r(x) u_1(x) u_2(x)$$

Multiply across by -1:

$$u_1(x) L[u_2](x) - u_2(x) L[u_1](x) = -(\lambda_2 - \lambda_1)\, r(x) u_1(x) u_2(x)$$

Integrate from $\underline{a}$ to a general point $x \in (a,b)$:

$$\int_a^x \left\{ \begin{array}{l} u_1(x) \dfrac{d}{dx}\left[p(x)\dfrac{du_2}{dx}\right] + \cancel{u_1(x)\, q(x)\, u_2(x)} \\ - u_2(x) \dfrac{d}{dx}\left[p(x)\dfrac{du_1}{dx}\right] - \cancel{u_2(x)\, q(x)\, u_1(x)} \end{array} \right\} dx$$

$$= -(\lambda_2 - \lambda_1) \int_a^x r(x) u_1(x) u_2(x)\, dx$$

$$\to \int_a^x \left\{ u_1(x) \frac{d}{dx}\left[p(x)\frac{du_2}{dx}\right] - u_2(x) \frac{d}{dx}\left[p(x)\frac{du_1}{dx}\right] \right\} dx$$

$$\Rightarrow \int_a^x \left\{ u_1(x)\frac{d}{dx}\left[p(x)\frac{du_2}{dx}\right] - u_2(x)\frac{d}{dx}\left[p(x)\frac{du_1}{dx}\right]\right\}dx$$

$$= -(\lambda_2-\lambda_1)\int_a^x r(x)u_1(x)u_2(x)\,dx$$

IBP:

$$\int_a^x \left\{ \begin{aligned} &\underline{\frac{d}{dx}\left[u_1 \overbrace{p\frac{du_2}{dx}}\right]} - \cancel{\frac{du_1}{dx}\, p\,\frac{du_2}{dx}} \\ &- \underline{\frac{d}{dx}\left[u_2 \overbrace{p\frac{du_1}{dx}}\right]} + \cancel{\frac{du_2}{dx}\, p\,\frac{du_1}{dx}} \end{aligned} \right\} dx$$

$$= -(\lambda_2-\lambda_1)\int_a^x r(x)u_1(x)u_2(x)\,dx$$

$$\Rightarrow \left[p u_1 u_2' - p u_2 u_1'\right]_a^x = -(\lambda_2-\lambda_1)\int_a^x r(x)u_1(x)u_2(x)\,dx$$

Look at $x = a$:

$$u_1(a)u_2'(a) - u_2(a)u_1'(a)$$

$$= u_1(a)\left[-\frac{\alpha_a}{\beta_a}u_2(a)\right] - u_2(a)\left[-\frac{\alpha_a}{\beta_a}u_1(a)\right]$$

$$= 0.$$

Robin BC:

$$\alpha_a u_1(a) + \beta_a u_1'(a) = 0$$

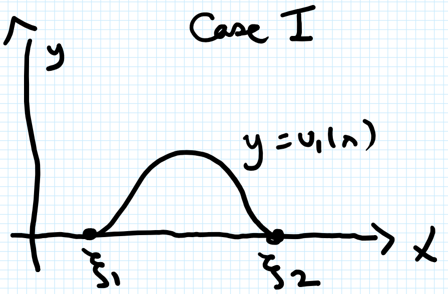
$$\Rightarrow -\frac{\alpha_a}{\beta_a}u_1(a) = u_1'(a)$$

Same for $u_2(a)$.

Result becomes:

$$(p\,\cancel{u_1}u_2' - p u_2 u_1')(x) = -(\lambda_2-\lambda_1)\int_a^x r(x)u_1(x)u_2(x)\,dx. \qquad \underline{(2)}$$

Let ξ_1 and ξ_2 be two successive roots of u_1:



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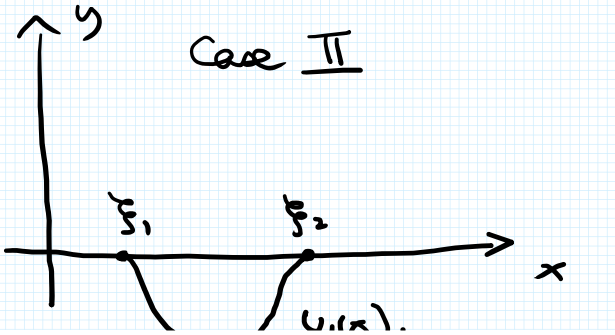
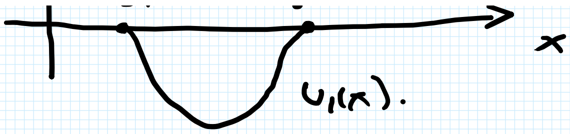




We can assume Case I.

Otherwise, if in Case II, take $u_1 \to -u_1$. This is still a λ_1-eigenfunction, but it takes us back to Case I.

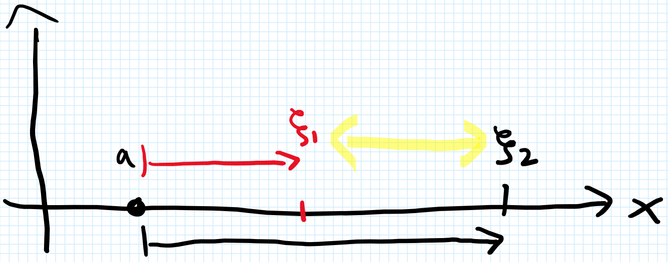
$$u_1(\xi_1) = 0 \quad , \quad u_1'(\xi_1) \geq 0$$

$$u_1(\xi_2) = 0 \quad , \quad u_1'(\xi_2) \leq 0 .$$

Evaluate the result (2) at ξ_1 and ξ_2 :

$$-p(\xi_1)u_2(\cancel{\xi_1})u_1'(\xi_1) = -(\lambda_2-\lambda_1)\int_a^{\xi_1} \underline{r(x)u_1(x)u_2(x)}\,dx \qquad (3i)$$

$$-p(\xi_2)u_2(\cancel{\xi_2})u_1'(\xi_2) = -(\lambda_2-\lambda_1)\int_a^{\xi_2} \underline{r(x)u_1(x)u_2(x)}\,dx \qquad (3ii)$$



$$\int_a^{\xi_2}(\ldots)\,dx - \int_a^{\xi_1}(\ldots)\,dx = \int_{\xi_1}^{\xi_2}(\ldots)\,dx$$

Subtract: (3ii) $-$(3i) :

$$-\overbrace{p(\xi_2)}^{\text{pos}}\overbrace{u_2(\xi_2)}^{\text{pos or zero}}\overbrace{u_1'(\xi_2)}^{\text{neg or zero}} + \underbrace{p(\xi_1)}_{\text{pos}}\underbrace{u_2(\xi_1)}_{\text{pos or zero}}\underbrace{u_1'(\xi_1)}_{\text{pos or zero}} = -(\lambda_2-\lambda_1)\int_{\xi_1}^{\xi_2} r(x)u_1(x)u_2(x)\,dx \qquad (4)$$

Assume for contradiction that u_2 has no zeros in (ξ_1, ξ_2).

W.l.o.g. we can assume for contradiction that:

W.l.o.g. we can assume for contradiction that:

$$u_2(x) > 0 \quad \forall \quad x \in (\xi_1, \xi_2).$$

We look at the signs of the different terms in Eq. (4)

$$\text{LHS} \geq 0$$

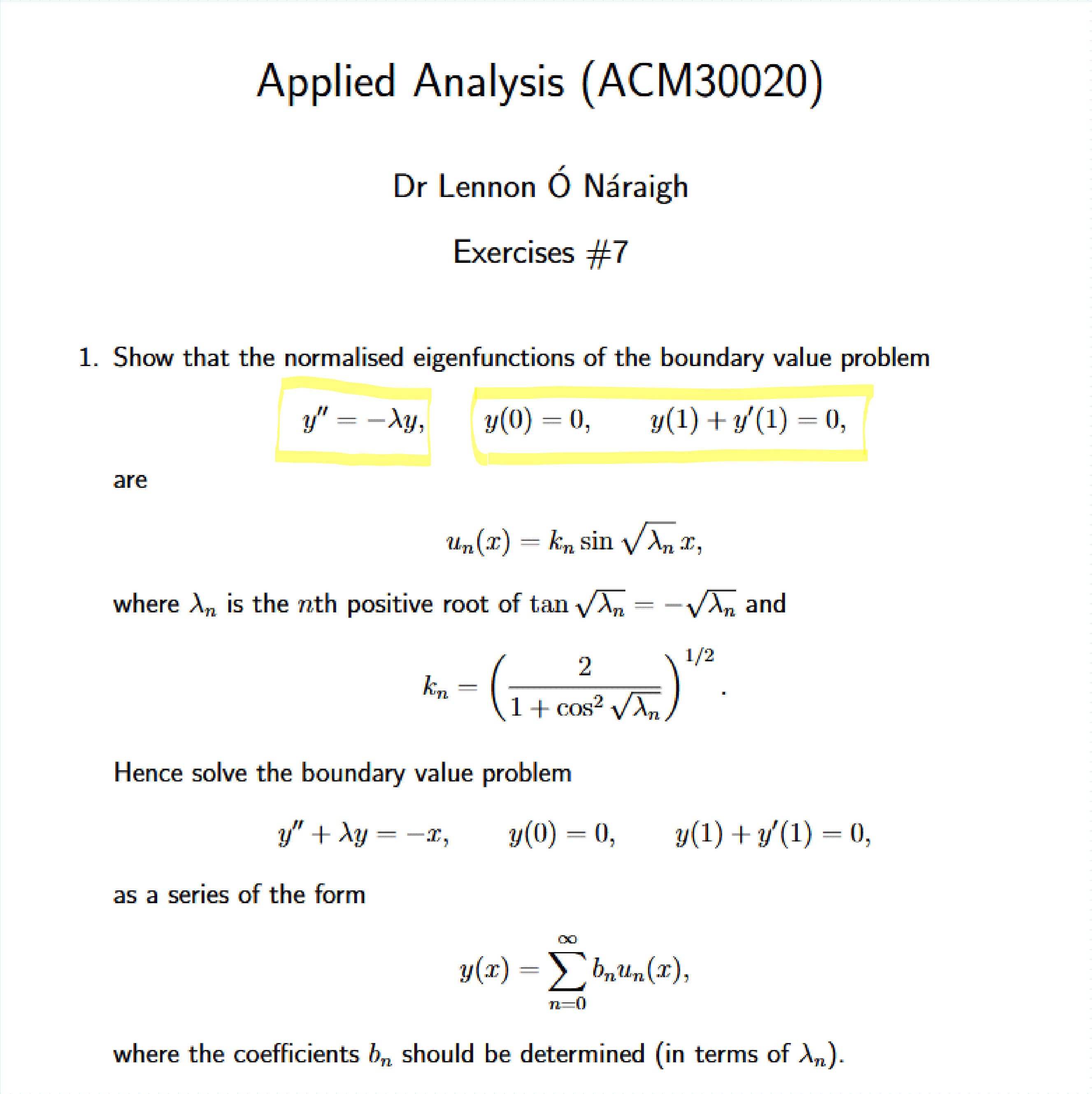
$$\text{RHS}: \quad \underbrace{- \overbrace{(\lambda_2 - \lambda_1)}^{\text{pos}} \int_{\xi_1}^{\xi_2} \overbrace{r(x)}^{\text{pos}} \overbrace{u_1(x)}^{\text{pos}} \overbrace{u_2(x)}^{\text{pos}} \, dx}_{\text{strictly neg.}} < 0$$

Contradiction. Conclude u_2 can't be positive for all $x \in (\xi_1, \xi_2)$. Hence, there must exist some $\xi \in (\xi_1, \xi_2)$ such that:

$$u_2(\xi) = 0 \qquad \blacksquare$$

Plan :

- Wednesday : Look at some weekly exercises (revision)
- Friday : { Applications of this module in Fluid Mechanics
  - Structure of Final Exam



Applied Analysis (ACM30020)

Dr Lennon Ó Náraigh

Exercises #7

1. Show that the normalised eigenfunctions of the boundary value problem

$$y'' = -\lambda y, \qquad y(0) = 0, \qquad y(1) + y'(1) = 0,$$

are

$$u_n(x) = k_n \sin\sqrt{\lambda_n}\, x,$$

where λ_n is the nth positive root of $\tan\sqrt{\lambda_n} = -\sqrt{\lambda_n}$ and

$$k_n = \left(\frac{2}{1+\cos^2\sqrt{\lambda_n}}\right)^{1/2}.$$

Hence solve the boundary value problem

$$y'' + \lambda y = -x, \qquad y(0) = 0, \qquad y(1) + y'(1) = 0,$$

as a series of the form

$$y(x) = \sum_{n=0}^{\infty} b_n u_n(x),$$

where the coefficients b_n should be determined (in terms of λ_n).

$$y'' = -\lambda y$$

$$\Rightarrow \quad y'' + \lambda y = 0.$$

Solution: $y = \sin(\sqrt{\lambda}x), \quad \cos(\sqrt{\lambda}x).$

Requires: $\lambda > 0$. Rule out $\lambda \leq 0$.

LHBC: $y(0) = 0$. Rules out the cosine solution.

$$\Rightarrow \quad \boxed{y \propto \sin(\sqrt{\lambda}x)} \longleftarrow$$

Rule out other options as they lead to trivial

$$\Rightarrow \boxed{y \propto \sin(\sqrt{\lambda}\, x)}$$

Rule out other options as they lead to trivial sl^{n}s.

RHBC: $y(1) + y'(1) = 0$

$$\Rightarrow \sin\sqrt{\lambda} + \sqrt{\lambda}\cos\sqrt{\lambda} = 0.$$

Divide across by $\cos\sqrt{\lambda}$, re-arrange:

$$\boxed{\tan\sqrt{\lambda} = -\sqrt{\lambda}}$$

Sl^{n}s are λ_n, infinitely many.

Solution:

$$y(x) = u_n(x) \propto \sin(\sqrt{\lambda_n}\, x)$$

$$\Rightarrow y(x) = u_n(x) = k_n \sin(\sqrt{\lambda_n}\, x)$$

From the eq$^{\underline{n}}$ in the box, λ_n solves

$$\tan\sqrt{\lambda_n} = -\sqrt{\lambda_n}.$$

Rule out $\lambda \leq 0$.

- $\lambda = 0 \Rightarrow y'' = 0 \Rightarrow y = Ax + B.$

  $y(0) = 0 \Rightarrow B = 0 \Rightarrow y = Ax.$

  $y(1) + y'(1) = 0 \Rightarrow 2A = 0 \Rightarrow A = 0.$

  Trivial sl$^{\underline{n}}$.

- $\lambda < 0 \Rightarrow y = \cosh(\sqrt{-\lambda}\, x),\ \sinh(\sqrt{-\lambda}\, x)$

  LHBC $\Rightarrow y \propto \sinh\sqrt{-\lambda}\, x$.

  RHBC $\Rightarrow \sinh\sqrt{-\lambda} = -\sqrt{-\lambda}\cosh\sqrt{-\lambda}$

  $$\Rightarrow \tanh\sqrt{-\lambda} = -\sqrt{-\lambda}.$$

  This eq$^{\underline{n}}$ has only one root: $\lambda = 0$.

  Trivial sl$^{\underline{n}}$

This eq$^{\underline{n}}$ has only one root: $\lambda = 0$.

Trivial sl$^{\underline{n}}$.

Hence, $\lambda \leq 0$ is ruled out.

Normalization (k_n):

$$\int_0^1 |u_n|^2 dx = 1 .$$

$$\int_0^1 |u_n|^2 dx = k_n^2 \int_0^1 \sin^2(\sqrt{\lambda_n}\, x)\, dx$$

Trig ID: $\sin^2\theta = \frac{1}{2}(1 - \cos 2\theta)$

$$\Rightarrow \int_0^1 |u_n|^2 dx = \frac{k_n^2}{2} \int_0^1 [1 - \cos(2\sqrt{\lambda_n}\, x)]\, dx$$

$$\downarrow$$

$$= \frac{k_n^2}{2}\left[1 - \frac{1}{2\sqrt{\lambda_n}} \sin(2\sqrt{\lambda_n}\, x) \Big|_0^1 \right]$$

$$= \frac{1}{2}k_n^2\left[1 - \frac{1}{2\sqrt{\lambda_n}} \sin(2\sqrt{\lambda_n})\right]$$

$\downarrow$ Trig ID

$$= \frac{1}{2}k_n^2\left[1 - \frac{1}{\cancel{2}\sqrt{\lambda_n}}\, \cancel{2} \sin\sqrt{\lambda_n} \cos\sqrt{\lambda_n} \right]$$

Use: $\sin\sqrt{\lambda_n} = -\sqrt{\lambda_n}\cos\sqrt{\lambda_n}$.

$$\therefore \int_0^1 |u_n|^2 dx = \frac{1}{2}k_n^2\left[1 \overset{+}{-} \frac{1}{\cancel{\sqrt{\lambda_n}}}\left(-\cancel{\sqrt{\lambda_n}} \cos\sqrt{\lambda_n}\right) \cos\sqrt{\lambda_n}\right]$$

$$= \frac{1}{2}k_n^2\left(1 + \cos^2\sqrt{\lambda_n}\right)$$

$$= 1 \qquad \text{(normalization)}$$

$$\left(\quad 2 \quad \right)^{1/2}$$

$$\therefore \quad k_n = \left(\frac{2}{1+\cos^2\sqrt{\lambda_n}} \right)^{1/2}$$

Summarize :

$$\boxed{\begin{aligned} y(x) &= k_n \sin(\sqrt{\lambda_n}\, x) \\ k_n &= \left(\frac{2}{1+\cos^2\sqrt{\lambda_n}} \right)^{1/2} \\ \sin\sqrt{\lambda_n} &= -\sqrt{\lambda_n}\cos\sqrt{\lambda_n} \end{aligned}} \longleftarrow$$

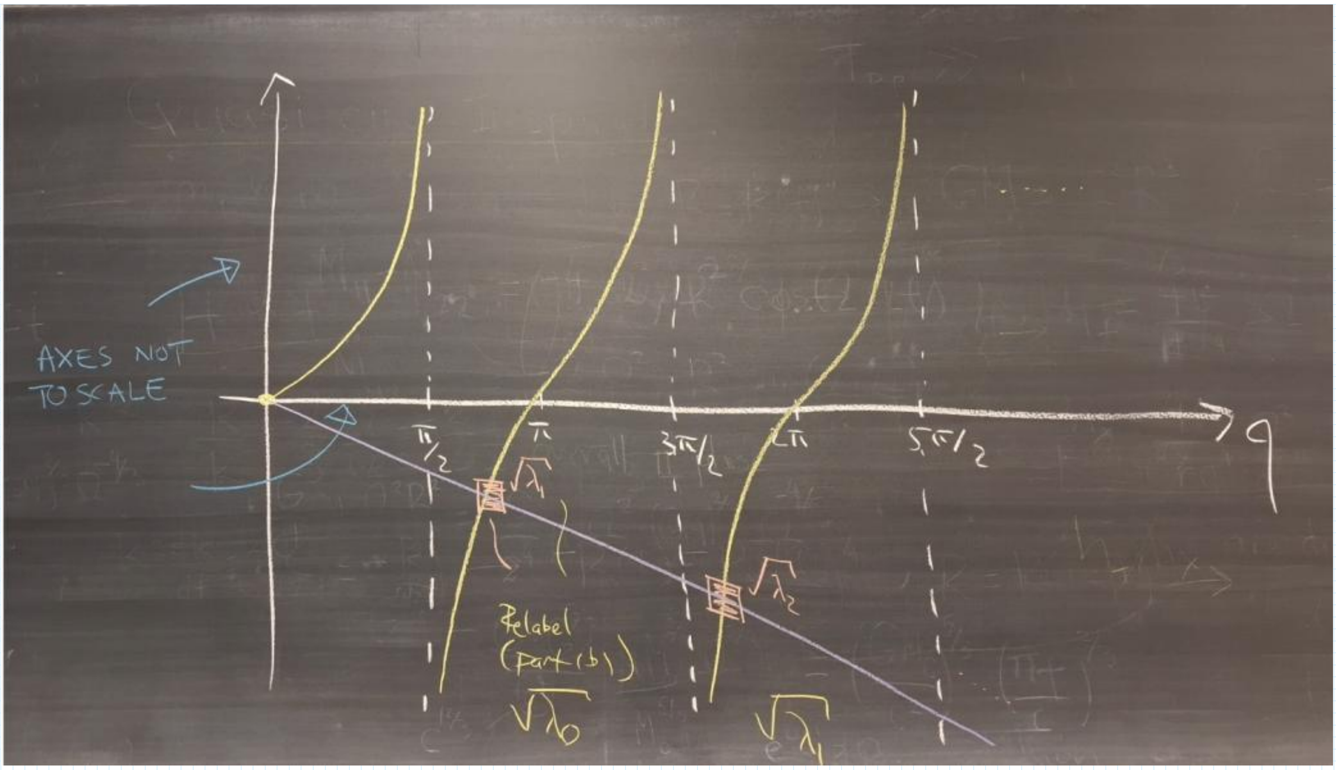
Roots of $\sin\sqrt{\lambda} = -\sqrt{\lambda}\cos\sqrt{\lambda}$.

Let $q = \sqrt{\lambda}$, $q \in \mathbb{R}$. Take $q \geq 0$.

Look at : $\sin q = -q\cos q$

$$\Rightarrow \quad \underbrace{\tan q}_{y_1(q)} = \underbrace{-q}_{y_2(q)} .$$

Remark : The roots for standard eigenvalue problems are tabulated, e.g. for Bessel equations.



Second part:

(1) $\quad y'' + \lambda y = -x \quad , \quad y(0) = 0, \quad y(1) + y'(1) = 0 .$

Attempt a sln (given):

$$y(x) = \sum_{n=0}^{\infty} b_n u_n(x) . \qquad (2)$$

- Determine the b_n's
- Conditions on the free parameter λ.

Sub (2) into (1):

$$\sum_{n=0}^{\infty} \left(b_n u_n'' + \lambda b_n u_n \right) = -x .$$

From the first part: $u_n'' = -\lambda_n u_n$.

$$\therefore \sum_{n=0}^{\infty} \left(b_n (-\lambda_n u_n) + \lambda b_n u_n \right) = -x$$

$$\therefore \sum_{n=0} \left(b_n(-\lambda_n u_n) + \lambda b_n u_n \right) = -x$$

$$\Rightarrow \sum_{n=0}^{\infty} b_n (\lambda - \lambda_n) u_n = -x .$$

Multiply across by $u_m(x)$:

$$\int_0^1 \sum_{n=0}^{\infty} b_n (\lambda - \lambda_n) u_n u_m \, dx = \int_0^1 -x \, u_m(x) \, dx$$

Use: $\displaystyle\int_0^1 u_n u_m \, dx = \delta_{nm}$

Only term that survives on LHS: when $n=m$.

$$\therefore \quad b_m (\lambda - \lambda_m) = \int_0^1 (-x) u_m(x) \, dx .$$

When λ is not an eigenvalue ($\lambda \notin S$) there is a solution for b_m:

$$b_m = \frac{1}{\lambda - \lambda_m} \int_0^1 (-x) u_m(x) \, dx$$

$$\therefore \quad y(x) = \sum_{m=0}^{\infty} \frac{1}{\lambda - \lambda_m} \left[\int_0^1 (-x) u_m(x) dx \right] u_m(x)$$

where:

- $\lambda \notin S = \{ \lambda_0, \lambda_1, \dots \}$.

What happens when $\lambda \in S$? From the Fredholm alternative:

- Either option 1, infinitely many solutions

Fredholm alternative:

- Either option 1, infinitely many solutions
- ~~Fd~~ Option 2, no solutions.

Depends on:

$$-\underbrace{\int_0^1 (x)\, u_m(x)\, dx}_{I} \to 0 \text{, for some eigenvalue.}$$

If $I \neq 0$, then Option 2 holds.

$$I = k_m \int_0^1 \underset{u}{x}\, \underbrace{\sin(\sqrt{\lambda_m}\, x)\, dx}_{dv}$$

LIATES

$u = x$
$du = dx$
$dv = \sin \sqrt{\lambda_m}\, x$
$v = -\frac{1}{\sqrt{\lambda_m}} \cos(\sqrt{\lambda_m} x)$

$$\propto -x \frac{\cos \sqrt{\lambda_m}\, x}{\sqrt{\lambda_m}} \Big|_0^1 + \int_0^1 \frac{1}{\sqrt{\lambda_m}} \cos(\sqrt{\lambda_m} x)\, dx$$

$$\Rightarrow I \propto -\frac{\cos \sqrt{\lambda_m}}{\sqrt{\lambda_m}} + \frac{1}{\sqrt{\lambda_m}} \frac{\sin \sqrt{\lambda_m}}{\sqrt{\lambda_m}}$$

$$= \frac{1}{\sqrt{\lambda_m}} \left[-\cos \sqrt{\lambda_m} + \frac{1}{\sqrt{\lambda_m}} \sin \sqrt{\lambda_m} \right]$$

Use: $\sin \sqrt{\lambda_m} = -\sqrt{\lambda_m} \cos \sqrt{\lambda_m}$

$$\Rightarrow I \propto \frac{1}{\sqrt{\lambda_m}} \left[\frac{\sin \sqrt{\lambda_m}}{\sqrt{\lambda_m}} + \frac{\sin \sqrt{\lambda_m}}{\sqrt{\lambda_m}} \right]$$

$$\Rightarrow I \propto \frac{2}{\lambda_m} \sin\sqrt{\lambda_m} .$$

Can't satisfy:

$$\left\{ \begin{array}{l} \sin\sqrt{\lambda_m} = -\sqrt{\lambda_m}\cos\sqrt{\lambda_m} \\ \sin\sqrt{\lambda_m} = 0 \end{array} \right\}$$

simultaneously. Therefore, $\sin\sqrt{\lambda_m} \neq 0$, hence

$$I \neq 0$$

so option 2 in the fredholm alternative applies:

- $\lambda \notin S = \{\lambda_0, \lambda_1, \dots\}$, BVP has a unique sl^{n}
- $\lambda \in S$, BVP has no sl^{n} .