Advanced Mathematical Methods (ACM30020)

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Exercises #7

1. Show that the normalised eigenfunctions of the boundary value problem

$$y'' = -\lambda y, \qquad y(0) = 0, \qquad y(1) + y'(1) = 0,$$

are

$$u_n(x) = k_n \sin\sqrt{\lambda_n}\, x,$$

where λ_n is the nth positive root of $\tan\sqrt{\lambda_n} = -\sqrt{\lambda_n}$ and

$$k_n = \left(\frac{2}{1+\cos^2\sqrt{\lambda_n}}\right)^{1/2}.$$

Hence solve the boundary value problem

$$y'' + \lambda y = -x, \qquad y(0) = 0, \qquad y(1) + y'(1) = 0,$$

as a series of the form

$$y(x) = \sum_{n=0}^{\infty} b_n u_n(x),$$

where the coefficients b_n should be determined (in terms of λ_n).

2. Use the properties of the **Legendre polynomials** to do the following:

   (a) Find the solution of $(1-x^2)y'' - 2xy' + by = f(x)$ that is valid on the range $[-1, 1]$ and finite at $x = 0$, in terms of Legendre polynomials.

   (b) Find the explicit solution if $b = 14$ and $f(x) = 5x^2$. Verify it by direct substitution.

3. Let $f(x)$ be a differentiable function on $-\infty < x < \infty$, vanishing at least as quickly as x^{-1} as $|x| \to \infty$, and consider the linear operator

$$L = \frac{\mathrm{d}}{\mathrm{d}x} + x,$$

acting on such functions. Is L self-adjoint?

4. (a) Suppose that u and v are solutions of the following two homogeneous linear second order differential equations in self-adjoint form:

$$(p_1(x)u')' + q_1(x)u = 0, \tag{1}$$

and

$$(p_2(x)v')' + q_2(x)v = 0. \tag{2}$$

By direct computation, show that:

$$\begin{aligned}\left(\frac{u}{v}(p_1u'v - p_2uv')\right)' &= \left(up_1u' - p_2v'u^2\frac{1}{v}\right)' \\ &= (p_1 - p_2)\,u'^2 + p_2\left(u' - v'\frac{u}{v}\right)^2 + (q_2 - q_1)\,u^2.\end{aligned}$$

(b) Using part (a), prove the Sturm–Picone comparison theorem:

Theorem: Let p_i and q_i for $i = 1, 2$ be real-valued continuous functions on the interval $[a, b]$ and let

$$\begin{aligned}(p_1(x)y')' + q_1(x)y &= 0, && \text{(3a)}\\ (p_2(x)y')' + q_2(x)y &= 0, && \text{(3b)}\end{aligned}$$

be two homogeneous linear second order differential equations in self-adjoint form with

$$0 < p_2(x) \leq p_1(x), \tag{4}$$

and

$$q_1(x) \leq q_2(x). \tag{5}$$

Let u be a non-trivial solution of Equation (3a) with successive roots at z_1 and z_2 and let v be a non-trivial solution of Equation (3b). Then one of the following properties holds:

- There exists an x in (z_1, z_2) such that $v(x) = 0$,
- there exists a $\mu \in \mathbb{R}$ such that $v(x) = \mu u(x)$.