Applied Analysis (ACM30020)

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Exercises #6

1. Consider the FIE

$$f(\theta) = h(\theta) + \int_0^{2\pi} K(\theta - \varphi) f(\varphi) \mathrm{d}\varphi, \tag{1}$$

where K is the kernel function

$$K(\varphi) = k_0 + k_1 \cos\varphi, \tag{2}$$

and where k_0 and k_1 are constant.

   (a) Using the trial solution

$$f(\theta) = h(\theta) + p + q\cos\theta + r\sin\theta,$$

   solve the FIE (1).

   (b) Hence or otherwise, find the eigenvalues of the FIE

$$y(\theta) = \lambda \int_0^{2\pi} K(\theta - \varphi) y(\varphi) \mathrm{d}\varphi,$$

   for the kernel function 2.

By direct substitution, we have:

$$\begin{aligned} f(\theta) &= \cancel{h(\theta)} + p + q\cos\theta + r\sin\theta \\ &\overset{\text{FIE}}{=} \cancel{h(\theta)} + \int_0^{2\pi} \left[k_0 + k_1\cos(\theta - \varphi)\right]\left[h(\varphi) + p + q\cos\varphi + r\sin\varphi\right] \mathrm{d}\varphi. \end{aligned}$$

We use a trig identity to re-write this as:

$$\begin{aligned} &p + q\cos\theta + r\sin\theta \\ &= \int_0^{2\pi} \left[k_0 + k_1\cos\theta\cos\varphi + k_1\sin\theta\sin\varphi\right]\left[h(\varphi) + p + q\cos\varphi + r\sin\varphi\right] \mathrm{d}\varphi. \end{aligned}$$

We expand out the different terms on the RHS and do the various integrals to get:

$$p = k_0 H + 2\pi k_0 p, \qquad H = \int_0^{2\pi} h(s)\mathrm{d}s,$$

hence

$$p = \frac{k_0 H}{1 - 2\pi k_0}.$$

Similarly,

$$q = k_1 H_c + k_1 \pi q, \qquad H_c = \int_0^{2\pi} h(s)\cos(s)\mathrm{d}s,$$

hence

$$q = \frac{k_1 H_c}{1 - \pi k_1}.$$

And finally,

$$r = k_1 H_c + k_1 \pi r, \qquad H_s = \int_0^{2\pi} h(s)\sin(s)\mathrm{d}s,$$

hence

$$r = \frac{k_1 H_s}{1 - \pi k_1}.$$

Thus, the full solution for $f(\theta)$ is given by:

$$f(\theta) = h(\theta) + \frac{k_0 H}{1 - 2\pi k_0} + \frac{k_1 H_c}{1 - \pi k_1}\cos\theta + \frac{k_1 H_s}{1 - \pi k_1}\sin\theta. \tag{3}$$

Next, we compute the eigenfunctions y_i

$$y_i = \lambda_i \int_0^{2\pi} \left[k_0 + k_1 \cos\theta\cos\varphi + k_1 \sin\theta\sin\varphi\right] y(\varphi)\mathrm{d}\varphi. \tag{4}$$

We recognize that the kernel is separable and hence only a finite number of eigenfunctions exist. From the first part, we recognize these as:

$$y_0(\theta) \propto \text{Const.}, \qquad y_1(\theta) \propto \cos\theta, \qquad y_2(\theta) \propto \sin\theta.$$

We substitute these trial forms into Equation (4) and obtain:

$$\lambda_0 = \frac{1}{2\pi k_0}, \qquad \lambda_1 = \lambda_2 = \frac{1}{k_1 \pi}.$$

This is the final answer for part (b) and is perfectly acceptable. However, to understand this solution in more detail, we revisit the general FIE:

$$f(\theta) = h(\theta) + \lambda \int_0^{2\pi} \left[k_0 + k_1 \cos\theta\cos\varphi + k_1 \sin\theta\sin\varphi\right] f(\varphi)\mathrm{d}\varphi.$$

Following the lecture notes, we propose

$$f(\theta) = h(\theta) + \sum_i a_i y_i(\theta), \tag{5}$$

(recall, the difference $f - h$ is expanded in terms of eigenfunctions y_i). Hence,

$$\begin{aligned} f(\theta) - h(\theta) &= \sum_i a_i y_i(\theta), \\ &= \lambda K * f. \end{aligned}$$

Project on to the y_i eigenfunction, and use $\langle y_i, y_j \rangle = \delta_{ij}$:

$$\begin{aligned} a_i &= \lambda \langle y_i, K * f \rangle, \\ &\overset{K \text{ symmetric}}{=} \lambda \langle K * y_i, f \rangle, \\ &= \frac{\lambda}{\lambda_i} \langle y_i, f \rangle, \\ &= \frac{\lambda}{\lambda_i} \langle y_i, h(\theta) + \sum_j a_j y_j(\theta) \rangle. \end{aligned}$$

Hence,

$$a_i = \frac{\lambda}{\lambda_i} \left[\langle y_i, h \rangle + a_i \right],$$

hence

$$a_i = \frac{\lambda}{\lambda_i - \lambda} \langle y_i, h \rangle.$$

Sub in to Equation (5):

$$f(\theta) = h(\theta) + \sum_i \frac{\lambda}{\lambda_i - \lambda} \langle y_i, h \rangle.$$

$$f(\theta) = h(\theta) + \sum_i \frac{\lambda}{\lambda_i - \lambda} \langle h, y_i \rangle y_i(\theta), \qquad \lambda \notin \{\lambda_0, \lambda_1, \lambda_2\}.$$

Hence,

$$\begin{aligned} f(\theta) &= h(\theta) + \left(\frac{\lambda}{\frac{1}{2\pi k_0} - \lambda} \right) \frac{1}{2\pi} H + \left(\frac{\lambda}{\frac{1}{\pi k_1} - \lambda} \right) \frac{1}{\pi} H_c \cos\theta + \left(\frac{\lambda}{\frac{1}{\pi k_1} - \lambda} \right) \frac{1}{\pi} H_s \cos\theta, \\ &= h(\theta) + \frac{\lambda k_0}{1 - \lambda(2\pi k_0)} H + \frac{\lambda k_1}{1 - \lambda(\pi k_1)} H_c \cos\theta + \frac{\lambda k_1}{1 - \lambda(\pi k_1)} H_s \cos\theta \end{aligned}$$

This is exactly Equation (3), with $\lambda = 1$.

2. Consider the FIE

$$f(x) = g(x) + \lambda \int_{-\infty}^{\infty} K(x,y) f(y) \mathrm{d}y, \tag{6}$$

where $K(x,y) = 1$ inside the square $\{|x| < a, |y| < a\}$ and zero elsewhere. Using a trial solution

$$f(x) = g(x) + (\cdots),$$

find the solution of Equation (6).

For $|x| > a$, we have $K(x,y) = 0$, hence

$$f(x) = g(x), \qquad |x| > a.$$

For $|x| < a$ we have $K(x,y) = 1$, hence

$$f(x) = g(x) + \lambda \int_{-a}^{a} f(y) \mathrm{d}y = g(x) + C, \qquad |x| \leq a. \tag{7}$$

where C is a constant. Hence, overall,

$$f(x) = \begin{cases} g(x), & |x| > a, \\ g(x) + C, & |x| \leq a. \end{cases}$$

So it remains to find the constant C.

We integrate Equation (7) from $x = -a$ to $x = a$ to get:

$$\begin{aligned} \int_{-a}^{a} f(x)\mathrm{d}x &= \int_{-a}^{a} g(x)\mathrm{d}x + 2a\lambda \int_{-a}^{a} f(y)\mathrm{d}y, \\ (1 - 2\lambda a) \int_{-a}^{a} f(x)\mathrm{d}x &= \int_{-a}^{a} g(x)\mathrm{d}x = G. \end{aligned}$$

Hence,

$$f(x) = \frac{G}{1 - 2\lambda a}, \qquad 2\lambda a \neq 1.$$

Back-substitution into Equation (7) gives:

$$f(x) = g(x) + \frac{\lambda G}{1 - 2\lambda a}, \qquad |x| \leq a.$$

So the general solution is thus:

$$f(x) = \begin{cases} g(x), & |x| > a, \\ g(x) + \frac{\lambda G}{1 - 2\lambda a}, & |x| \leq a. \end{cases}$$

3. The kernel of the FIE

$$\psi(x) = \lambda \int_a^b K(x,y)\psi(y)\mathrm{d}y$$

has the form

$$K(x,y) = \sum_{n=1}^{\infty} h_n(x) g_n(y),$$

where the $h_n(x)$ form a complete orthonormal set of functions over the interval $[a,b]$.

(a) Show that the eigenvalues λ_i are given by:

$$\left| M - \lambda^{-1}\mathbb{I} \right| = 0,$$

where M is the matrix with elements

$$M_{k\ell} = \int_a^b g_k(s) h_\ell(s)\mathrm{d}s.$$

If the corresponding solutions are $\psi^{(i)}(x) = \sum_{n=1}^{\infty} a_n^{(i)} h_n(x)$, find an expression for the $a_n^{(i)}$.

(b) Obtain the eigenvalues and eigenfunctions over the interval $[0, 2\pi]$ if

$$K(x,y) = \sum_{n=1}^{\infty} \frac{1}{n} \cos(nx)\cos(ny).$$

We have:

$$y_i = \lambda_i K * y_i. \tag{8}$$

By completeness of the h_n's we have:

$$y_i = \sum_{n=1}^{\infty} a_n^{(i)} h_n(x). \tag{9}$$

Hence:

$$y_i = \lambda_i \int_a^b \sum_{n=1}^{\infty} h_n(x) g_n(s) y_i(s)\mathrm{d}s,$$

and

$$\begin{aligned}\sum_{n=1}^{\infty} a_n^{(i)} h_n(x) &= \lambda_i \int_a^b \sum_{n=1}^{\infty} h_n(x) g_n(s) \sum_{p=1}^{\infty} a_p^{(i)} h_p(s)\mathrm{d}s, \\ &= \lambda_i \sum_{n=1}^{\infty}\sum_{p=1}^{\infty} h_n(x)\langle g_n, h_p\rangle a_n^{(i)}, \\ &= \lambda_i \sum_{n=1}^{\infty} h_n(x)\left(\sum_{p=1}^{\infty}\langle g_n, h_p\rangle a_n^{(i)}\right).\end{aligned}$$

Equate coefficients of the linearly independent functions h_n:

$$a_n^{(i)} = \lambda_i \sum_{p=1}^{\infty}\langle g_n, h_p\rangle a_p^{(i)}.$$

We identify $M_{np} = \langle g_n, h_p\rangle$. Hence, we recover the infinite-dimensional eigenvalue problem required in part (a):

$$a_n^{(k)} = \lambda_k \sum_{p=1}^{\infty} M_{np} a_p^{(k)}$$

(we switch to using k for the eigenvalue label).

For part (b), we work out:

$$M_{np} = \int_0^{2\pi} \frac{1}{n}\cos(nx)\cos(px)\mathrm{d}x = \frac{\pi}{n}\delta_{np}.$$

The EVP is thus:

$$a_n^{(k)} = \lambda_k \frac{\pi}{n} a_n^{(k)},$$

By inspection, the solution is:

$$a_n^{(k)} = \delta_{nk},$$

hence

$$\lambda_k = \frac{k}{\pi}.$$

From (9), the eigenfunctions are:

$$y_k \propto \sum_{n=1}^{\infty} a_n^{(i)} h_n(x).$$

(note the starting-point in the sum). Hence,

$$\begin{aligned} y_k &\propto \sum_{n=1}^{\infty}\delta_{nk}\cos(nx), \\ &\propto \cos(kx), \\ &= \frac{1}{\sqrt{\pi}}\cos(kx).\end{aligned}$$

4. Use Fredholm theory to show that, for the kernel

$$K(x,z) = (x+z)\mathrm{e}^{x-z},$$

over the interval $[0,1]$, the resolvent kernel is:

$$\Gamma(x,z;\lambda) = \frac{\mathrm{e}^{x-z}\left[(x+z) - \lambda\left(\frac{1}{2}x + \frac{1}{2}z - xz - \frac{1}{3}\right)\right]}{1 - \lambda - \frac{1}{12}\lambda^2}.$$

Hence, solve

$$y(x) = x^2 + 2\int_0^1 (x+z)\mathrm{e}^{x-z}y(z)\mathrm{d}z,$$

Leave your answer in terms of I_n, where $I_n = \int_0^1 u^n \mathrm{e}^{-u}\mathrm{d}u$.

We have:

$$\begin{aligned}
K_2(x,t) &= K * K, \\
&= \int_0^1 K(x,s)K(s,t)\mathrm{d}s, \\
&= \int_0^1 \mathrm{e}^{x-s}\mathrm{e}^{s-t}(x+s)(s+t)\mathrm{d}s, \\
&= \mathrm{e}^{x-t}\left(\tfrac{1}{2}x + xt + \tfrac{1}{3} + \tfrac{1}{2}t\right), \\
&= \tfrac{1}{2}(x+t)\mathrm{e}^{x-t} + \mathrm{e}^{x-t}\left(xt + \tfrac{1}{3}\right), \\
&= \tfrac{1}{2}K + \mathrm{e}^{x-t}\left(xt + \tfrac{1}{3}\right).
\end{aligned}$$

Furthermore,

$$\begin{aligned}
K_3(x,t) &= K_2 * K, \\
&= \int_0^1 K_2(x,s)K(s,t)\mathrm{d}s, \\
&= \int_0^1 \left[\tfrac{1}{2}K(x,s) + \mathrm{e}^{x-s}\left(xs + \tfrac{1}{3}\right)\right]K(s,t)\mathrm{d}s, \\
&= \tfrac{1}{2}K * K + \int_0^1 \mathrm{e}^{x-s}\left(xs + \tfrac{1}{3}\right)K(s,t)\mathrm{d}s, \\
&= \tfrac{1}{2}K_2 + \mathrm{e}^{x-t}\int_0^1 (xs + \tfrac{1}{3})(s+t)\mathrm{d}s, \\
&= \tfrac{1}{2}K_2 + \mathrm{e}^{x-t}\left[\tfrac{1}{3}(x+t) + \tfrac{1}{2}\left(xt + \tfrac{1}{3}\right)\right], \\
&= \tfrac{1}{2}K_2 + \tfrac{1}{3}K(x,t) + \tfrac{1}{2}\mathrm{e}^{x-t}\left(xt + \tfrac{1}{3}\right), \\
&= \tfrac{1}{2}K_2 + \tfrac{1}{3}K(x,t) + \tfrac{1}{2}\left[K_2 - \tfrac{1}{2}K\right], \\
&= K_2 + \left(\tfrac{1}{3} - \tfrac{1}{4}\right)K, \\
&= K_2 + \tfrac{1}{12}K.
\end{aligned}$$

Thus, at next order, we have:

$$K_4 = K_3 + \tfrac{1}{12}K_2.$$

Hence, for $n \geq 2$, we have a difference equation:

$$K_{n+2} = K_{n-1} + \tfrac{1}{12}K_n, \qquad n \geq 2.$$

This is the key step which now enables us to sum up the resolvent operator. We have:

$$\begin{aligned}
\Gamma &= \sum_{n=0}^{\infty} \lambda^n K_{n+1}, \\
&= K + \lambda K_2 + \sum_{n=2} \lambda^n K_{n+1}, \\
&\overset{p+1=n}{=} K + \lambda K_2 + \lambda \sum_{p=1} \lambda^p K_{p+2}, \\
&\overset{\text{Diff. Eq.}}{=} K + \lambda K_2 + \lambda \left[\sum_{p=1}^{\infty} \lambda^p K_{p+1} + \tfrac{1}{12} \sum_{p=1}^{\infty} \lambda^p K_p \right], \\
&= K + \lambda K_2 + \lambda \left[\left(\sum_{p=0}^{\infty} \lambda^p K_{p+1} - K \right) + \tfrac{1}{12} \sum_{p=1}^{\infty} \lambda^p K_p \right], \\
&= K(1-\lambda) + \lambda K_2 + \lambda\Gamma + \tfrac{1}{12}\lambda \sum_{p=1}^{\infty} \lambda^p K_p, \\
&\overset{q+1=p}{=} K(1-\lambda^2) + \lambda K_2 + \Gamma + \tfrac{1}{12}\lambda^2 \sum_{q=0}^{\infty} \lambda^q K_{q+1}, \\
&= K(1-\lambda) + \lambda K_2 + \Gamma + \tfrac{1}{12}\lambda^2 \Gamma.
\end{aligned}$$

Hence, the resolvent is:

$$\Gamma = \frac{K(1-\lambda) + \lambda K_2}{1 - \lambda - \frac{1}{12}\lambda^2}, \tag{10}$$

or

$$\Gamma(x, z; \lambda) = \frac{K(x, z) + \lambda \left[xz + \frac{1}{3} - \frac{1}{2}(x+z) \right] \mathrm{e}^{x-z}}{1 - \lambda - \frac{1}{12}\lambda^2}, \tag{11}$$

For the particular integral equation, we have $\lambda = 2$ (not an eigenvalue). Hence:

$$\begin{aligned}
\Gamma(x, z; \lambda) &\overset{\text{Eq. (10)}}{=} \frac{K(x, z)(1-\lambda) + \lambda K_2(x, z)}{1 - \lambda - \frac{1}{12}\lambda^2}, \\
&\overset{\lambda=2}{=} \frac{-K + 2K_2(x, z)}{-4/3}, \\
&= \frac{-K(x, z) + 2\left[\frac{1}{2}K(x, z) + \mathrm{e}^{x-t}\left(xt + \frac{1}{3}\right)\right]}{-4/3}, \\
&= \frac{(2xz + \frac{2}{3})\mathrm{e}^{x-z}}{-4/3}.
\end{aligned}$$

Hence, the solution of the FIE is:

$$\begin{aligned}
y(x) &= f(x) + \lambda \int_0^1 \Gamma(x, z; \lambda) f(z) \mathrm{d}z, \\
&= x^2 + 2 \int_0^1 \frac{(2xz + \frac{2}{3}) \mathrm{e}^{x-z} z^2}{-4/3} \mathrm{d}z, \\
&= x^2 - \int_0^1 (3xz^3 + z^2) \mathrm{e}^{x-z} \mathrm{d}z, \\
&= x^2 - (3xI_3 + I_2) \mathrm{e}^x.
\end{aligned}$$