Chapter 9, FIEs:

$$y(x) = f(x) + \lambda \int_a^b k(x,s)\, y(s)\, ds \qquad (1)$$

Sl$^{\underline{n}}$ as a series:

$$y = (\mathbb{I} - \lambda \mathcal{K})^{-1} f = \sum_{n=0}^{\infty} \lambda^n \mathcal{K}^n f \qquad (2)$$

The sum of the series in (2) is called the resolvent kernel (§ 9.2).

$$\mathcal{K} f(x) = \int_a^b k(x,s) f(s)\, ds$$

$$\mathcal{K}^2 f(x) = \int_a^b k(x,s) \left[\int_a^b k(s,s') f(s')\, ds' \right] ds$$

$$= \int_a^b \Big[\underbrace{\int_a^b k(x,s) k(s,s')\, ds}_{= k_2(x,s')} \Big] f(s')\, ds'$$

Pattern:

$$\mathcal{K}^n f(x) = \int_a^b K_n(x,s) f(s)\, ds$$

K_n's defined inductively:

$$\begin{cases} K_n(x,s) = \int_a^b K(x,\tilde{s}) K_{n-1}(\tilde{s},s)\, d\tilde{s} \\ k_1(x,s) = K(x,s) \end{cases}$$

Hence: $k_n(x,s) = \int_a^b k_m(x,\tilde{s}) k_{n-m}(\tilde{s},s)\, d\tilde{s}$

Also:

$$|k_n(x,s)| \leq M^n |b-a|^n$$

Back to:

$$y(x) = f(x) + \sum_{n=1}^{\infty} \lambda^n \mathcal{K}^n f(x) \qquad \text{(last week)}.$$

$$y(x) = f(x) + \sum_{n=1}$$

$$= f(x) + \sum_{n=1}^{\infty} \lambda^n \int_a^b K_n(x,s) f(s)\, ds$$

$$= f(x) + \lambda \sum_{n=1}^{\infty} \lambda^{n-1} \int_a^b K_n(x,s) f(s)\, ds \qquad p = n-1, \; p_{start} = 0$$

$$= f(x) + \lambda \sum_{p=0}^{\infty} \lambda^p \int_a^b K_{p+1}(x,s) f(s)\, ds \qquad p+1 = n$$

$$= f(x) + \lambda \sum_{n=0}^{\infty} \lambda^n \int_a^b K_{n+1}(x,s) f(s)\, ds$$

$$= f(x) + \lambda \int_a^b \underbrace{\left[\sum_{n=0}^{\infty} \lambda^n K_{n+1}(x,s) \right]}_{\Gamma(x,s,\lambda)} f(s)\, ds$$

$\Gamma(x,s,\lambda) = \sum_{n=0}^{\infty} \lambda^n K_{n+1}(x,s)$ is the resolvent kernel.

Hence:

$$y(x) = f(x) + \underbrace{\lambda \int_a^b \Gamma(x,s,\lambda) f(s)\, ds}_{\text{no } y\text{-dependence; it has been resolved.}}$$

Theorem 9.1 The resolvent kernel satisfies the FIE.

Proof:

$$\Gamma(x,t,\lambda) = \sum_{n=0}^{\infty} \lambda^n K_{n+1}(x,t)$$

$$= K_1(x,t) + \sum_{n=1}^{\infty} \lambda^n K_{n+1}(x,t) \qquad p = n$$

-1

$$= k_1(x,t) + \sum_{n=1} \lambda^n k_{n+1}(x,t)$$

$$= \underbrace{k(x,t)} + \lambda \sum_{n=1}^{\infty} \lambda^{n-1} k_{n+1}(x,t)$$

p = n
p start
p+1
p →

$$= k(x,t) + \lambda \sum_{p=0}^{\infty} \lambda^p k_{p+2}(x,t)$$

$$= k(x,t) + \lambda \sum_{p=0}^{\infty} \lambda^p \int_a^b k(x,s) k_{p+1}(s,t)\, ds$$

$$= k(x,t) + \lambda \int_a^b \Bigg[\underbrace{\sum_{p=0}^{\infty} \lambda^p k_{p+1}(s,t)}_{\Gamma(s,t,\lambda)} \Bigg] k(x,s)$$

$$\therefore \Gamma(x,t,\lambda) = \underbrace{k(x,t)} + \lambda \int_a^b k(x,s)\, \Gamma(s,t,\lambda)\, ds$$

Worked example: (§9.2.1)

$$k(x,t) = 1 - 3xt \quad , \quad x \in [0,1] \ , \ t \in [0,1]$$

$$\Gamma(x,t,\lambda) = \sum_{n=0}^{\infty} \lambda^n k_{n+1}(x,t)$$

$$k_1(x,t) = 1 - 3xt$$

$$k_2(x,t) = \int_0^1 k(x,s)\, k_1(s,t)\, ds$$

$$= \int_0^1 (1 - 3xs)(1 - 3st)\, ds$$

$n-1$

$= 0$

$= n$

$2 = n+1$

s

$)\, ds$

$\blacksquare$

$$= \int_0^1 (1-3xs)(1-3st)\,ds$$

$$= 1 - \frac{3}{2}(x+t) + 3xt$$

$$k_3(x,t) = \int_0^1 k(x,s)\, k_2(s,t)\,ds$$

$$= \int_0^1 (1-3xs)\left[1 - \frac{3}{2}(s+t) + 3st\right] ds$$

$$= \frac{1}{4}(1-3xt) = \frac{1}{4} K_1(x,t)$$

The problem is closed:

$$k_n(x,t) = \frac{1}{4} k_{n-2}(x,t) \qquad n \geq 3 .$$

$$\Gamma(x,t,\lambda) = \sum_{n=0}^{\infty} \lambda^n k_{n+1}(x,t)$$

$$= k(x,t) + \sum_{n=1}^{\infty} \lambda^n k_{n+1}(x,t)$$

$$= k(x,t) + \lambda^2 K_3(x,t) + \lambda^4 k_5(x,t) + \cdots$$

$$+ \lambda K_2(x,t) + \lambda^3 K_4(x,t) + \cdots$$

$$= k(x,t) + \lambda\, k_2(x,t) + \lambda^3 k_4(x,t) + \cdots$$

$$+ \lambda^2 k_3(x,t) + \lambda^4 k_5(x,t) + \cdots$$

$$= k(x,t) + \lambda\, k_2(x,t) + \lambda^3 \frac{1}{4} k_2(x,t) + \cdots$$

$$+ \lambda^2 \frac{1}{4} k(x,t) + \lambda^4 \frac{1}{4}\cdot\frac{1}{4} k(x,t)$$

$$= \left[1 + \frac{\lambda^2}{4} + \frac{\lambda^4}{4^2} + \dots \right] k(x,t)$$

$$+ \lambda \left[1 + \frac{\lambda^2}{4} + \dots \right] k_2(x,t)$$

$$\overset{\text{G.P.}}{=} \frac{1}{1 - \lambda^2/4} k(x,t) + \lambda \cdot \frac{1}{1 - \frac{\lambda^2}{4}} k_2(x,t)$$

Requires: $\lambda^2/4 < 1$

Hence:

$$\Gamma(x,t,\lambda) = \frac{1}{1 - \frac{\lambda^2}{4}} \left[(1 - 3xt) + \lambda \left(1 - \tfrac{3}{2}(x+t) + 3xt \right) \right]$$

Resolvent kernel valid for all $\lambda \neq \pm 2$:

- G.P. requires $\lambda^2 < 4$, but final answer is more general and can be applied for all $\lambda^2 \neq 2$.
- Resolvent kernel is valid provided λ is not an eigenvalue of the FIE.

General method for computing the resolvent kernel:

- Try and find a <u>difference equation</u> linking k_n to k_{n-1}, k_{n-2} etc. the problem is <u>closed</u> and the series in the resolvent kernel can be computed explicitly.

can be computed explicitly.

Friday – we will apply this method to Exerises # 6

IUTAM

Week 10, Lecture 3

Fri 10th April 2026

FIEs — done.

Remark — FIEs give rise to a reproducing kernel Hilbert Space, which is a concept that is used a lot in Data Science .....

Today: Exercises #6 Question 4.

FIE:

$$y(x) = f(x) + \lambda \int_0^1 K(x,s)\,y(s)\,ds \quad \underline{(1)}$$

Given: $K(x,s) = (x+s)\,e^{x-s}$. $\underline{(2)}$

Aim: Compute

$$\Gamma(x,z,\lambda) = \sum_{n=0}^{\infty} \lambda^n K_{n+1}(x,z) .$$

<u>Solution</u>: Compute $K_1 = K$, K_2, K_3, etc. and see if a pattern emerges.

$$K_2 = K * K$$

$$\Rightarrow K_2(x,t) = \int_0^1 K(x,\underline{s})\,K(\underline{s},t)\,ds$$

$$= \int_0^1 (x+s)\,e^{x-\cancel{s}}\,(s+t)\,e^{\cancel{s}-t}\,ds$$

$$= e^{x-t} \int_0^1 (x+s)(s+t)\,ds$$

Hence:

$$K_2(x,t) = e^{x-t}\left[\tfrac{1}{2}x + xt + \tfrac{1}{3} + \tfrac{1}{2}t\right]$$

$$= e^{x-t}\left[\tfrac{1}{2}(x+t) + \left(xt + \tfrac{1}{3}\right)\right]$$

$$\Rightarrow \boxed{K_2(x,t) = \tfrac{1}{2}K(x,t) + e^{x-t}\left(xt+\tfrac{1}{3}\right)} \quad (3)$$

$$K_3 = K_2 * K$$

$$= \int_0^1 K_2(x,s)\,K(s,t)\,ds$$

$$= \int_0^1 \left[\tfrac{1}{2}K + e^{x-s}\left(xs+\tfrac{1}{3}\right)\right]\underbrace{e^{s-t}\,(s+t)}_{K}\,ds$$

$$= \tfrac{1}{2}K_2 + \int_0^1 e^{x-\cancel{s}}\,e^{\cancel{s}-t}\left(xs+\tfrac{1}{3}\right)(s+t)\,ds$$

$$= \tfrac{1}{2}K_2 + e^{x-t}\int_0^1 \left(xs+\tfrac{1}{3}\right)(s+t)\,ds$$

$$= \tfrac{1}{2}K_2 + e^{x-t}\left[\tfrac{1}{3}x + \tfrac{1}{2}xt + \tfrac{1}{3}\cdot\tfrac{1}{2} + \tfrac{1}{3}t\right]$$

$$= \tfrac{1}{2}K_2 + \tfrac{1}{3}e^{x-t}(x+t) + \tfrac{1}{2}\boxed{e^{x-t}\left(xt+\tfrac{1}{3}\right)}$$

$$\Rightarrow K_3 \overset{\text{Eq. (3)}}{=} \tfrac{1}{2}K_2 + \tfrac{1}{3}K + \tfrac{1}{2}\left[K_2 - \tfrac{1}{2}K\right]$$

$$\underline{\tfrac{1}{3}K - \tfrac{1}{4}K}$$

Hence:

$$k_3 = k_2 + \frac{1}{12} k .$$

This closes the problem.

$$\boxed{k_{n+2} = k_{n+1} + \frac{1}{12} k_n , \quad n \geq 1} \quad (4)$$

Back to

$$\Gamma = \sum_{n=0}^{\infty} \lambda^n k_{n+1}$$

$$= k + \lambda k_2 + \sum_{n=2}^{\infty} \lambda^n k_{n+1}$$

$$= k + \lambda k_2 + \lambda \sum_{n=2}^{\infty} \lambda^{n-1} k_{n+1}$$

$$= k + \lambda k_2 + \lambda \sum_{p=1}^{\infty} \lambda^p k_{p+2}$$

$p = n-1$

$p_{start} = 1$

$p+1 = n$

$p+2 = n+1$

$$\overset{\text{Eq. (4)}}{=} k + \lambda k_2 + \lambda \sum_{p=1}^{\infty} \lambda^p \left[k_{p+1} + \frac{1}{12} k_p \right]$$

$$= k + \lambda k_2 + \lambda \Big[\sum_{p=0}^{\infty} \lambda^p k_{p+1} - k \Big]$$

$$+ \frac{1}{12} \lambda \sum_{p=1}^{\infty} \lambda^p k_p$$

Γ

$$\Gamma = \underline{k} + \lambda k_2 + \lambda\left[\Gamma - \underline{k}\right] + \frac{1}{12}\lambda \sum_{p=1}^{\infty} \lambda^{p} k_p$$

$$= (1-\lambda)k + \lambda k_2 + \lambda\Gamma + \frac{1}{12}\lambda\cdot\lambda \sum_{p=1}^{\infty} \lambda^{p-1} k_p$$

Re-index: $q = p-1$, $q_{start} = 0$ | $q+1 = p$.

$$\Rightarrow \Gamma = (1-\lambda)k + \lambda k_2 + \lambda\Gamma + \frac{1}{12}\lambda^2 \underbrace{\sum_{q=0}^{\infty} \lambda^{q} k_{q+1}}_{\Gamma}$$

$$= (1-\lambda)k + \lambda k_2 + \left(\lambda + \frac{1}{12}\lambda^2\right)\Gamma.$$

$$\Rightarrow \left(1 - \lambda - \frac{1}{12}\lambda^2\right)\Gamma = (1-\lambda)k + \lambda k_2.$$

Eigenvalues: $1 - \lambda - \frac{1}{12}\lambda^2 = 0.$

One λ is not an eigenvalue, Γ exists and is equal to:

$$\Gamma = \frac{(1-\lambda)k + \lambda k_2}{1 - \lambda - \frac{1}{12}\lambda^2}.$$

4 b) Solve the FIE for:

$$f(x) = x^2$$
$$\lambda = 2.$$

Leave answer in terms of

$$I_n = \int_0^1 u^n e^{u}\, du.$$

<u>Check</u>: 2 is not an eigenvalue.

$$Q(\lambda) = 1 - \lambda - \frac{1}{12}\lambda^2.$$

$$Q(2) = 1 - 2 - \frac{1}{12}\cdot 4$$
$$= -1 - \frac{1}{3} \neq 0.$$

$\therefore$ $\lambda = 2$ not an eigenvalue.

Solution of FIE from lecture notes:

$$y(x) = \underbrace{x^2}_{=f} + \underbrace{2}_{=\lambda} \int_0^1 \Gamma(x,z;2)\underbrace{z^2}_{f}\, dz.$$

Look at:

$$\Gamma(x,z,2) = \left(\frac{(1-\lambda)K + \lambda k_2}{1 - \lambda - \frac{1}{12}\lambda^2} \right)_{\lambda=2}$$

$$\Gamma(x,z,2) = \frac{-\cancel{k} + 2\left(\cancel{\tfrac{1}{2}k} + e^{x-z}\left(xz + \tfrac{1}{3}\right)\right)}{(-4/3)}$$

Leave as:

$$\Gamma(x,z,2) = \frac{e^{x-z}\left(2xz + \frac{2}{3}\right)}{-4/3}$$

Back to:

$$y(x) = x^2 + 2\int_0^1 \frac{e^{x-z}\left(2xz + \frac{2}{3}\right)}{-4/3}\, \overset{f(z)}{z^2}\, dz$$

$$= x^2 + \frac{4xe^x}{-4/3} \overbrace{\int_0^1 e^{-z} z^3\, dz}^{I_3}$$

$$+ \frac{(4/3)\, e^x}{-4/3} \overbrace{\int_0^1 e^{-z} z^2\, dz}^{I_2}$$

$$\therefore \boxed{y(x) = x^2 - 3xe^x I_3 - e^x I_2}$$