This week: Sturm–Liouville Theory (Ch. 11)

Ties together:

- BVPs (new: a free parameter λ)
- BVPs can be turned into an integral equation (Ch. 7)
- The free parameter λ now means that our BVP maps directly on to a FIE.
- Gives physical applications of FIEs.

Standard Sturm–Liouville (SL) problem:

$$\frac{d}{dx}\left[p(x)\frac{du}{dx}\right] + q(x)\,u(x) = -\lambda\, r(x)u(x) \qquad \underline{(1)}$$

Here, λ is an eigenvalue TBC. For (1) to be a SL problem, we have some conditions:

- $p(x) > 0$
- $r(x) > 0$

Equation (1) is solved as a BVP on an interval $[a,b]$ with BCs:

$$\left.\begin{aligned} \alpha_a u(a) + \beta_a u'(a) &= 0 \\ \alpha_b u(b) + \beta_b u'(b) &= 0 \end{aligned}\right\} \quad (2)$$

Other BCs are OK too, e.g. periodic BCs:

$$u(a) = u(b)$$

$$u(a) = u(b)$$
$$u'(a) = u'(b)$$

Periodic BCs apply provided the coefficient functions p, q, and r are $(b-a)$ - periodic.

<u>Another possibility for BCs</u>: We allow for p to vanish at an end-point, e.g. $p(a) = 0$, but $p(x) > 0 \ \forall \ x \in (a,b]$.

Then we impose a <u>behavioural BC</u> at $x = a$, i.e. u and u' are continuous at $x = a$.

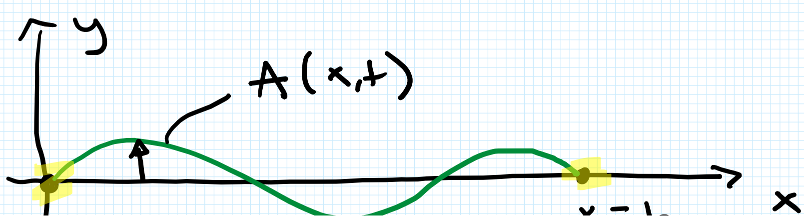
We will mostly look at the Robin BCs (2).

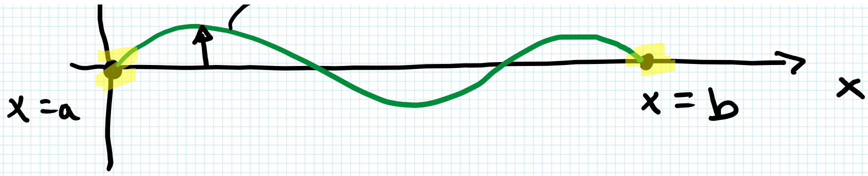
Example – §11.1 <u>Where SL problems arise</u>.

1D wave equation for a vibrating string; string is non-uniform, with mass density and string tension varying along the length of the string:

$$\rho(x) \frac{\partial^2 A}{\partial t^2} = \frac{\partial}{\partial x}\left(T(x) \frac{\partial A}{\partial x} \right)$$

- $\rho(x) > 0$, mass density
- $T(x) > 0$, string tension
- $A(x,t)$ is the string's displacement away from equilibrium.





- Clamped BCs at $x=a$, $x=b$:

$$A = 0 \quad @ \quad x=a,\ x=b.$$

If instead we have a restoring force acting on the string, we have:

$$\underline{(3)}\quad \rho(x)\frac{\partial^2 A}{\partial t^2} = \frac{\partial}{\partial x}\left(T(x)\frac{\partial A}{\partial x}\right) - \underbrace{K(x)A(x,t)}_{\text{"Hooke's Law", } K>0}$$

Trial solution: $A(x,t) = y(x)\cos(\omega t - \varphi)$

Sub into wave equation (3):

$$-\omega^2 \rho y = \frac{d}{dx}\left[T\frac{dy}{dx}\right] - Ky$$

Re-arrange:

$$\frac{d}{dx}\left[\underbrace{T}_{p(x)>0\ \checkmark}\frac{dy}{dx}\right] - \underbrace{K}_{q(x),\text{ any sign}}y = -\underbrace{\omega^2}_{\lambda}\underbrace{\rho}_{r(x)>0\ \checkmark}y$$

$$y(a)=0$$
$$y(b)=0$$

Hence, the wave eq^n (3) takes the form of a SL problem, with eigenvalue $\lambda = \omega^2$.

It is with no great loss of generality that we look at BVPs in SL form. Reason: every ODE look at BVPs in SL form. Reason: every ODE of the following type:

$$u''(x) + P(x)u'(x) + Q(x)u(x) = 0 \qquad (4)$$

can be brought to SL form

$$\frac{d}{dx}\left(p u'\right) + q u = 0 . \qquad (5)$$

Multiply (4) by an integrating factor I:

$$I u'' + P I u' + Q I u = 0$$

$$\frac{d}{dx}\left(I u'\right) - I' u' + P I u' + Q I u = 0$$

Demand $= 0$.

$$\Rightarrow \frac{d}{dx}(I u') + u'\left(\cancel{PI - I'}\right) + Q I u = 0 .$$

Demand: $I' = P I \Rightarrow I = e^{\int P(x)dx}$

Equation becomes:

$$\frac{d}{dx}(I u') + Q I u = 0 .$$

Compare with SL form (5):

- $p = I = e^{\int P(x)dx} > 0$ ☑
- $q = Q p$ ☑

Example (§ 11.3.1)

Example (§ 11.3.1)

Bessel's eqn:

$$x^2 y'' + xy' + (x^2 - \nu^2) = 0, \quad \nu = \text{const}.$$

Divide by x^2:

$$y'' + \overbrace{\frac{1}{x}}^{P=\frac{1}{x}} y' + \overbrace{\left(1 - \frac{\nu^2}{x^2}\right)}^{Q = 1 - \frac{\nu^2}{x^2}} = 0$$

$$I = e^{\int P\,dx} = e^{\int \frac{1}{x}dx} = e^{\ln(x)} = x$$

$$\Rightarrow \quad I = x.$$

$$p(x) = I = x$$

$$q(x) = pQ = x\left(1 - \frac{\nu^2}{x^2}\right) = x - \frac{\nu^2}{x}.$$

SL. form:

$$\frac{d}{dx}\left(x\frac{du}{dx}\right) + \left(x - \frac{\nu^2}{x}\right)u = 0.$$

<u>Remark</u>: Work on an interval where $x > 0$ (or $x < 0$).

Adjoint operator, SL problem (§ 11.4)

<u>SL operator</u>

$$L[u](x) = \frac{d}{dx}\left(p(x)\frac{du}{dx}\right) + q(x)u(x).$$

<u>SL equation</u> (Eq. (1)):

$$L[u](x) = -\lambda\, r(x)\, u(x), \quad x \in (a,b).$$

Boundary conditions :

$$\alpha_a u(a) + \beta_a u'(a) = 0$$

$$\alpha_b u(b) + \beta_b u'(b) = 0$$

$$\int_a^b u(x)\, L[v](x)\, dx = \langle u, L[v] \rangle$$

$$\langle u, L[v] \rangle = \int_a^b u(x) \left[\frac{d}{dx}\left(p \frac{dv}{dx}\right) + q(x) v(x) \right] dx$$

$$= \int_a^b u(x) \frac{d}{dx}\left(p \frac{dv}{dx}\right) dx + \int_a^b u(x) v(x) q(x)\, dx$$

$$= \int_a^b \left[\frac{d}{dx}\left(u\, p \frac{dv}{dx} \right) - \frac{du}{dx} \cdot p \cdot \frac{dv}{dx} \right] dx + \int_a^b u(x) v(x) q(x)\, dx$$

$$= (p u v')_a^b - \int_a^b p \frac{du}{dx} \cdot \frac{dv}{dx}\, dx + \int_a^b u(x) v(x) q(x)\, dx$$

$$= (p u v')_a^b - \left\{ \int_a^b \left[\frac{d}{dx}\left(p \frac{du}{dx} \cdot v \right) - v \frac{d}{dx}\left(p \frac{du}{dx} \right) \right] dx \right\} + \int_a^b u(x) v(x) q(x)\, dx$$

$$= (puv')_a^b - (pvu')_a^b$$

$$+ \int_a^b v \frac{d}{dx}\left[p\frac{du}{dx} \right] dx + \int_a^b u(x)v(x)q(x)\,dx$$

$$= \Big\{ p(x)\left[u(x)v'(x) - v(x)u'(x) \right] \Big\}_a^b$$

$$+ \langle v, L[u] \rangle$$

Summarizing:

$$\langle u, L[v] \rangle = \langle v, L[u] \rangle + \Big\{ p(x)\left[u(x)v'(x) - v(x)u'(x) \right] \Big\}_a^b$$

This result is called <u>Lagrange's Identity</u>.

We now focus exclusively on functions that satisfy the Robin BCs at $x=a$ and $x=b$.

E.g. at $x=a$ we have:

$$\alpha_a u(a) + \beta_a u'(a) = 0$$

$$\alpha_a v(a) + \beta_a v'(a) = 0$$

$$\Rightarrow v'(a) = -\frac{\alpha_a}{\beta_a} v(a) \quad \Big| \quad u'(a) = -\frac{\alpha_a}{\beta_a} u(a)$$

Sub in to

$$p(a)\left[u(a)v'(a) - v(a)u'(a) \right]$$

$$p(a)\left[u(a)v'(a) - v(a)u'(a)\right]$$

$$= p(a)\,u(a)\left(-\frac{\alpha_a}{\beta_a}v(a)\right) - p(a)\,v(a)\left(-\frac{\alpha_a}{\beta_a}u(a)\right)$$

$$= p(a)u(a)v(a)\left(-\frac{\alpha_a}{\beta_a}\right) + p(a)v(a)u(a)\left(\frac{\alpha_a}{\beta_a}\right)$$

$$= 0.$$

The same for $x=b$.

Hence, for any two functions u and v satisfying the Robin BCs at $x=a$ and $x=b$, we have:

$$\langle u, L[v]\rangle = \langle v, L[u]\rangle.$$

Hence, restricted to the functions satisfying the Robin BCs, L is <u>self-adjoint</u>.

Sturm-Liouville Problem (SL)

$$\left.\begin{aligned} &\frac{d}{dx}\left[p(x)\frac{du}{dx}\right] + q(x)u(x) = -\lambda r(x)u(x) \\ &x \in (a,b) \\ &p(x)>0, \quad r(x)>0 \\ \text{BC}: \quad &\alpha_a u(a) + \beta_a u'(a) = 0 \\ &\alpha_b u(b) + \beta_b u'(b) = 0 \end{aligned}\right\} \quad (1)$$

Properties of the SL system (1):

1. Eigenvalues are real

2. Eigenfunctions corresponding to different eigenvalues are orthogonal.

3. When (a,b) is a finite interval, the eigenvalues are <u>discrete</u> (can be labelled by the integers).

Furthermore, the eigenfunctions span the Hilbert space

$$\mathcal{H} = \left\{ u(x) \,\middle|\, \int_a^b r(x)[u(x)]^2 dx < \infty \,; \begin{array}{l} \alpha_a u(a) + \beta_a u'(a) = 0 \\ \alpha_b u(b) + \beta_b u'(b) = 0 \end{array} \right\}$$

4. The eigenfunctions oscillate

Applications in Quantum Mechanics

5. The larger the eigenvalues the faster the oscillation.

6. The moduli of the eigenvalues have a lowest value and increases without limit. Specifically,

$$S = \{ \lambda_1, \lambda_2, \ldots, \lambda_n, \ldots \}$$

$$\min_n |\lambda_i| < \infty$$

$$\lim_{n \to \infty} |\lambda_n| = \infty$$

Today: We prove #1 and #2.

<u>Proof</u>: Write the SL problem (1) as:

$$\boxed{L[u](x) = -\lambda \underbrace{r(x)\, u(x)}_{\text{RHS}}} \qquad (2)$$

where

$$L[u](x) = \frac{d}{dx}\left[p(x)\frac{du}{dx}\right] + q(x)\,u(x)$$

We treat (2) like an inhomogeneous problem:

$$L[u](x) = -\lambda(\text{RHS})$$

Let $G(x,s)$ be the Green's Function for L.

Therefore:

$$u(x) = -\lambda \int_a^b G(x,s) \overbrace{r(s)\, u(s)}^{\text{RHS}}\, ds$$

Multiply both sides by $\sqrt{r(x)}$:

$$\sqrt{r(x)}\, u(x) = \lambda \int^b \left[-\sqrt{r(x)}\, G(x,s) \sqrt{r(s)}\right] \underline{\sqrt{r(s)}\, u(s)}\, ds$$

$$\underbrace{\sqrt{r(x)}\, v(x)}_{y(x)} = \lambda \int_a \underbrace{\left[-\sqrt{r(x)}\, G(x,s)\sqrt{r(s)}\right]}_{K(x,s)} \underbrace{\sqrt{r(s)}\, v(s)}_{y(s)}\, ds$$

$$\therefore \quad y(x) = \lambda \int_a^b K(x,s)\, y(s)\, ds$$

$$K(x,s) = -\underbrace{\sqrt{r(x)}\ \sqrt{r(s)}}_{\text{symmetric in } x \text{ and } s}\ \underbrace{G(x,s)}_{\text{symmetric, since } L \text{ is self-adjoint}}$$

(§ 6.2, from Def. 6.1)

Thus, K is a symmetric kernel, so by Theorem 8.1 and Theorem 8.2 :

i The eigenvalues λ_i are real.

ii The eigenfunctions corresponding to different eigenvalues are orthogonal.

Result (i) here establishes Property # 1 of SL systems.

Result (ii) :

$$\int_a^b y_i(x)\, y_j(x)\, dx = 0, \quad i \neq j$$

$$\Rightarrow \int_a^b \sqrt{r(x)}\, v_i(x) \sqrt{r(x)}\, v_j(x)\, dx = 0, \quad i \neq j$$

$$\int^b$$

$$\Rightarrow \int_a^b r(x)\, U_i(x) U_j(x)\, dx = 0, \quad i \neq j$$

This establishes property #2: the eigenfunctions are orthogonal with respect to the weight function r, $r(x) > 0$.

Next week:

Th$^{\underline{m}}$ 11.3 Suppose $U_1(x)$ and $U_2(x)$ are eigenfunctions of the SL problem on a finite domain and have the same eigenvalue. Then U_1 and U_2 are proportional.

Remark:

- u_1 and u_2 are proportional $\Rightarrow$ eigenspaces are 1D
- True for Robin BCs, not true for periodic BCs.
- Applications in QM. For the 1D Schrödinger Eq$^{\underline{n}}$ eigenspaces are one-dimensional.

Th$^{\underline{m}}$ 11.4 (Sturm Comparison Theorem) Let $u_i(x)$ and $u_j(x)$ be eigenfunctions of the SL system (1). Suppose $\lambda_j > \lambda_i$. Then, between any two zeros of U_i there exists at least one zero of u_j.

Remark:

- The bigger the eigenvalue, the faster the oscillation.

- The bigger the eigenvalue, the faster the oscillation.
- Applications in QM, in the 1D Schrödinger eq$^{\underline{n}}$: the higher the energy level, the faster the wave function oscillates (in x).

<u>Worked Example</u> (§ 11.4.1)

Take Eq$^{\underline{n}}$ (1), $p(x) = 1$, $q(x) = 0$, $r(x) = 1$, $(a,b) = (0, \pi)$. Dirichlet BCs $\begin{pmatrix} \alpha_a = 1, & \beta_a = 0 \\ \alpha_b = 1, & \beta_b = 0 \end{pmatrix}$,

hence $u(0) = 0$, $u(\pi) = 0$.

Eq$^{\underline{n}}$ (1) becomes:

$$\frac{d^2u}{dx^2} = -\lambda u \quad , \quad x \in (0, \pi)$$

$$u = e^{\sqrt{-\lambda}\, x} \quad , \quad u = e^{-\sqrt{-\lambda}\, x}$$

Different options for λ (λ is real).

1. $\lambda < 0 \implies \lambda = -\omega^2$, $\omega \in \mathbb{R}$.

$$u = e^{\omega x} \, , \quad e^{-\omega x}$$

General sl$^{\underline{n}}$: $u = Ae^{\omega x} + Be^{-\omega x}$

<u>BCs</u>:
$$u(0) = A + B = 0$$
$$u(\pi) = Ae^{\omega\pi} + Be^{-\omega\pi} = 0$$

$$\underbrace{\begin{pmatrix} 1 & 1 \\ e^{\omega\pi} & e^{-\omega\pi} \end{pmatrix}}_{M} \begin{pmatrix} A \\ B \end{pmatrix} = 0 .$$

To get a non-trivial sl^n for A and B, we require

$$\det(M) = 0$$

$$\Rightarrow \quad \underbrace{e^{-\omega\pi} - e^{\omega\pi}}_{\propto \sinh(\omega\pi)} = 0$$

$$\therefore \quad \sinh(\omega\pi) = 0 \Rightarrow \omega = 0 \quad \text{Contradiction} ,$$

since $\lambda = -\omega^2 < 0$. Option 1 can't apply.

In other words, only the trivial sl^n $u = 0$ exists under Option 1.

2. $\lambda = 0 \Rightarrow \dfrac{d^2u}{dx^2} = 0 \Rightarrow u = Ax + B$

$$u(0) = B = 0 \Rightarrow \boxed{B = 0}$$

$$u(\pi) = A\pi = 0 \Rightarrow \boxed{A = 0}$$

Hence, under Option 2, only the trivial sl^n exists.

3. $\lambda > 0 \Rightarrow \lambda = \omega^2, \quad \omega \in \mathbb{R}$.

$$u = e^{\sqrt{-\lambda}\, x} , \quad e^{-\sqrt{-\lambda}\, x}$$

$$u = e^{\sqrt{-\lambda}\,x}, \quad e^{-\sqrt{-\lambda}\,x}$$

$$\sqrt{-\lambda} = \sqrt{-\omega^2} = i\omega$$

$$u = e^{i\omega x}, \quad e^{-i\omega x}$$

Take linear combinations to get real-valued functions:

$$u(x) = \cos(\omega x), \quad \sin(\omega x).$$

General s$\underline{\text{ln}}$ is a linear combination:

$$u(x) = \cancel{A\cos(\omega x)} + B\sin(\omega x)$$

Boundary conditions:

$$u(0) = A \cdot 1 + B \cdot 0 = 0 \Rightarrow A = 0$$

$$u(\pi) = B\sin(\omega\pi) = 0, \quad B \neq 0 \left(\begin{matrix}\text{non-trivial} \\ \text{s}\underline{\text{ln}}\end{matrix}\right)$$

For a non-trivial s$\underline{\text{ln}}$ we require

$$\sin(\omega\pi) = 0$$

Therefore, $\omega \in \mathbb{Z} / \{0\}$.

E.g. $\omega = -n \quad (n > 0)$

$$\sin(\omega x) = \sin(-nx) = -\sin(nx)$$

To obtain linearly independent eigenfunctions, take

$$\omega \in \mathbb{N}.$$

Solution :

$$\left.\begin{aligned} u_n(x) &\propto \sin(nx) \\ \lambda = \omega^2 = n^2 \quad &\therefore \quad \lambda_n = n^2 \end{aligned}\right\} \quad n \in \mathbb{N}$$

- Infinitely many eigenfunctions, forming a basis for

$$\mathcal{H} = \left\{ u : [0,\pi] \to \mathbb{R} \;\middle|\; \int_0^\pi |u(x)|^2 dx < \infty , \; u(0) = u(\pi) = 0 \right\}$$

- $\min(\lambda_n) = 1$ ☑
- $\lim_{n \to \infty} \lambda_n = \infty$ ☑

In agreement with Properties # 1 — # 6 .

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Week 11, Lecture 3

S.L. problem:

$$L[u](x) = -\lambda r(x) u(x), \quad x \in (a,b)$$

&

where

$$L[u](x) = \frac{d}{dx}\left[p(x) \frac{du}{dx} \right] + q(x) u(x)$$

and where $p(x), r(x) > 0$ on $[a,b]$,

BCs:

$$\left.\begin{aligned} \alpha_a u(a) + \beta_a u'(a) &= 0 \\ \alpha_b u(b) + \beta_b u'(b) &= 0 \end{aligned}\right\} \text{Robin}$$

(1)

<u>Theorem 11.3</u>: Suppose u_1 and u_2 solve the SL problem 1 with Robin BCs, and have the same eigenvalue. Then u_1 and u_2 are proportional.

<u>Remark</u>:

- Eigenspaces are all one-dimensional
- Eigenvalues are non-degenerate.

II

Proof: Since v_1 and v_2 solve the same SL problem, we have:

$$\alpha_a v_1(a) + \beta_a v_1'(a) = 0$$

$$\alpha_a v_2(a) + \beta_a v_2'(a) = 0$$

$$\boxed{\Rightarrow \underbrace{\begin{pmatrix} v_1(a) & v_1'(a) \\ v_2(a) & v_2'(a) \end{pmatrix}}_{M} \underbrace{\begin{pmatrix} \alpha_a \\ \beta_a \end{pmatrix}}_{\text{not the zerovector!}} = 0 .}$$

Therefore, $\det(M) = 0$.

But $W(a) = \det(M)$.

Therefore, $W(a) = 0$.

We apply Abel's Lemma:

$$\overset{p_n}{p}\, u'' + \overset{p_{n-1}}{p'}\, u' + \cancel{q(x)+r} + (q + r\lambda)\, u = 0 .$$

... This is eq$^{\underline{n}}$ (1) re-written, in the form given in Ch. 3.

$$W(x) = W(x_0)\, e^{-\int_{x_0}^{\cancel{x_0}} \frac{p'}{p} dx}$$

$$= W(x_0)\, e^{-\ln p \big|_{x_0}^{x}}$$

$$= W(x_0)\, e^{-\ln\left[p(x)/p(x_0)\right]}$$

$$\Rightarrow W(x) = W(x_0)\, e^{+\ln[p(x_0)/p(x)]}$$

$$= W(x_0)\,\frac{p(x_0)}{p(x)}$$

$$\Rightarrow \quad p(x)\,W(x) = p(x_0)\,W(x_0)$$

But, take $x = a$:

$$p(a)\,W(a) = p(x_0)\,W(x_0),$$

true for any $x_0 \in (a,b)$.

Since p is never zero,

$$W(x_0) = 0, \quad \text{for all } x_0 \in (a,b).$$

Definition of Wronskian:

$$W(x_0) = \begin{vmatrix} v_1(x_0) & v_2(x_0) \\ v_1'(x_0) & v_2'(x_0) \end{vmatrix}$$

$$\therefore \begin{vmatrix} v_1(x_0) & v_2(x_0) \\ v_1'(x_0) & v_2'(x_0) \end{vmatrix} = 0 \quad \forall\, x_0 \in (a,b).$$

$$\frac{v_1(x)\,v_2'(x)}{v_1 v_2} = \frac{v_1'(x)\,v_2(x)}{v_1 v_2} \quad \forall\, x \in (a,b).$$

$$\Rightarrow \quad \frac{v_2'}{v_2} = \frac{v_1'}{v_1}$$

$$\Rightarrow \frac{d}{dx} \log v_2 = \frac{d}{dx} \log v_1$$

$$\Rightarrow \frac{d}{dx}\left(\log v_2 - \log v_1\right) = 0 .$$

$$\Rightarrow \log v_2 - \log v_1 = \text{Const} .$$

$$\vdots$$

$$u_1 = C \cdot u_2$$

Therefore, u_1 and u_2 are proportional. $\blacksquare$

<u>Special case</u>: This result does not hold if we have periodic boundary conditions (BCs) instead of Robin BCs.

<u>Example</u>: $u'' + \lambda^2 u = 0$, $(a,b) = (0, 2\pi)$.

Periodic BCs: $u(x+2\pi) = u(x)$.

<u>Solution</u>: $u = e^{\sqrt{-\lambda}\, x}$, $u = e^{-\sqrt{-\lambda}\, x}$.

$$\Rightarrow u'' = -\lambda^2 u$$

$$\Rightarrow u'' + \lambda^2 u = 0 .$$

To enforce the BCs, we take λ to be real

$\longrightarrow$

$\lambda = k^2$.

$u = e^{ikx}, \ e^{-ikx}$.

Linear combinations:

$u = \sin(kx)$

$u = \cos(kx)$.

Require: $u(0) = u(2\pi)$.

<u>sin</u> $\quad 0 = \sin(2\pi k)$.

This gives: $k \in \{\frac{1}{2}, 1, \frac{3}{2}, 2, \dots\}$.

Odd eigenfunctions:

$u_n(x) \propto \sin(k_n x)$

$k_n \in \{\frac{1}{2}, 1, \frac{3}{2}, \dots\}$

<u>cos</u> $\quad u(0) = u(2\pi)$.

$1 = \cos(k \cdot 2\pi)$.

This gives $k \in \{1, 2, \dots\}$.

Even eigenfunctions:

$u_n(x) \propto \cos(k_n x)$

$k_n \in \{1, 2, \dots\}$

Ground state - the state with the minimum eigenvalue.

$$k_n = 1/2, \quad u_n = \sin(x/2)$$

Ground state is non-degenerate.

First excited state : $k = 1$.

$$u(x) = \sin(x) \qquad \underline{OR}$$

$$u(x) = \cos(x)$$

One eigenvalue $(k=1)$, two linearly independent eigenvectors. The first excited state is degenerate.

Applications: Schrödinger's Equation for 1D systems in Quantum Mechanics.