Applied Analysis (ACM30020)

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Exercises #7

1. Show that the normalised eigenfunctions of the boundary value problem

$$y'' = -\lambda y, \qquad y(0) = 0, \qquad y(1) + y'(1) = 0,$$

are

$$u_n(x) = k_n \sin \sqrt{\lambda_n}\, x,$$

where λ_n is the nth positive root of $\tan\sqrt{\lambda_n} = -\sqrt{\lambda_n}$ and

$$k_n = \left(\frac{2}{1+\cos^2\sqrt{\lambda_n}}\right)^{1/2}.$$

Hence solve the boundary value problem

$$y'' + \lambda y = -x, \qquad y(0) = 0, \qquad y(1) + y'(1) = 0,$$

as a series of the form

$$y(x) = \sum_{n=0}^{\infty} b_n u_n(x),$$

where the coefficients b_n should be determined (in terms of λ_n).

We first of all compute the **eigenvalues** of the homogeneous problem $y'' = -\lambda y$, with $y(0) = 0$ and $y(1) + y'(1) = 0$. We look first at the case $\lambda > 0$. A solution is clearly:

$$y(x) \propto \sin\left(\sqrt{\lambda}x\right). \tag{1}$$

This automatically satisfies the LHBC[1]. For the RHBC we require:

$$\sin\sqrt{\lambda} + \sqrt{\lambda}\cos\sqrt{\lambda} = 0. \tag{2}$$

[1]We use LHBC and RHBC for the left-hand and right-hand boundary conditions, respectively

Divide both sides by $\cos\sqrt{\lambda}$:

$$\tan\sqrt{\lambda} = -\sqrt{\lambda}.$$

This is a root-finding problem. We replace $\sqrt{\lambda}$ with q and we look for roots of $\tan q = -q$. To find the roots, we plot two curves $y_1(q) = \tan q$ and $y_2(q) = -q$. The points of intersection $y_1(q) = y_2(q)$ are the roots of the equation $\tan q = -q$. From such a plot we would see that there infinitely many positive roots. We label the corresponding values of $\sqrt{\lambda}$ as λ_n:

$$\tan\sqrt{\lambda_n} = -\sqrt{\lambda_n}, \qquad n \in \{0, 1, 2, \cdots\}. \tag{3}$$

We do not consider the root $q = 0$ as this corresponds to $\lambda = 0$. Also, we do not consider negative roots, since $q = \sqrt{\lambda}$ can't be negative.

We also rule out other possibilities for λ:

- $\lambda = 0$. This would give $y'' = 0$, hence $y = Ax + B$. With the LHBC and the RHBC this forces $y = 0$, hence, the trivial solution.
- $\lambda < 0$. This would give $y \propto \sinh(\sqrt{-\lambda_n}x)$, which satisfies the LHBC automatically. To satisfy the RHBC, we would require:

  $$\tanh\sqrt{-\lambda} = -\sqrt{-\lambda}$$

  With $q = \sqrt{-\lambda}$ as before, we would plot $y_1(q) = \tanh q$ and $y_2(q) = -q$. These curves intersect only at $q = 0$, which is not allowed since $\lambda < 0$. Thus, this case is ruled out also.

We next compute the **normalization factor** in the eigenfunctions of Equation (1). Hence, we write $y(x) = u_n(x) = k_n \sin\left(\sqrt{\lambda_n}x\right)$, and we require:

$$\int_0^1 |u_n|^2 \mathrm{d}x = 1.$$

We have:

$$\int_0^1 |u_n|^2 \mathrm{d}x = k_n^2 \int_0^1 \sin^2\left(\sqrt{\lambda_n}x\right) \mathrm{d}x.$$

We use the trig ID

$$\sin^2\theta = \tfrac{1}{2}\left[1 - \cos(2\theta)\right],$$

valid for all θ. Hence:

$$\begin{aligned}
\int_0^1 |u_n|^2 \mathrm{d}x &= \tfrac{1}{2}k_n^2 \int_0^1 \left[1 - \cos\left(2\sqrt{\lambda_n}x\right)\right] \mathrm{d}x, \\
&= \tfrac{1}{2}k_n^2 \left[1 - \frac{1}{2\sqrt{\lambda_n}} \sin\left(2\sqrt{\lambda_n}x\right)\Big|_0^1\right], \\
&= \tfrac{1}{2}k_n^2 \left[1 - \frac{1}{2\sqrt{\lambda_n}} \sin(2\sqrt{\lambda_n})\right], \\
&= \tfrac{1}{2}k_n^2 \left[1 - \frac{1}{2\sqrt{\lambda_n}} 2\sin(\sqrt{\lambda_n})\cos(\sqrt{\lambda_n})\right], \\
&\overset{\text{Eq. (3)}}{=} \tfrac{1}{2}k_n^2 \left\{1 - \frac{1}{\sqrt{\lambda_n}} \cos(\sqrt{\lambda_n}) \left[-\sqrt{\lambda_n}\cos(\sqrt{\lambda_n})\right]\right\}, \\
&= \tfrac{1}{2}k_n^2 \left[1 + \cos^2(\sqrt{\lambda_n})\right].
\end{aligned}$$

All of this must be equal to 1, so we have:

$$k_n^2 = \frac{2}{1+\cos^2(\sqrt{\lambda_n})}$$

and hence finally,

$$k_n = \sqrt{\frac{2}{1+\cos^2(\sqrt{\lambda_n})}},$$

as required.

To solve the inhomogeneous problem $y + \lambda y = -x$, we propose:

$$y(x) = \sum_{n=0}^{\infty} c_n u_n(x),$$

where

$$u_n(x) = k_n \sin(\sqrt{\lambda_n} x).$$

We sub in to the inhomogeneous problem and we get:

$$\sum_{n=0}^{\infty} b_n(u_n'' + \lambda u_n) = -x.$$

But $u_n'' = -\lambda_n u_n$, hence:

$$\sum_{n=0}^{\infty} b_n(\lambda - \lambda_n)u_n = -x.$$

Since $\langle u_n, u_m \rangle = \delta_{nm}$ in the usual inner product on the interval $[0, 1]$, we get:

$$c_n = -\frac{\langle u_n, x \rangle}{\lambda - \lambda_n}.$$

Hence:

$$y(x) = \sum_{n=0}^{\infty} \frac{\langle u_n, x \rangle}{\lambda - \lambda_n} \int_0^1 u_n(s)(-s)\mathrm{d}s. \tag{4}$$

This is the required solution. However, we comment on what happens when $\lambda \to \lambda_n$, for some n. By Fredholm theory, the only way for a solution to exist in this limit is if the corresponding coefficient $\langle u_n, x \rangle$ goes to zero. Therefore, we consider the integral

$$I = \int_0^1 u_n(s)s\,\mathrm{d}s.$$

We have:

$$\begin{aligned}
I &\propto k_n \int_0^1 \underbrace{\sin(\sqrt{\lambda_n}x)}_{=\mathrm{d}v} \underbrace{x}_{=u} \,\mathrm{d}x, \\
&= k_n \left[-\frac{x\cos(\sqrt{\lambda_n}x)}{\sqrt{\lambda_n}}\bigg|_0^1 + \int_0^1 \frac{1}{\sqrt{\lambda_n}}\cos(\sqrt{\lambda_n}x)\mathrm{d}x \right], \\
&\propto -\frac{\cos(\sqrt{\lambda_n})}{\sqrt{\lambda_n}} + \frac{\sin(\sqrt{\lambda_n})}{\lambda_n}, \\
&= \frac{1}{\sqrt{\lambda_n}} \left[-\cos(\sqrt{\lambda_n}) + \frac{1}{\sqrt{\lambda_n}}\sin(\sqrt{\lambda_n}) \right], \\
&\overset{\text{Eq. (3)}}{=} \frac{1}{\sqrt{\lambda_n}} \left[\frac{1}{\sqrt{\lambda_n}}\sin(\sqrt{\lambda_n}) + \frac{1}{\sqrt{\lambda_n}}\sin(\sqrt{\lambda_n}) \right], \\
&= \frac{1}{\sqrt{\lambda_n}} \times \left[2\frac{1}{\sqrt{\lambda_n}}\sin(\sqrt{\lambda_n}) \right], \\
&= \frac{2\sin(\sqrt{\lambda_n})}{\lambda_n}, \\
&\neq 0.
\end{aligned}$$

Thus, $I \neq 0$ and in the limit as $\lambda \to \lambda_n$, Equation (4) there breaks down, and there is no solution to the BVP.

This completes the solution. However, it can be noted that the BVP can be solved without eigenfunction expansions, by using the method of variation of parameters instead. In this approach, we take $u(x)$ to be a solution of the ODE $y'' + \lambda y = 0$ satisfying the LHBC:

$$u(x) = \sin\sqrt{\lambda}x.$$

Similarly, we take $v(x)$ to be a solution of the ODE satisfying the RHBC:

$$v(x) = \sin\sqrt{\lambda}(1-x) + \sqrt{\lambda}\cos\sqrt{\lambda}(1-x).$$

The Wronskian is thus:

$$W(x) = -\sqrt{\lambda}(\sqrt{\lambda}\cos\sqrt{\lambda} + \sin\sqrt{\lambda}).$$

Then, by variation of parameters, the solution to the BVP is:

$$y(x) = \frac{u(x)}{W}\int_0^x v(s)(-s)\,\mathrm{d}s + \frac{v(x)}{W}\int_x^1 u(s)(-s)\,\mathrm{d}s.$$

After doing the various trigonometric integrals here, this becomes:

$$y(x) = -\frac{x}{\lambda} + \frac{2}{\lambda(\sqrt{\lambda}\cos\sqrt{\lambda} + \sin\sqrt{\lambda})}\sin\sqrt{\lambda}x. \tag{5}$$

Although not immediately obvious, the series solution (4) must necessarily converge to the solution (5) except when $\lambda = \lambda_n$ for some eigenvalue. A sceptic can also verify by direct substitution that Equation (5) satisfies the BVP.

2. Use the properties of the **Legendre polynomials** to do the following:
   (a) Find the solution of $(1-x^2)y''-2xy'+by=f(x)$ that is valid on the range $[-1,1]$ and finite at $x=0$, in terms of Legendre polynomials.
   (b) Find the explicit solution if $b=14$ and $f(x)=5x^2$. Verify it by direct substitution.

We propose a solution of the form:

$$y(x)=\sum_{n=0}^{\infty}a_nP_n(x) \tag{6}$$

valid on the interval $[-1,1]$. Here, the P_n's are the Legendre polynomials; each P_n satisfies:

$$(1-x^2)P_n''-2xP_n'+n(n+1)P_n=0, \qquad n\in\{0,1,2,\cdots\}.$$

Also, the solution (6) is valid because the $P_n's$ are a complete set. Furthermore, in the standard form, the $P_n's$ satisfy an orthogonality relation

$$\int_{-1}^{1}P_nP_m=\frac{2}{2n+1}\delta_{nm}.$$

Finally, and again because of completeness of the Legendre polynomials on the interval $[-1,1]$, we can write:

$$f(x)=\sum_{n=0}^{\infty}f_nP_n(x), \qquad f_n=\frac{2n+1}{2}\int_{-1}^{1}f(x)P_n(x)\mathrm{d}x.$$

We substitute the trial solution (6) into the ODE:

$$\sum_{n=0}^{\infty}\left\{(1-x^2)P_n''-2xyP_n'+n(n+1)P_n+[b-n(n+1)]P_n\right\}a_n=\sum_{n=0}^{\infty}f_nP_n.$$

We use the properties of the Legendre polynomials to conclude:

$$[b-n(n+1)]\,a_n=f_n. \tag{7}$$

Case 1: Provided b is never equal to $n(n+1)$ for n a positive integer or zero, we have:

$$a_n=\frac{f_n}{b-n(n+1)},$$

and the solution to the ODE can be written as:

$$y(x)=\sum_{n=0}^{\infty}\frac{f_n}{b-n(n+1)}f_nP_n(x).$$

Case 2: On the other hand, if b is equal to $n_0(n_0+1)$, for n_0 zero or a positive integer, then a_{n_0} is undetermined. Furthermore, for consistency $(0 = 0)$, we require $f_{n_0} = 0$. All the other $a_n's$ are still given by Equation (7), in which case we have:

$$y(x) = \sum_{\substack{n=0 \\ n\neq n_0}}^{\infty} \frac{f_n}{b - n(n+1)} f_n P_n(x) + A P_{n_0}(x), \qquad f_{n_0} = 0.$$

where A is an arbitrary constant.

For Part (b), we first of all make sure that $14 \neq n(n+1)$ (it isn't). Therefore, Case 1 applies, and we express $f(x) = 5x^3$ in terms of standard Legendre polynomials:

$$5x^3 = 2\left[\tfrac{1}{2}\left(5x^3 - 3x\right)\right] + 3\,(x) = 2P_3(x) + 3P_1(x).$$

We conclude that, because of the mutual orthogonality of the Legendre polynomials, only a_3 and a_1 in the series solution will be non-zero. To find them we need to evaluate:

$$\int_{-1}^{1} f(x)P_3(x)\mathrm{d}x = 2\frac{2}{2\times 3+1} = \tfrac{4}{7}.$$

Similarly, $\int_{-1}^{1} f(x)P_1(x)\mathrm{d}x = 3 \times (2/3) = 2.$

Inserting these values gives:

$$a_3 = \frac{7}{2(14-12)} \times \frac{4}{7} = 1, \qquad a_1 = \frac{3}{2(14-2)} = \tfrac{1}{4}.$$

Thus, the solution is:

$$\begin{aligned} y(x) &= \tfrac{1}{4}P_1(x) + P_3(x), \\ &= \tfrac{1}{4} + \tfrac{1}{2}(5x^3 - 3x), \\ &= \tfrac{5}{4}(2x^3 - x). \end{aligned}$$

We verify:

$$\begin{aligned} (1-x^2)\left(\tfrac{60}{4}x\right) - 2x\left[\tfrac{1}{4}\left(30x62 - 5\right)\right] + \tfrac{1}{4}\left[140x^3 - 70x\right] &= 5x^3 \\ \Longrightarrow\ 60x - 60x^3 - 60x^3 + 10x + 140x^3 - 70x &= 20x^3, \end{aligned}$$

which is satisfied.

3. Let $f(x)$ be a differentiable function on $-\infty < x < \infty$, vanishing at least as quickly as x^{-1} as $|x| \to \infty$, and consider the linear operator

$$L = \frac{\mathrm{d}}{\mathrm{d}x} + x,$$

acting on such functions. Is L self-adjoint?

Let f and g be functions with the properties given in the question. We have:

$$\begin{aligned}
\langle f, Lg \rangle &= \int_{-\infty}^{\infty} f \frac{\mathrm{d}g}{\mathrm{d}x}\mathrm{d}x + \int_{-\infty}^{\infty} xfg\mathrm{d}x, \\
&\stackrel{\text{IBP}}{=} \cancel{(fg)_{-\infty}^{\infty}} - \int_{-\infty}^{\infty} g\frac{\mathrm{d}f}{\mathrm{d}x}\mathrm{d}x + \int_{-\infty}^{\infty} xfg\mathrm{d}x, \\
&= \langle \left(-\frac{\mathrm{d}}{\mathrm{d}x} + x \right) f, g \rangle.
\end{aligned}$$

Thus,

$$L^{\dagger} = -\frac{\mathrm{d}}{\mathrm{d}x} + x \neq L,$$

and L is not self-adjoint.

4. (a) Suppose that u and v are solutions of the following two homogeneous linear second order differential equations in self-adjoint form:

$$(p_1(x)u')' + q_1(x)u = 0, \tag{8a}$$

and

$$(p_2(x)v')' + q_2(x)v = 0. \tag{8b}$$

By direct computation, show that:

$$\begin{aligned}
\left(\frac{u}{v}(p_1u'v - p_2uv')\right)' &= \left(up_1u' - p_2v'u^2\frac{1}{v}\right)' \\
&= (p_1 - p_2)\,u'^2 + p_2\left(u' - v'\frac{u}{v}\right)^2 + (q_2 - q_1)\,u^2.
\end{aligned}$$

(b) See below.

We have:

$$
\begin{aligned}
\left(\frac{u}{v}(p_1u'v - p_2uv')\right)' &= \left[(p_1u')u - (p_2v')\frac{u^2}{v}\right]', \\
&= p_1(u')^2 + (p_1u')'u \\
&\qquad - (p_2v')'\frac{u^2}{v} - p_2v'\left(\frac{2uu'v - u^2v'}{v^2}\right), \\
&= \left[(p_1u')'u - (p_2v')'\frac{u^2}{v}\right] \\
&\qquad + p_1(u')^2 + p_2\frac{(v')^2}{v^2}u^2 - 2p_2uu'\frac{v'}{v}, \\
&\overset{\text{SL Eqn}}{=} (q_2 - q_1)u^2 + (p_1 - p_2)(u')^2 \\
&\qquad + p_2\left[(u')^2 - 2p_2uu'\frac{v'}{v} + \frac{(v')^2}{v^2}u^2\right], \\
&= (q_2 - q_1)u^2 + (p_1 - p_2)(u')^2 + p_2\left(u' - \frac{v'}{v}u\right)^2.
\end{aligned}
$$

We next look at 4(b) where we are required to prove the **Sturm–Picone comparison theorem**: Let p_i and q_i for $i = 1, 2$ be real-valued continuous functions on the interval $[a, b]$ and let

$$
\begin{aligned}
(p_1(x)y')' + q_1(x)y &= 0, &\text{(9a)}\\
(p_2(x)y')' + q_2(x)y &= 0, &\text{(9b)}
\end{aligned}
$$

be two homogeneous linear second order differential equations in self-adjoint form with

$$0 < p_2(x) \le p_1(x), \tag{10}$$

and

$$q_1(x) \le q_2(x). \tag{11}$$

Let u be a non-trivial solution of Equation (9a) with successive roots at z_1 and z_2 and let v be a non-trivial solution of Equation (9b). Then one of the following properties holds:

- There exists an x in (z_1, z_2) such that $v(x) = 0$,
- there exists a $\mu \in \mathbb{R}$ such that $v(x) = \mu u(x)$.

Suppose for contradiction that $v(x) \neq 0$ for all $x \in [z_1, z_2]$. Without loss of generality, we can look at the case where $u(x) \ge 0$ for all $x \in [z_1, z_2]$. Then we also have that $v(x) \neq 0$ in the same interval and again without loss of generality we assume that $v(x) > 0$ in $[z_1, z_2]$.

We look at Picone's identity from Part (a),

$$\left(\frac{u}{v}(p_1u'v - p_2uv')\right)' = (q_2 - q_1)u^2 + (p_1 - p_2)(u')^2 + p_2\left(u' - \frac{v'}{v}u\right)^2.$$

We integrate from $x = z_1$ to an arbitrary value $x < z_2$ to obtain:

$$\left(\frac{u}{v}(p_1u'v - p_2uv')\right)(x) - \underbrace{\left(\frac{u}{v}(p_1u'v - p_2uv')\right)(z_1)}_{=0} \geq 0.$$

But the term with the underbrace is zero since $u(z_1) = 0$. We can tidy up the above expression and we are left with:

$$p_1u'v - p_2uv' \geq 0, \qquad x \in [z_1, z_2].$$

We have $uv > 0$ in (z_1, z_2), so we divide across by uv without breaking the inequality. This gives:

$$p_1\frac{u'}{u} \geq p_2\frac{v'}{v}, \qquad x \in (z_1, z_2) \tag{12}$$

(notice the open interval here). Thus, as in class notes u is $\cap$-shaped in $[z_1, z_2]$. Thus, there exists an $\alpha \in (z_1, z_2)$ such that:

$$\begin{aligned} u'(x) &> 0, \qquad x \in (z_1, \alpha), \\ u'(x) &< 0, \qquad x \in (\alpha, z_2). \end{aligned}$$

Thus, Equation (12) can be re-written as:

$$-p_1\left|\frac{u'}{u}\right| \geq p_2\frac{v'}{v}, \qquad \text{for } x \in (\alpha, z_2).$$

We have $p_1 \geq p_2$. Also, p_1 and p_2 are positive. So we have:

$$-p_1 \leq -p_2, \text{ or } -p_2 \geq -p_1.$$

Hence:

$$-p_2\left|\frac{u'}{u}\right| \geq -p_1\left|\frac{u'}{u}\right| \geq p_2\frac{v'}{v}, \qquad x \in (\alpha, z_2).$$

Cancel the p_2's (these are strictly positive) and restore the sign of u' to get:

$$\frac{u'}{u} \geq \frac{v'}{v}, \qquad x \in (\alpha, z_2),$$

or

$$\frac{u'}{u} \geq \frac{v'}{v}, \qquad x \in (\alpha, z_2).$$

Use Gronwall to conclude:

$$v(x) \leq Cu(x), \qquad x \in (\alpha, z_2).$$

where C is a positive constant. Now either we have equality, in which case $C = \mu$, and $v(x) = \mu u(x)$, in which case Option 1 in the Theorem is shown. Or we have a strict inequality, and

$$v(x) < Cu(x), \qquad x \in (\alpha, z_2).$$

We take $x \to z_2$. By continuity,

$$v(z_2) \leq 0.$$

But this is a contradiction, since $v(x) > 0$ for all $[a, b]$ is assumed. Hence, we have Option 2 of the theorem, and we are forced to conclude that in fact, there is an $x \in (z_1, z_2)$ such that $v(x) = 0$.