This week: Sturm–Liouville Theory (Ch. 11)

Ties together:

- BVPs (new: a free parameter λ)
- BVPs can be turned into an integral equation (Ch. 7)
- The free parameter λ now means that our BVP maps directly on to a FIE.
- Gives physical applications of FIEs.

Standard Sturm–Liouville (SL) problem:

$$\frac{d}{dx}\left[p(x)\frac{du}{dx}\right] + q(x)u(x) = -\lambda r(x)u(x) \qquad \underline{(1)}$$

Here, λ is an eigenvalue TBC. For (1) to be a SL problem, we have some conditions:

- $p(x) > 0$
- $r(x) > 0$

Equation (1) is solved as a BVP on an interval $[a,b]$ with BCs:

$$\left.\begin{aligned} \alpha_a u(a) + \beta_a u'(a) &= 0 \\ \alpha_b u(b) + \beta_b u'(b) &= 0 \end{aligned}\right\} \quad (2)$$

Other BCs are OK too, e.g. periodic BCs:

$$u(a) = u(b)$$

$$u(a) = u(b)$$
$$u'(a) = u'(b)$$

Periodic BCs apply provided the coefficient functions p, q, and r are $(b-a)$-periodic.

<u>Another possibility for BCs</u>: We allow for p to vanish at an end-point, e.g. $p(a) = 0$, but $p(x) > 0 \ \forall x \in (a,b]$.

Then we impose a <u>behavioural BC</u> at $x = a$, i.e. u and u' are continuous at $x = a$.

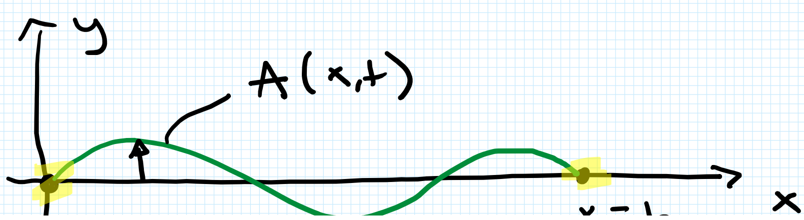
We will mostly look at the Robin BCs (2).

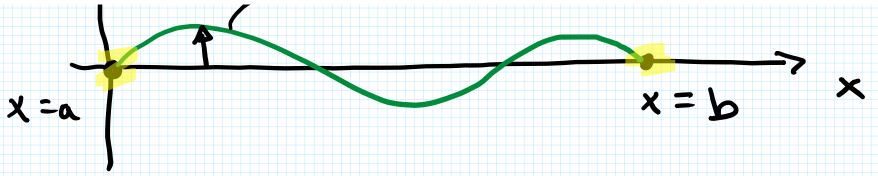
Example – §11.1 <u>Where SL problems arise</u>.

1D wave equation for a vibrating string; string is non-uniform, with mass density and string tension varying along the length of the string:

$$\rho(x) \frac{\partial^2 A}{\partial t^2} = \frac{\partial}{\partial x}\left(T(x) \frac{\partial A}{\partial x} \right)$$

- $\rho(x) > 0$, mass density
- $T(x) > 0$, string tension
- $A(x,t)$ is the string's displacement away from equilibrium.





- Clamped BCs at $x = a$, $x = b$:

$$A = 0 \quad @ \quad x = a, \; x = b.$$

If instead we have a restoring force acting on the string, we have:

$$\underline{(3)} \quad \rho(x) \frac{\partial^2 A}{\partial t^2} = \frac{\partial}{\partial x}\left(T(x) \frac{\partial A}{\partial x} \right) - K(x) A(x,t)$$

"Hooke's law", $k > 0$

Trial solution: $A(x,t) = y(x) \cos(\omega t - \varphi)$

Sub into wave equation (3):

$$-\omega^2 \rho y = \frac{d}{dx}\left[T \frac{dy}{dx} \right] - k y$$

Re-arrange:

$$\frac{d}{dx}\left[T \frac{dy}{dx} \right] - k y = -\underbrace{\omega^2}_{\lambda} \rho \, y \qquad \begin{aligned} y(a) &= 0 \\ y(b) &= 0 \end{aligned}$$

$p(x) > 0$ ✓ $\qquad q(x)$, any sign $\qquad r(x) > 0$ ✓

Hence, the wave eq^n (3) takes the form of a SL problem, with eigenvalue $\lambda = \omega^2$.

It is with no great loss of generality that we look at BVPs in SL form. Reason: every ODE look at BVPs in SL form. Reason: every ODE of the following type:

$$u''(x) + P(x)u'(x) + Q(x)u(x) = 0 \qquad (4)$$

can be brought to SL form

$$\frac{d}{dx}\left(p u'\right) + q u = 0 . \qquad (5)$$

Multiply (4) by an integrating factor I:

$$I u'' + P I u' + Q I u = 0$$

$$\frac{d}{dx}\left(I u'\right) - I' u' + P I u' + Q I u = 0$$

Demand $= 0$.

$$\Rightarrow \frac{d}{dx}(I u') + u' \cancel{\left(P I - I'\right)} + Q I u = 0 .$$

Demand: $\quad I' = P I \Rightarrow I = e^{\int P(x)\,dx}$

Equation becomes:

$$\frac{d}{dx}(I u') + Q I u = 0 .$$

Compare with SL form (5):

- $p = I = e^{\int P(x)\,dx} > 0$ ☑
- $q = Q p$ ☑

Example (§ 11.3.1)

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Bessel's eqⁿ:

$$x^2 y'' + x y' + (x^2 - \nu^2) = 0, \qquad \nu = \text{const.}$$

Divide by x^2:

$$y'' + \overbrace{\frac{1}{x}}^{P = \frac{1}{x}} y' + \overbrace{\left(1 - \frac{\nu^2}{x^2}\right)}^{Q = 1 - \frac{\nu^2}{x^2}} = 0$$

$$I = e^{\int P\,dx} = e^{\int \frac{1}{x} dx} = e^{\ln(x)} = x$$

$$\Rightarrow \quad I = x.$$

$$p(x) = I = x$$

$$q(x) = pQ = x\left(1 - \frac{\nu^2}{x^2}\right) = x - \frac{\nu^2}{x}.$$

SL. form:

$$\frac{d}{dx}\left(x \frac{du}{dx}\right) + \left(x - \frac{\nu^2}{x}\right) u = 0.$$

<u>Remark</u>: Work on an interval where $x > 0$ (or $x < 0$).

Adjoint operator, SL problem (§ 11.4)

<u>SL operator</u>

$$L[u](x) = \frac{d}{dx}\left(p(x) \frac{du}{dx}\right) + q(x) u(x).$$

<u>SL equation</u> (Eq. (1)):

$$L[u](x) = -\lambda\, r(x)\, u(x), \quad x \in (a,b).$$

Boundary conditions:

$$\alpha_a u(a) + \beta_a u'(a) = 0$$
$$\alpha_b u(b) + \beta_b u'(b) = 0$$

$$\int_a^b u(x)\, L[v](x)\, dx = \langle u, L[v] \rangle$$

$$\langle u, L[v] \rangle = \int_a^b u(x) \left[\frac{d}{dx}\left(p \frac{dv}{dx} \right) + q(x) v(x) \right] dx$$

$$= \int_a^b u(x) \frac{d}{dx}\left(p \frac{dv}{dx} \right) dx + \int_a^b u(x) v(x) q(x)\, dx$$

$$= \int_a^b \left[\frac{d}{dx}\left(u\, p \frac{dv}{dx} \right) - \frac{du}{dx} \cdot p \cdot \frac{dv}{dx} \right] dx + \int_a^b u(x) v(x) q(x)\, dx$$

$$= (p u v')_a^b - \int_a^b p \frac{du}{dx} \cdot \frac{dv}{dx}\, dx + \int_a^b u(x) v(x) q(x)\, dx$$

$$= (p u v')_a^b - \left\{ \int_a^b \left[\frac{d}{dx}\left(p \frac{du}{dx} \cdot v \right) - v \frac{d}{dx}\left(p \frac{du}{dx} \right) \right] dx \right\} + \int_a^b u(x) v(x) q(x)\, dx$$

$$= (puv')_a^b - (pvu')_a^b$$

$$+ \int_a^b v \frac{d}{dx}\left[p\frac{du}{dx}\right]dx + \int_a^b u(x)v(x)q(x)\,dx$$

$$= \Big\{ p(x)\left[u(x)v'(x) - v(x)u'(x)\right]\Big\}_a^b$$

$$+ \langle v, L[u] \rangle$$

Summarizing:

$$\langle u, L[v] \rangle = \langle v, L[u] \rangle$$

$$+ \Big\{ p(x)\left[u(x)v'(x) - v(x)u'(x)\right]\Big\}_a^b$$

This result is called <u>Lagrange's Identity</u>.

We now focus exclusively on functions that satisfy the Robin BCs at $x=a$ and $x=b$.

E.g. at $x=a$ we have:

$$\alpha_a u(a) + \beta_a u'(a) = 0$$

$$\alpha_a v(a) + \beta_a v'(a) = 0$$

$$\Rightarrow v'(a) = -\frac{\alpha_a}{\beta_a} v(a) \quad \Big| \quad u'(a) = -\frac{\alpha_a}{\beta_a} u(a)$$

Sub in to

$$p(a)\left[u(a)v'(a) - v(a)u'(a)\right]$$

$$p(a)\left[u(a)v'(a) - v(a)u'(a)\right]$$

$$= p(a)\,u(a)\left(-\frac{\alpha_a}{\beta_a}v(a)\right) - p(a)\,v(a)\left(-\frac{\alpha_a}{\beta_a}u(a)\right)$$

$$= p(a)u(a)v(a)\left(-\frac{\alpha_a}{\beta_a}\right) + p(a)u(a)v(a)\left(\frac{\alpha_a}{\beta_a}\right)$$

$$= 0.$$

The same for $x=b$.

Hence, for any two functions u and v satisfying the Robin BCs at $x=a$ and $x=b$, we have:

$$\langle u, L[v]\rangle = \langle v, L[u]\rangle .$$

Hence, restricted to the functions satisfying the Robin BCs, L is <u>self-adjoint</u>.